Bosonic/Fermionic Fock Space: quick recap

(from "Operator Algebras and Quantum Statistical Mechanics" vol. 2 by O. Bratteli and D. W. Robinson)

Assume that the states of each particle form a complex Hilbert space $\mathfrak{h}$ and let $\mathfrak{h}^n = \mathfrak{h} \otimes \mathfrak{h} \otimes \cdots \otimes \mathfrak{h}$ denote the n-fold tensor product of $\mathfrak{h}$ with itself. Further introduce the *Fock space* $\mathfrak{F}(\mathfrak{h})$ by

$$\mathfrak{F}(\mathfrak{h}) = \bigoplus_{n \geq 0} \mathfrak{h}^n \ ,$$

where $\mathfrak{h}^0 = \mathbb{C}$. Thus a vector $\psi \in \mathfrak{F}(\mathfrak{h})$ is a sequence $\{\psi^{(n)}\}_{n \geq 0}$ of vectors $\psi^{(n)} \in \mathfrak{h}^n$ and $\mathfrak{h}^n$ can be identified as the closed subspace of $\mathfrak{F}(\mathfrak{h})$ formed by the vectors with all components except the nth equal to zero.

In order to introduce the subspaces relevant to the description of bosons and fermions we first define operators $P_{\pm}$ on $\mathfrak{F}(\mathfrak{h})$ by

$$P_+(f_1 \otimes f_2 \otimes \cdots \otimes f_n) = (n!)^{-1} \sum_{\pi} f_{\pi_1} \otimes f_{\pi_2} \otimes \cdots \otimes f_{\pi_n} \ ,$$

$$P_-(f_1 \otimes f_2 \otimes \cdots \otimes f_n) = (n!)^{-1} \sum_{\pi} \varepsilon_\pi f_{\pi_1} \otimes f_{\pi_2} \otimes \cdots \otimes f_{\pi_n}$$

for all $f_1, \ldots, f_n \in \mathfrak{h}$. The sum is over all permutations $\pi; (1, 2, \ldots, n) \mapsto (\pi_1, \pi_2, \ldots, \pi_n)$ of the indices and ε_π is one if π is even and minus one if π is odd. Extension by linearity yields two densely defined operators with $\|P_{\pm}\| = 1$ and the $P_{\pm}$ extend by continuity to bounded operators of norm one. The P_+ and P_- restricted to $\mathfrak{h}^n$, are the projections onto the subspaces of $\mathfrak{h}^n$ corresponding to the one-dimensional unitary representations $\pi \mapsto 1$ and $\pi \mapsto \varepsilon_\pi$ of the permutation group of n elements, respectively. The *Bose-Fock space* $\mathfrak{F}_+(\mathfrak{h})$ and the *Fermi-Fock space* $\mathfrak{F}_-(\mathfrak{h})$ are then defined by

$$\mathfrak{F}_{\pm}(\mathfrak{h}) = P_{\pm}\, \mathfrak{F}(\mathfrak{h})$$

and the corresponding n-particle subspaces $\mathfrak{h}_{\pm}^n$ by $\mathfrak{h}_{\pm}^n = P_{\pm}\, \mathfrak{h}^n$. We also define a *number operator* N on $\mathfrak{F}(\mathfrak{h})$ by

$$D(N) = \left\{ \psi;\ \psi = \{\psi^{(n)}\}_{n \geq 0},\ \sum_{n \geq 0} n^2 \|\psi^{(n)}\|^2 < +\infty \right\}$$

and

$$N\psi = \{n\psi^{(n)}\}_{n \geq 0}$$

for each $\psi \in D(N)$. It is evident that N is selfadjoint since it is already given in its spectral representation. Note that e^{itN} leaves the subspaces $\mathfrak{F}_\pm(\mathfrak{h})$ invariant. We will also use N to denote the selfadjoint restrictions of the number operator to these subspaces.

The peculiar structure of Fock space allows the amplification of operators on $\mathfrak{h}$ to the whole spaces $\mathfrak{F}_\pm(\mathfrak{h})$ by a method commonly referred to as *second quantization*. This is of particular interest for selfadjoint operators and unitaries.

