Functional Analysis II – Problem sheet 7

Mathematisches Institut der LMU – SS2009
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Due: Tuesday 16.06.2009 by 1 p.m. in the "Funktionalanalysis II" box
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Grader: Ms. S. Sonner – Übungen on Wednesdays, 4,30 - 6 p.m., room C-111

Exercise 18. Let A be a self-adjoint $n \times n$ matrix on the complex numbers ($n > 1$). Assume that A has a *degenerate* eigenvalue λ, i.e., there are at least two linearly independent vectors $e_1, e_2 \in \mathbb{C}^n$ such that $Ae_1 = \lambda e_1$ and $Ae_2 = \lambda e_2$. Explain whether A admits a cyclic vector or not.

Exercise 19. Let A be an operator on a Hilbert space $\mathcal{H}$ that is unitarily equivalent to the multiplication by x acting on the L^2-functions over a compact subset of $\mathbb{R}$. In other words, assume that there exists a compact set $K \subset \mathbb{R}$, a Borel measure μ on K, and a unitary map $U : \mathcal{H} \to L^2(K)$ such that $UAU^* : L^2(K, \mathrm{d}\mu(x)) \to L^2(K, \mathrm{d}\mu(x))$ is the multiplication operator $\psi(x) \mapsto x\psi(x)$. Prove that A is bounded and self-adjoint. Construct a cyclic vector for A. (*Please:* construct a *not too complicated* cyclic vector…!) Note that here you are considering the reverse than the situation in Lemma 1.48.

Exercise 20. Let $\mathcal{H}$ be an infinite dimensional separable Hilbert space and let $\{\psi_n\}_{n=1}^{\infty}$ be an orthonormal basis of $\mathcal{H}$. Let $\{a_n\}_{n=1}^{\infty} \subset \ell^{\infty}(\mathbb{R})$ where the a_n's are pairwise distinct. Let $A : \mathcal{H} \to \mathcal{H}$ be the linear operator defined to act as $A\psi_n := a_n\psi_n$ on the basis and extended by linearity. Prove that A is bounded and self-adjoint. Prove that A admits a cyclic vector, for example the vector $\psi := \sum_{n=1}^{\infty} 2^{-n/2}\psi_n$. (*Hint:* connect this problem with Exercise 19 above and use the thesis stated there. To this aim, you need to exhibit a compact $K \subset \mathbb{R}$, a measure μ, and a unitary isomorphism $\mathcal{H} \cong L^2(K, \mathrm{d}\mu(x))$ and you need to *prove* that A acts on $L^2(K, \mathrm{d}\mu(x))$ as the multiplication by x. To be sure to have fixed all the details, check the role played in this construction by the assumption that the a_n's are real, uniformly bounded, and distinct.)