Functional Analysis II – Problem sheet 12

Mathematisches Institut der LMU – SS2009
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Handout: 14.07.2009
Due: Tuesday 21.07.2009 by 1 p.m. in the "Funktionalanalysis II" box
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Grader: Ms. S. Sonner – Übungen on Wednesdays, 4,15 - 6 p.m., room C-111

Exercise 32. Let A be a symmetric operator on a (dense domain of a) Hilbert space $\mathcal{H}$. Prove that A is essentially self-adjoint *if and only if* A has one and only one self-adjoint extension.

NOTICE! MODIFIED EXERCISE! (16/07/2009) Since proving one of the implications requires tools that have not been discussed yet, the new version of Exercise 32 is: Prove that A is essentially self-adjoint *implies* that A has one and only one self-adjoint extension.

Exercise 33. Consider on the Hilbert space $L^2(\mathbb{R})$ the operator

$$\begin{array}{rccc} D \ : & H^1(\mathbb{R}) & \to & L^2(\mathbb{R}) \\ & \psi & \longmapsto & i\psi' \end{array}$$

where ψ' is the *weak derivative* of ψ. Prove that D is self-adjoint.

Exercise 34. Consider the Laplacian operator $\Delta : H^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ (d positive integer). As a special case of the "Spectral Theorem in multiplication form" discussed in the class, prove that Δ is isomorphically equivalent to a multiplication operator. In other words, exhibit

- a new Hilbert space $L^2(\Omega, \mathrm{d}\mu)$
- an isomorphism $U : L^2(\mathbb{R}^d, \mathrm{d}x) \xrightarrow{\cong} L^2(\Omega, \mathrm{d}\mu)$
- the action of $U^*\Delta U$ as a multiplication operator on $L^2(\Omega, \mathrm{d}\mu)$
- the domain of $U^*\Delta U$.

(*Hint:* use the Fourier transform.)