If H is selfadjoint operator on $\mathfrak{h}$, one can define H_n on $\mathfrak{h}_\pm^n$ by setting $H_0 = 0$ and

$$H_n(P_\pm(f_1 \otimes \cdots \otimes f_n)) = P_\pm\left(\sum_{i=1}^{n} f_1 \otimes f_2 \otimes \cdots \otimes Hf_i \otimes \cdots \otimes f_n\right)$$

for all $f_i \in D(H)$, and then extending by continuity. The direct sum of the H_n is essentially selfadjoint because (1) it is symmetric and hence closable, (2) it has a dense set of analytic vectors formed by finite sums of (anti-) symmetrized products of analytic vectors of H. The selfadjoint closure of this sum is called the second quantization of H and is denoted by $d\Gamma(H)$. Thus

$$d\Gamma(H) = \overline{\bigoplus_{n\geq 0} H_n} \ .$$

The simplest example of this second quantization is given by choosing $H = \mathbb{1}$, one then has

$$d\Gamma(\mathbb{1}) = N \ .$$

If U is unitary, U_n is defined by $U_0 = \mathbb{1}$ and by setting

$$U_n(P_\pm(f_1 \otimes f_2 \otimes \cdots \otimes f_n)) = P_\pm(Uf_1 \otimes Uf_2 \otimes \cdots \otimes Uf_n)$$

and extending by continuity. The second quantization of U is denoted by $\Gamma(U)$, where

$$\Gamma(U) = \bigoplus_{n\geq 0} U_n \ .$$

Note that $\Gamma(U)$ is unitary. The notation $d\Gamma$ and Γ is chosen because if $U_t = e^{itH}$ is a strongly continuous one-parameter unitary group, then

$$\Gamma(U_t) = e^{itd\Gamma(H)} \, .$$

Next we wish to describe two C^*-algebras of observables associated with bosons and fermions, respectively. Both algebras are defined with the aid of particle *"annihilation"* and *"creation" operators* which are introduced as follows. For each $f \in \mathfrak{h}$ we define operators $a(f)$, and $a^*(f)$, on $\mathfrak{F}(\mathfrak{h})$ by initially setting $a(f)\psi^{(0)} = 0, a^*(f)\psi^{(0)} = f$, $f \in \mathfrak{h}$, and

$$a(f)(f_1 \otimes f_2 \otimes \cdots \otimes f_n) = n^{1/2}(f, f_1) f_2 \otimes f_3 \otimes \cdots \otimes f_n \ ,$$

$$a^*(f)(f_1 \otimes f_2 \otimes \cdots \otimes f_n) = (n+1)^{1/2} f \otimes f_1 \otimes \cdots \otimes f_n \ .$$

Extension by linearity again yields two densely defined operators and if $\psi^{(n)} \in \mathfrak{h}^n$, one easily calculates that

$$\|a(f)\psi^{(n)}\| \leq n^{1/2}\|f\|\|\psi^{(n)}\|, \qquad \|a^*(f)\psi^{(n)}\| \leq (n+1)^{1/2}\|f\|\,\|\psi^{(n)}\| \ .$$

Thus $a(f)$ and $a^*(f)$ have well-defined extensions to the domain $D(N^{1/2})$ of $N^{1/2}$ and

$$\|a^{\#}(f)\psi\| \leq \|f\|\,\|(N+1)^{1/2}\psi\|$$

for all $\psi \in D(N^{1/2})$, where $a^{\#}(f)$ denotes either $a(f)$ or $a^*(f)$. Moreover, one has the adjoint relation

$$(a^*(f)\varphi, \psi) = (\varphi, a(f)\psi)$$

for all $\varphi, \psi \in D(N^{1/2})$. Finally, we define annihilation and creation operators $a_\pm(f)$ and $a_\pm^*(f)$ on the Fock spaces $\mathfrak{F}_\pm(\mathfrak{h})$ by

$$a_\pm(f) = P_\pm a(f)P_\pm\,, \qquad a_\pm^*(f) = P_\pm a^*(f)P_\pm \ .$$

The relations

$$(a_\pm^*(f)\varphi, \psi) = (\varphi, a_\pm(f)\psi), \qquad \|a_\pm^*(f)\psi\| \ \leq \ \|f\|\,\|(N+1)^{1/2}\psi\|$$

follow from the corresponding relations for $a(f)$ and $a^*(f)$. Moreover,

$$a_\pm(f) = a(f)P_\pm, \qquad a_\pm^*(f) = P_\pm a^*(f)$$

because $a(f)$ leaves the subspaces $\mathfrak{F}_\pm(\mathfrak{h})$ invariant. Note that the maps $f \mapsto a_\pm(f)$ are anti-linear but the maps $f \mapsto a_\pm^*(f)$ are linear.

