LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN
MATHEMATISCHES INSTITUT

Spring term 2017

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Riemannian Geometry

Sheet 3

Exercise 1. Consider $S^n \subset \mathbb{R}^{n+1}$ with the round metric, i.e. the metric it inherits as a submanifold of $\mathbb{R}^{n+1}$. Fix $p = (1, 0, \ldots, 0)$ and consider $B_R(p)$ the open ball of radius R.

1. For which R is $B_R(p)$ convex?
2. For which R is $\exp_p : B_R(0) \subset T_pS^n \to S^n$ a diffeomorphism onto its image?

Exercise 2. Consider the 2-torus $T^2 = \mathbb{R}^2/\Gamma_a$, where Γ_a is the lattice $\mathbb{Z}^2 \subset \mathbb{R}^2$ acting by isometries (translations) on $\mathbb{R}^2$. Give T^2 the metric it inherits from $\mathbb{R}^2$. Fix $p = [(0, 0)]$.

1. Determine a maximal convex neighbourhood of p.
2. Consider instead of Γ_a the lattice $\Gamma_b = \langle (1, 0), (\frac{1}{2}, \frac{\sqrt{3}}{2}) \rangle$. What is now a maximally convex neighbourhood of p?
3. For Γ_a, find a necessary and sufficient condition for a geodesic γ with $\gamma(0) = p$ and $\dot{\gamma}(0) = v \neq 0$ to be closed.

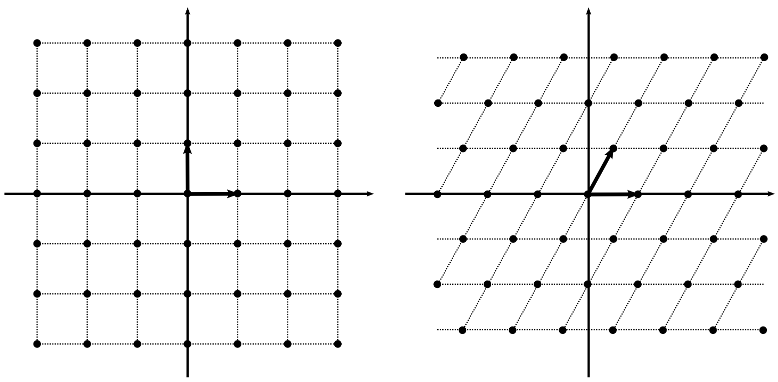

Figure 1: The lattices Γ_a and Γ_b.

Exercise 3 (Clairaut's theorem)**.** Let $f, g \in C^\infty(\mathbb{R})$ and consider $\varphi : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$\varphi(u, v) = (f(v)\sin(u), f(v)\cos(u), g(v)),$$

where U is the open set $U = \{(u, v) \in \mathbb{R}^2 \,|\, u_0 < u < u_1,\, v_0 < v < v_1\}$.

1. Show that if $f'(v)^2 + g'(v)^2 \neq 0$ and $f(v) \neq 0$, then φ is an immersion (i.e. its differential is injective). The surface $\varphi(U) =: S$ is generated by the rotation of the curve $(f(v), g(v))$ around the zz axis, and called a *surface of revolution*. Curves with constant u, resp. constant v, are called *meridians*, resp. *parallels*.

2. Show that the metric induced by the immersion is given in the coordinates (u, v) by

$$g = \begin{pmatrix} f^2 & 0 \\ 0 & (f')^2 + (g')^2 \end{pmatrix}.$$

3. Show that the equations of a geodesic γ are

$$\frac{d^2u}{dt^2} + \frac{2ff'}{f^2}\frac{du}{dt}\frac{dv}{dt} = 0,$$

$$\frac{d^2v}{dt^2} - \frac{ff'}{(f')^2 + (g')^2}\left(\frac{du}{dt}\right)^2 + \frac{f'f'' + g'g''}{(f')^2 + (g')^2}\left(\frac{dv}{dt}\right)^2 = 0.$$

4. Let $\beta(t) < \pi$ denote the (oriented) angle of γ with a parallel P intersecting γ at $\gamma(t)$. Show that $r\cos(\beta(t))$ is constant, where r denotes the radius of the parallel P. This is *Clairaut's relation.*

Exercise 4. Let (M, g) be a Riemannian manifold, and let $f \in C^\infty(M)$.

1. Recall that the gradient of f is the vector field defined by

$$g(\mathrm{grad}(f), X) = df(X), \text{ for all } X \in \mathfrak{X}(M).$$

Express $\mathrm{grad}(f)_p$ in normal coordinates centered at p.

2. The *Hessian* of f is the bilinear form defined by

$$H_f(X, Y) = g(\nabla_X(\mathrm{grad}(f)), Y).$$

Compute the Hessian at a critical point of f.

Hand in until 2:00pm of Thursday, May 18th in the appropriate box on the 1st floor.