Functional Analysis II – Problem sheet 3

Mathematisches Institut der LMU – SS2009
Prof. Dr. P. Müller, Dr. A. Michelangeli

Handout: 13.05.2009
Due: Tuesday 19.05.2009 by 1 p.m. in the "Funktionalanalysis II" box
Questions and infos: Dr. A. Michelangeli, office B-334, michel@math.lmu.de
Grader: Ms. S. Sonner – Übungen on Wednesdays, 4,30 - 6 p.m., room C-111

Exercise 6. Let $\mathcal{B}(\mathbb{R})$ be the family of Borel sets in $\mathbb{R}$. Recall that a *projection-valued measure* is a family $\{P_\Omega\}_{\Omega\in\mathcal{B}(\mathbb{R})}$ of bounded operators on a Hilbert space satisfying the following properties:

(a) Each P_Ω is an orthogonal projection.

(b) $P_\emptyset = \mathbf{0}$ (the projection associated to the empty set is the null operator) and $P_\mathbb{R} = \mathbb{1}$.

(c) If $\Omega = \bigcup_{n=1}^\infty \Omega_n$ with $\Omega_n \cup \Omega_m = \emptyset$ for all $n \neq m$ then

$$P_\Omega = s\text{-}\lim_{N\to\infty} \sum_{n=1}^{N} P_{\Omega_n}$$

(d) $P_{\Omega_1\cap\Omega_2} = P_{\Omega_1}P_{\Omega_2}$.

The goal of this problem is to prove that actually property (d) follows from (a) and (c) by abstract considerations.

6.1) Prove the operator identity $P_{\Omega_1\cup\Omega_2} + P_{\Omega_1\cap\Omega_2} = P_{\Omega_1} + P_{\Omega_2}$.

6.2) Prove that $\Omega_1 \cap \Omega_2 = \emptyset \Rightarrow P_{\Omega_1}P_{\Omega_2} = \mathbf{0}$. (*Hint:* square the identity in the previous point and prove by algebraic manipulations only that $P_{\Omega_1}P_{\Omega_2}$ has to be at the same time self-adjoint and anti-self-adjoint.)

6.2) Use the above points to deduce (d) from (a) and (c).

Exercise 7. Recall from the class that given a projection-valued measure $\{P_\Omega\}_{\Omega\in\mathcal{B}(\mathbb{R})}$ on $\mathcal{B}(\mathbb{R})$, the Borel sets of $\mathbb{R}$, one defines the integral of a bounded Borel function f with respect to the measure $\{P_\Omega\}_{\Omega\in\mathcal{B}(\mathbb{R})}$ as that bounded operator

$$\int_\mathbb{R} f(\lambda)\,\mathrm{d}P(\lambda) \;:=\; \lim_{n\to\infty}\int_\mathbb{R} f_n(\lambda)\,\mathrm{d}P(\lambda) \tag{$*$}$$

(the above limit being meant in the norm operator sense) for some sequence $\{f_n\}_{n=1}^\infty$ of simple functions approximating f uniformly, i.e., such that $\|f - f_n\|_\infty \to 0$. In the class the well-posedness of $\int_\mathbb{R} g(\lambda)\,\mathrm{d}P(\lambda)$ was proved for *simple* functions, together with the estimate

$$\Big\| \int_\mathbb{R} g(\lambda)\,\mathrm{d}P(\lambda) \Big\|_{BL(\mathcal{H})} \;\leqslant\; \|g\|_\infty\,. \tag{$**$}$$

Prove that, due to $(**)$, the limit operator defined in $(*)$ is unique (i.e., independent of the approximating sequence of simple functions one considers) and $(**)$ holds for any bounded Borel function.

Exercise 8. Given a projection-valued measure $\{P_\Omega\}_{\Omega\in\mathcal{B}(\mathbb{R})}$ on $\mathcal{B}(\mathbb{R})$, the Borel sets of $\mathbb{R}$, consider the map

$$\begin{aligned}\Psi \;:\; \{\text{bounded Borel functions}\} &\longrightarrow BL(\mathcal{H}) = \{\text{bounded operators on } \mathcal{H}\}\\ f &\longrightarrow \int_{\mathbb{R}} f(\lambda)\,\mathrm{d}P(\lambda)\end{aligned}$$

A key point in the proof of the Spectral Theorem is that Ψ is a $*$-homomorphism between the two spaces above and reproduces the functional calculus for f. In particular, in this exercise prove that Φ preserves the product, i.e.,

$$\Psi(fg) \;=\; \Psi(f)\Psi(g) \tag{$\bullet$}$$

for any two bounded Borel functions f and g. (*Hint:* use the result of Exercise 4 to approximate f and g with simple functions and prove $(\bullet)$ for simple functions. To this aim, the relation $P_{\Omega_1\cap\Omega_2} = P_{\Omega_1}P_{\Omega_2}$ proved in Exercise 6.2 will be crucial [if you did not solve Exercise 6, just grab such an identity from there]. Last, use $(*)$ of Exercise 7 to perform a limiting argument and lift your result from simple functions to generic bounded Borel functions.)

Exercise 9. Let $T\in BL(\mathcal{H})$, that is, a bounded operator on the Hilbert space $\mathcal{H}$. Assume that T is invertible with bounded inverse. Find a condition on *how close* to T another bounded operator S has to be, *in the norm operator sense*, so that S too is invertible with bounded inverse. (*Hint:* deduce a "Neumann series" for S^{-1}. The condition on $\|T-S\|$ will immediately follow when you impose the convergence of the series.)