The physical interpretation of these operators is the following. Let $\Omega = (1, 0, 0, \ldots)$, then Ω corresponds to the zero-particle state, the *vacuum*. The vectors

$$\psi_\pm(f) = a_\pm^*(f)\Omega$$

identify with elements of the one-particle space $\mathfrak{h}$ and hence $a_\pm^*(f)$ "creates" a particle in the state *f*. The vectors

$$\begin{aligned}\psi_\pm(f_1, \ldots, f_n) &= (n!)^{-1/2}a_\pm^*(f_1)\cdots a_\pm^*(f_n)\Omega \\ &= P_\pm(f_1 \otimes \cdots \otimes f_n)\end{aligned}$$

are n-particle states which arise from successive "creation" of particles in the states $f_n, f_{n-1}, \ldots, f_1$. Similarly the $a_\pm(f)$ reduce the number of particles, i.e., they annihilate particles. Note that if $f_i = f_j$ for some pair i, j with $1 \leq i < j \leq n$, then

$$\psi_-(f_1, \ldots, f_n) = P_-(f_1 \otimes \cdots \otimes f_n) = 0$$

by anti-symmetry. Thus it is impossible to create two fermions in the same state. This is the celebrated *Pauli principle* which is reflected by the operator equation

$$a_-^*(f)a_-^*(f) = 0 \ .$$

This last relation is the simplest case of the commutation relations which link the annihilation and creation operators.

One computes straightforwardly that

$$[a_+(f), a_+(g)] = 0 = [a_+^*(f),\, a_+^*(g)] \ ,$$

$$[a_+(f),\, a_+^*(g)] = (f, g)\mathbb{1} \ ,$$

and

$$\{a_-(f),\, a_-(g)\} = 0 = \{a_-^*(f), a_-^*(g)\} \ ,$$

$$\{a_-(f),\, a_-^*(g)\} = (f, g)\mathbb{1} \ ,$$

where we have again used the notation $\{A, B\} = AB + BA$. The first relations are called the *canonical commutation relations* (CCRs) and the second the *canonical anti-commutation relations* (CARs).

Although there is a superficial similarity between these two sets of algebraic rules, the properties of the respective operators are radically different. In applications to physics these differences are thought to be at the root of the fundamentally disparate behaviors of Bose and Fermi systems at low temperatures. In order to emphasize these differences we separate the subsequent discussion of the CARs and CCRs but before the general analysis we give an example of the creation and annihilation operators for point particles.

EXAMPLE If $\mathfrak{h} = L^2(\mathbb{R}^\nu)$, then $\mathfrak{F}_\pm(\mathfrak{h})$ consists of sequences $\{\psi^{(n)}\}_{n\geq 0}$ of functions of n variables $x_i \in \mathbb{R}^\nu$ which are totally symmetric ($+$ sign) or totally antisymmetric ($-$ sign). The action of the annihilation and creation operators is given by

$$(a_\pm(f)\psi)^{(n)}(x_1, \ldots, x_n) = (n+1)^{1/2} \int dx\, \overline{f(x)}\psi^{(n+1)}(x, x_1, \ldots, x_n) \ ,$$

$$(a_\pm^*(f)\psi)^{(n)}(x_1, \ldots, x_n) = n^{-1/2} \sum_{i=1}^{n} (\pm 1)^{i-1} f(x_i)\psi^{(n-1)}(x_1, \ldots, \hat{x}_i, \ldots, x_n) \ ,$$

where $\hat{x}_i$ denotes that the i th variable is to be omitted. Note that as the maps

$$f \mapsto a_\pm(f), \qquad f \mapsto a_\pm^*(f)$$

are anti-linear and linear, respectively, one may introduce operator-valued distributions, i.e., fields $a_\pm(x)$, and $a_\pm^*(x)$, such that

$$a_\pm(f) = \int dx\, \overline{f(x)}\, a_\pm(x) \ , \qquad a_\pm^*(f) = \int dx\, f(x) a_\pm^*(x) \ ,$$

and then the action of these fields is given by

$$(a_\pm(x)\psi)^{(n)}(x_1, \ldots, x_n) = (n+1)^{1/2}\, \psi^{(n+1)}(x, x_1, \ldots, x_n) \ ,$$

$$(a_\pm^*(x)\psi)^{(n)}(x_1, \ldots, x_n) = n^{-1/2} \sum_{i=1}^{n} (\pm 1)^{i-1} \delta(x - x_i)\psi^{(n-1)}(x_1, \ldots, \hat{x}_i, \ldots, x_n) \ .$$

In terms of these fields the number operator N is formally given by

$$N = \int dx\, a_\pm^*(x) a_\pm(x) \ .$$