Functional Analysis II – Problem sheet 4

Mathematisches Institut der LMU – SS2009
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Due: Tuesday 26.05.2009 by 1 p.m. in the "Funktionalanalysis II" box
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Exercise 10. [*Exponentiating a bounded self-adjoint operator*] Let A be a bounded self-adjoint operator acting on the separable Hilbert space $\mathcal{H}$.

10.1) Prove that the operator series $\displaystyle\sum_{n=0}^{\infty}\frac{A^n}{n!}$ converges in the norm operator topology (it is understood that $A^0 := \mathbb{1}$). This is a possible way to give meaning to the operator e^A. Call it temporarily $(e^A)_{\text{series}}$.

10.2) Now *define* the operator e^A by means of the functional calculus, i.e., as the operator $f(A)$ where $f : \text{Spec}(A) \to \mathbb{R}$ is the function $f(\lambda) = e^\lambda$ (f is bounded and continuous so the definition is well-posed). Do the operators e^A and $(e^A)_{\text{series}}$ coincide? Why?

10.3) For any $t \in \mathbb{R}$ define analogously the operator e^{tA}. Prove that (independently of t) e^{tA} is self-adjoint and give an upper bound to its norm $\|e^{tA}\|$. Also, prove that $\forall\, t, s \in \mathbb{R}$ one has $e^{tA}e^{sA} = e^{(t+s)A}$. Prove that e^{tA} is invertible and determine its inverse.

10.4) Prove that the operator-valued function $\mathbb{R} \ni t \mapsto e^{tA}$ is norm-continuous on $\mathbb{R}$ and *Lipschitz norm-continuous* on any *bounded* subset of $\mathbb{R}$. Recall that an operator-valued function $t \mapsto B_t$ is Lipschitz norm-continuous if $\|B_t - B_s\|_{BL(\mathcal{H})} \leqslant C|t-s|$ for some constant $C > 0$ independent of t, s. Estimate such a constant.

10.5) Prove that the operator-valued function $t \mapsto e^{tA}$ is differentiable in the norm operator topology and compute its derivative (which is an operator!) in $t = 0$.

Exercise 11. [*The Stone's formula*] Let A be a bounded self-adjoint operator acting on the separable Hilbert space $\mathcal{H}$.

11.1) Let $a, b \in \mathbb{R}$ (for simplicity assume $a < b$) and let $\varepsilon > 0$. Show that by the functional calculus the integral

$$\frac{1}{2\pi i}\int_a^b \Big((A-\lambda-i\varepsilon)^{-1} - (A-\lambda+i\varepsilon)^{-1} \Big)\,\mathrm{d}\lambda \qquad (*)$$

is well defined and is a bounded self-adjoint operator on $\mathcal{H}$.

11.2) Let $\{P_\Omega\}_\Omega$ be the spectral measure associated with the operator A. Prove that

$$\frac{1}{2\pi i}\int_a^b \Big((A-\lambda-i\varepsilon)^{-1} - (A-\lambda+i\varepsilon)^{-1} \Big)\, \mathrm{d}\lambda \xrightarrow[\text{strongly}]{\varepsilon\to 0} \frac{1}{2}\Big(P_{[a,b]} + P_{(a,b)} \Big) .$$

This is the *Stone's formula.*

11.3) Does the above limit hold in norm?

Exercise 12. Consider the Hilbert space $L^2([0,1])$ and let M be the self-adjoint multiplication operator acting on every $\psi \in L^2([0,1])$ as $(M\psi)(x) := x\psi(x)\ \forall\, x \in [0,1]$. Let $f : [0,1] \to \mathbb{R}$ be a bounded Borel measurable function. Define the action of the operator $f(M)$.