Spring term 2017

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Riemannian Geometry

Sheet 8

Exercise 1. Let M be a complete, simply-connected Riemannian manifold with non-positive sectional curvature. Let $\gamma : \mathbb{R} \to M$ be a geodesic. Prove that for every point $p \in M$ with $p \notin \gamma(\mathbb{R})$ there exists a unique geodesic σ such that σ connects p to γ and σ is orthogonal to γ. Show by examples that all assumptions on M are necessary.

Exercise 2. Fix maps $K, \bar{K} : \mathbb{R} \to \mathbb{R}$ with $\bar{K}(t) \geq K(t)$, and consider the two ordinary differential equations

$$\begin{aligned} f''(t) + K(t)f(t) =& 0, \quad f(0) = 0, \quad f'(0) = 1, \quad t \in [0, l], \\ \bar{f}''(t) + \bar{K}(t)\bar{f}(t) =& 0, \quad \bar{f}(0) = 0, \quad \bar{f}'(0) = 1, \quad t \in [0, l]. \end{aligned}$$

1. Show that for all $t \in [0, l]$,

$$\begin{aligned} 0 &= \int_0^t \left(\bar{f}(f'' + Kf) - f(\bar{f}'' + \bar{K}\bar{f})\right) dt \\ &= [\bar{f}f' - f\bar{f}']_0^t + \int_0^t (K - \bar{K}) f\bar{f} dt. \end{aligned}$$

   Conclude that if $\bar{f}(t) > 0$ on $(0, t_0)$ and $\bar{f}(t_0) = 0$, then $f(t) > 0$ on $(0, t_0)$.

2. Suppose that $\bar{f}(t) > 0$ on $(0, l]$. Use the above and $f(t) > 0$ on $(0, l]$ to show that $f(t) \geq \bar{f}(t)$ for $t \in [0, l]$, and that equality holds for $t = t_1 \in (0, l]$ if, and only if, $K(t) = \bar{K}(t)$ for $t \in [0, t_1]$. Verify that this is the Rauch comparison theorem in dimension two.

Exercise 3. Let M be an orientable Riemannian manifold with positive sectional curvature and even dimension. Show that any closed geodesic γ is homotopic to a curve with length strictly smaller than the length of γ.

Hand in until 2:00pm of Thursday, June 29th in the appropriate box on the 1st floor.