Functional Analysis – Exercise sheet 11

Issued: Monday 12 July 2010
Due: Monday 19 July 2010 by 12 p.m. in the designated "Funktionalanalysis" box on the 1st floor
Students who will be attending the Mon 19 July tutorial have to hand in their solution sheets at 10:15 in class.
Info: `www.math.lmu.de/~michel/SS10_FA.html`

> *The full mark in each exercise is 10 points. Correct answers without proofs are not accepted. Each step should be justified. You can hand in the solutions either in German or in English.*

Exercise 41. Prove that the weak closure of the unit sphere in ℓ^2 is the unit ball in ℓ^2.

Exercise 42. Prove that the map $x = (x_1, x_2, x_3, \dots) \mapsto \sum_n x_n$ is

(i) weakly continuous on ℓ^1

(ii) but not *weakly-$*$* continuous.

Exercise 43. Let $(X, \| \ \|_X)$ and $(Y, \| \ \|_Y)$ be normed vector spaces and assume that X is reflexive (i.e., $X \cong X^{**}$ canonically – see Problem in class no. 38). Let $T : X \to Y$ a linear map such that

$$ x_n \xrightarrow[n\to\infty]{\text{(weakly in } X)} 0 \qquad \Rightarrow \qquad Tx_n \xrightarrow[n\to\infty]{\text{(weakly in } Y)} 0 \,. $$

Prove that T is bounded. (*Hint:* weak (sequential) compactness of the unit ball in a reflexive space.)

Exercise 44. Let X_1, X_2 be two subspaces in the Banach space X such that $X_1 \cap X_2 = \emptyset$ and $\mathrm{Span}\,\{X_1, X_2\} = X$. Let $P : X \to X$ be the projector to X_1 along X_2 (i.e., if $x = x_1 + x_2$ with $x_1 \in X_1$ and $x_2 \in X_2$ then $Px = x_1$). Prove that the operator P is bounded if and only if the subspaces X_1, X_2 are closed. (*Hint:* closed graph theorem.)

NOTICE: this is the LAST exercise sheet!