Functional Analysis II – Problem sheet 5

Mathematisches Institut der LMU – SS2009
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Handout: 27.05.2009
Due: Wednesday 3.06.2009 by 1 p.m. in the "Funktionalanalysis II" box
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Grader: Ms. S. Sonner – Übungen on Wednesdays, 4,30 - 6 p.m., room C-111

Exercise 13. Let A be a bounded self-adjoint operator on the Hilbert space $\mathcal{H}$. Prove that

$$\lambda \in \mathrm{Spec}_{\mathrm{dis}}(A) \quad \Leftrightarrow \quad \begin{cases} \lambda \text{ is an isolated point of } \mathrm{Spec}(A) \\ \lambda \text{ is an eigevalue with finite multiplicity} \end{cases}$$

Recall that λ being isolated means that, for some $\varepsilon > 0$, $(\lambda - \varepsilon, \lambda + \varepsilon) \cap \mathrm{Spec}(A) = \{\lambda\}$. By multiplicity of the eigenvalue λ one means the dimension of the corresponding eigenspace, so that *finite multiplicity* means that $\dim\{\psi \in \mathcal{H} \mid A\psi = \lambda\psi\} < \infty$. Can you provide concrete counterexamples when one of the two conditions on the right is relaxed?

Exercise 14. [*The Weyl's criterion for the essential spectrum*] Let A be a bounded self-adjoint operator acting on the Hilbert space $\mathcal{H}$. Show that $\lambda \in \mathrm{Spec}_{\mathrm{ess}}(A)$ if and only if there exists a sequence $\{\psi_n\}_{n=1}^{\infty}$ of *orthonormal* vectors (i.e., $\langle \psi_n, \psi_m \rangle = \delta_{m,n}$) such that $\|A\psi_n - \lambda\psi_n\|_{\mathcal{H}} \to 0$ as $n \to \infty$. Compare this result to the general statement of the Weyl's criterion ($\to$ Exercise 3): actually that statement does not exclude that the whole $\mathrm{Spec}(A)$, not only $\mathrm{Spec}_{\mathrm{ess}}(A)$, might be characterised by orthonormal Weyl's sequences (and the proof given there, being a proof by contradiction and not constructive, does not give any evidence on how the Weyl's sequence has to be). Can you exclude that? In other words, are there points in $\mathrm{Spec}(A)$ for which it is *not* possible to find an orthonormal Weyl's sequence?