Spring term 2017

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Riemannian Geometry

Sheet 5

Exercise 1. Let G be a Lie group, i.e. a group with a smooth manifold structure for which multiplication and inversion are smooth maps. G acts naturally on itself both by left and right multiplication, giving rise to left and right invariant vector fields. Recall that the commutator of left-invariant (resp. right-invariant) vector fields is again left-invariant (resp. right-invariant). Denote the linear space of left-invariant vector-fields by $\mathfrak{g} \subset \mathfrak{X}(G)$.

A metric $\langle\ ,\ \rangle$ on G is called *left-invariant* (resp. *right-invariant*) if left (resp. right) multiplication is an isometry. A metric which is both left and right-invariant is called *bi-invariant.*

1. Let $\langle\ ,\ \rangle$ be a left-invariant metric on G. For a left-invariant vector field X, define $\mathrm{ad}(X) : \mathfrak{g} \to \mathfrak{g}$ by
$$\mathrm{ad}(X)(Y) := [X, Y].$$
Denote by $\mathrm{ad}(X)^* : \mathfrak{g} \to \mathfrak{g}$ adjoint of $\mathrm{ad}(X)$ with respect to the metric, i.e.
$$\langle \mathrm{ad}(X)(Y), Z\rangle = \langle Y, \mathrm{ad}(X)^*(Z)\rangle \quad \text{for all } Y, Z \in \mathfrak{g}.$$
Show that the Levi-Civita connection ∇ of $\langle\ ,\ \rangle$ is given on left-invariant vector fields X, Y by
$$\nabla_X Y = \frac{1}{2}\left([X, Y] - \mathrm{ad}(X)^*Y - \mathrm{ad}(Y)^*X\right).$$

2. Suppose now that $\langle\ ,\ \rangle$ is bi-invariant. Show that the Levi-Civita connection ∇ is now given by
$$\nabla_X Y = \frac{1}{2}[X, Y]$$
on left-invariant vector-fields.

3. Determine the Riemann curvature tensor of a bi-invariant metric on a Lie group.

4. Show that a bi-invariant metric on a Lie group has non-negative sectional curvature.

Exercise 2. Let (M, g) be a Riemannian manifold. Fix $p \in M$ and let $\gamma : [0, a] \to M$ be a geodesic with $\gamma(0) = p$ and $\dot\gamma(0) = v$. Let $w \in T_v(T_pM)$ have unit norm and consider the Jacobi field J along γ given by
$$J(t) = (D\exp_p)_{tv}(tw).$$
Show that the Taylor expansion of $|J(t)|^2$ at $t = 0$ is given by
$$|J(t)|^2 = t^2 - \frac{1}{3} g(R(v, w)w, v)t^4 + R(t),$$
where $\lim_{t\to 0} \frac{R(t)}{t^4} = 0$.

Exercise 3. Consider $\mathbb{R}^{n+1}$ with $\langle\ ,\ \rangle$ the standard Euclidean metric.

1. Let $M \subset \mathbb{R}^{n+1}$ be a smooth submanifold of dimension n, and denote by g the induced metric on M. Define a connection on TM by

$$\nabla_X Y := \pi(\tilde{\nabla}_{\tilde{X}} \tilde{Y})\Big|_M ,$$

where $X, Y \in \mathfrak{X}(M)$, π is the orthogonal projection to TM, $\tilde{\nabla}$ is the Levi-Civita connection on $(\mathbb{R}^{n+1}, \langle\ ,\ \rangle)$, and $\tilde{X}, \tilde{Y}$ are local extensions of X, Y. Show that ∇ is the Levi-Civita connection of g.

2. Suppose that n is a unit normal to M, that is, a non-vanishing section of the normal bundle $\nu \to M$ with $\langle n, n\rangle = 1$. The *Weingarten map* L is defined on a vector field $X \in \mathfrak{X}(M)$ by

$$L(X) = -(\tilde{\nabla}_X \tilde{n})\Big|_M ,$$

where $\tilde{n}$ is a local extension of n. Show that $L(X) \in \mathfrak{X}(M)$.

3. Show that the Riemann curvature tensor of M is given by

$$R(X,Y)Z = \langle L(Y), Z\rangle L(X) - \langle L(X), Z\rangle L(Y).$$

Exercise 4. Consider $\mathbb{R}^{n+1} = \{(x_0, \dots, x_n)\}$ with the pseudo-Riemannian metric

$$h = -dx_0^2 + \sum_{i=1}^{n} dx_i^2.$$

Let H be the hypersurface $H = Q^{-1}(-1)$, where $Q : \mathbb{R}^{n+1} \to \mathbb{R}$ is the quadratic form

$$Q(x) = -x_0^2 + \sum_{i=1}^{n} x_i^2.$$

1. Show that the induced pseudo-Riemannian metric on H is positive definite, i.e. a genuine Riemannian metric.

2. Compute the sectional curvature of H with respect to the induced metric.

   Hint: adapt what you did in exercise 3.

Hand in until 2:00pm of Thursday, June 1st in the appropriate box on the 1st floor.