Spring term 2017

Prof. T. Vogel
R. Coelho

Riemannian Geometry

Sheet 7

Exercise 1. Prove that a subgroup Γ of translations of $\mathbb{R}^n$ is properly discontinuous if and only if there are m linearly independent vectors $v_i \in \mathbb{R}^n$, $0 \leq m \leq n$ such that

$$\Gamma = \{t_a \in G \,|\, a = \sum_i n_i v_i, \text{ with } n_i \in \mathbb{Z}\},$$

in which case Γ_1 is a free abelian group on m generators.

Exercise 2. Consider $\mathbb{R}^2$ with its standard Euclidean metric, and denote by G its isometry group. We denote an element in G by (A, t_a), where $A \in O(2)$ and t_a is a translation by $a \in \mathbb{R}^2$, so that $(A, t_a)(x) = A \cdot x + a$.

1. Let Γ be a subgroup of G acting freely on $\mathbb{R}^2$ and denote by $\Gamma_1 \subset \Gamma$ the elements of Γ which are translations. Show, after an appropriate choice of basis and origin, that either $\Gamma = \Gamma_1$ or $\Gamma = \Gamma_1 \cup \gamma\Gamma_1$ where
$$B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$
$\gamma = (B, t_b) \in G$ and $B(b) = b \neq 0$.
2. Show that $\Gamma = \Gamma_1 \cup \gamma\Gamma_1$ acts freely if and only if $2b \neq (B + I)a$ whenever $(I, t_a) \in \Gamma_1$.

Exercise 3. Let Γ be a subgroup of isometries of $\mathbb{R}^2$ which acts freely and properly discontinuously. In exercise 2 you proved that either $\Gamma = \Gamma_1$ or $\Gamma = \Gamma_1 \cup \gamma\Gamma_1$. Describe all possible quotients $\mathbb{R}^2/\Gamma$ up to diffeomorphism:

- $\Gamma_1 = \{1\}$;
- $\Gamma = \Gamma_1 \simeq \mathbb{Z}$;
- $\Gamma \neq \Gamma_1 \simeq \mathbb{Z}$;
- $\Gamma = \Gamma_1 \simeq \mathbb{Z}^2$;
- $\Gamma \neq \Gamma_1 \simeq \mathbb{Z}^2$.

Exercise 4. Let (M, g) be a connected Riemannian manifold. Show that if an isometry $\Phi : M \to M$ has $\Phi(p) = p$ and $D_p\Phi = Id_{T_pM}$ for some $p \in M$, then $\Phi = Id_M$.

Hand in until 2:00pm of Thursday, June 22nd in the appropriate box on the 1st floor.