Functional Analysis – Exercise sheet 4

Mathematisches Institut der LMU – SS2010
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Due: Monday 31 May 2010 by 12 p.m. in the designated "Funktionalanalysis" box on the 1st floor
Students who will be attending the Mon 31 May tutorial have to hand in their solution sheets at 10:15 in class.
No exercise sheet will be issued on 24 May.
Info: `www.math.lmu.de/~michel/SS10_FA.html`

The full mark in each exercise is 10 points. Correct answers without proofs are not accepted. Each step should be justified. You can hand in the solutions either in German or in English.

Exercise 13. Let $a : [0,1] \to \mathbb{C}$ be a measurable function. Let $T_a : L^p([0,1]) \to L^q([0,1])$, with $p, q \in [1,\infty]$, be the operator of pointwise multiplication by a, i.e., $(T_a f)(x) := a(x)f(x)$. Find the necessary and sufficient condition on a such that T_a is continuous

(i) when $p < q$,

(ii) when $p \geqslant q$.

Exercise 14. Let $K : [0,1] \times [0,1] \to \mathbb{C}$ be a continuous function. Consider the integral operator $f \stackrel{T}{\longmapsto} Tf$ given by $(Tf)(x) = \int_0^1 K(x,y)\, f(y)\, \mathrm{d}y$ and denote by $\|T\|_{p\to q}$ its norm as a $L^p([0,1]) \to L^q([0,1])$ map.

(i) Show that $\|T\|_{1\to 1} \;\leqslant\; \sup_{y\in\mathbb{R}} \int_0^1 |K(x,y)|\,\mathrm{d}x\,.$

(ii) Show that $\|T\|_{2\to 2} \;\leqslant\; \Big(\int_{[0,1]^2} |K(x,y)|^2\,\mathrm{d}x\,\mathrm{d}y\Big)^{1/2}.$

(iii) Show that $\|T\|_{\infty\to\infty} \;\leqslant\; \sup_{x\in\mathbb{R}} \int_0^1 |K(x,y)|\,\mathrm{d}y\,.$

(iv) Consider T as a $C([0,1]) \to C([0,1])$ map and take a sequence $\{f_n\}_{n=1}^\infty$ in $C([0,1])$ such that $\|f_n\|_\infty \leqslant 1$ for all n. Show that the sequence $\{Tf_n\}_{n=1}^\infty$ has a convergent subsequence in the $\|\ \|_\infty$-norm.

Exercise 15. Let $(X, \| \ \|_X)$ and $(Y, \| \ \|_Y)$ be two normed spaces and let $T : X \to Y$ be a linear map.

(i) Show that if $\dim X < \infty$ then T is bounded and $\exists\, x \in X$, $x \neq 0$, such that $\|Tx\|_Y = \|T\| \, \|x\|_X$.

(ii) Show that if T is bounded then $\ker T$ (the kernel of T) is closed in $(X, \| \ \|_X)$.

(iii) Is the converse of (ii) true? Give a proof or a counterexample.

(iv) Assume that $\ker T$ is closed and that $\dim(\operatorname{Im} T) < \infty$ ($\operatorname{Im} T$ is the image of T). Show that T is bounded. (*Hint:* use Problem 13 discussed in class.)

Exercise 16. Let $f \in C([0,1])$. Define the polynomials $p_{n,k}$ and $B_n^{(f)}$ by

$$p_{n,k}(x) := \binom{n}{k} x^k (1-x)^{n-k} \quad \text{and} \quad B_n^{(f)}(x) := \sum_{k=0}^{n} f\Big(\frac{k}{n}\Big)\, p_{n,k}(x)\,, \qquad \begin{array}{c} x \in [0,1]\,, \\ n,\, k \text{ integers} \\ \text{with } 0 \leqslant k \leqslant n. \end{array}$$

(i) Show that the following identity holds $\forall x \in [0,1]$ and $\forall n = 0, 1, 2, \ldots$:

$$\sum_{k=0}^{n} (k - nx)^2 p_{n,k}(x) \;=\; nx(1-x)\,. \tag{1}$$

(*Hint:* use the combinatorial properties of $p_{n,k}$ discussed in Problem 14 in class.)

(ii) Let $\delta > 0$. Using (1) prove the estimate

$$\sum_{k \text{ s.t. } |k-nx| \geqslant n\delta} p_{n,k}(x) \;\leqslant\; \frac{1}{4n\delta^2}\,. \tag{2}$$

(In the inequality selecting k it is understood that k is an integer between 0 and n.)

(iii) Using (2) show that

$$\lim_{n\to\infty} \big\| f - B_n^{(f)} \big\|_\infty \;=\; 0$$

i.e., the corresponding polynomial $B_n^{(f)}$'s approximate f uniformly.