LMU LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN

MATHEMATISCHES INSTITUT

Spring term 2017

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Riemannian Geometry

Sheet 6

Exercise 1. Let M be a Riemannian manifold. Show that M is complete if, and only if, its Riemannian universal covering is complete.

Exercise 2. Let (M, g) be a complete Riemannian manifold with non-positive sectional curvature, and let $p, q \in M$. Show that each homotopy class of paths from p to q contains exactly one geodesic.

Exercise 3. Let (M, g) be a simply connected Riemannian manifold with non-positive sectional curvature. Suppose that ϕ is an isometry with finite order, i.e. $\phi^k = Id$ for some k. Denote by $G \cong \mathbb{Z}_k$ the subgroup of isometries of M generated by ϕ.

1. Show that there exists a relatively compact convex normal neighbourhood U which contains an orbit of G.
2. Define $K = \{x \in M \mid G \cdot x \subset \overline{U}\}$. Show that K is convex and compact.
3. Consider the continuous function $f : K \to \mathbb{R}$ given by $f(p) = d(p, f(p))$. Show that a minimizer for f must be a fixed point of ϕ. Conclude that the fundamental group of a complete Riemannian manifold with non-positive sectional curvature has no torsion.

Exercise 4. A geodesic $\gamma : [0, \infty) \to M$ on a Riemannian manifold is called a *ray starting from* $\gamma(0)$ if it minimizes the distance between $\gamma(0)$ and $\gamma(s)$ for every positive s. Suppose M is non-compact and complete.

1. Show that for every $p \in M$, there exists a ray starting from p.
2. Let $p \in M$ be a given point. Must there be a geodesic $\gamma : \mathbb{R} \to M$ which is distance minimizing between $\gamma(0) = p$ and $\gamma(s)$ for all $s \in \mathbb{R}$?

Hand in until 2:00pm of Thursday, June 8th in the appropriate box on the 1st floor.