Elementary explicit types and polynomial time operations

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Honorary Colloquium on Occasion of Wilfried Bucholz' 60th Birthday,
April 5, 2008

Outline

Outline

Introduction

- Explicit Mathematics as introduced by Feferman.
- Weak theories exist for applicative part.
- Up to now, theories with types were of strength at least PRA.

Goal / Question

We want a theory with types, (full) type induction and of strength the polynomial time computable functions. Which types can be allowed to match these requirements?

Provably Total

Definition

A function $F : \mathbb{W}^n \to \mathbb{W}$ is called *provably total in an $\mathcal{L}$ theory* T, if there exists a closed $\mathcal{L}$ term t_F such that

(i) $\mathsf{T} \vdash t_F : \mathsf{W}^n \mapsto \mathsf{W}$ and, in addition,

(ii) $\mathsf{T} \vdash t_F \overline{w}_1 \cdots \overline{w}_n = \overline{F(w_1, \ldots, w_n)}$ for all $w_1, \ldots, w_n$ in $\mathbb{W}$.

$\overline{w}$ for $w \in \mathbb{W}$ means the corresponding standard term.
T is an applicative theory comprising combinatory algebra.

Outline

The Applicative Base

PT was introduced by Thomas Strahm. It is based on the Logic of Partial Terms with the binary words as basic elements.

Logic of Partial Terms

Language $\mathcal{L}$ *individual variables* $a, b, c, \ldots$; *individual constants* $\mathsf{k}, \mathsf{s}, \mathsf{p}, \mathsf{p}_0, \mathsf{p}_1$, d_W, ϵ, $\mathsf{s}_0, \mathsf{s}_1$, p_W, $\mathsf{c}_\subseteq$; constants $*$ and $\times$; binary function symbol $\cdot$, unary relation symbols $\downarrow$ and W, binary relation symbol $=$.

Terms Inductively by $\cdot$ from constants and variables.

Formulae Inductively by the usual connectives from relations.

Axioms/Rules Axioms and rules of Hilbert Calculus with equality plus axioms about definedness.

Important Abbreviations

$$0 := \mathsf{s}_0\epsilon \qquad 1 := \mathsf{s}_1\epsilon$$

$$(s_1, s_2) := \mathsf{p}st \qquad (s)_i := \mathsf{p}_i s \quad (i = 0, 1)$$

$$s \subseteq t := \mathsf{c}_{\subseteq} st = 0$$

$$s \leq t := \mathsf{l}_{\mathsf{W}} s \subseteq \mathsf{l}_{\mathsf{W}} t \qquad \mathsf{l}_{\mathsf{W}} s := 1 \times s$$

$$\begin{aligned}
\mathsf{W}_a(s) &:= (\mathsf{W}(s) \wedge s \leq a), \\
(\exists x \leq t)A &:= (\exists x \in \mathsf{W})(x \leq t \wedge A), \\
(\forall x \leq t)A &:= (\forall x \in \mathsf{W})(x \leq t \rightarrow A), \\
(t : \mathsf{W} \mapsto \mathsf{W}) &:= (\forall x \in \mathsf{W})(tx \in \mathsf{W}), \\
(t : \mathsf{W}^{m+1} \mapsto \mathsf{W}) &:= (\forall x \in \mathsf{W})(tx : \mathsf{W}^m \mapsto \mathsf{W}).
\end{aligned}$$

Axioms of Base Theory B

I Partial combinatory algebra and pairing
Axioms defining the behaviour of the well-known combinators k and *s* and of pairing p and projections p_0 and p_1

II Definition by cases on W

$$\mathrm{d_W}xyab = \begin{cases} x & a, b \in \mathrm{W} \wedge a = b \\ y & a, b \in \mathrm{W} \wedge a \neq b \end{cases}$$

III Closure, binary successors and predecessor
W contains the ϵ and is closed under successors $\mathrm{s}_0, \mathrm{s}_1$ and predecessor $\mathrm{p_W}$. Furthermore, $\mathrm{s}_0, \mathrm{s}_1$ and $\mathrm{p_W}$ behave as expected.

IV Initial subword relation
$\mathrm{c}_\subseteq$ is a total "predicate" on W. It behaves decently on W, deciding whether the first word is a initial subword of the second.

V Word concatenation and multiplication
$*$ concatenates two words as expected. $\times xy = \underbrace{x * \ldots * x}_{\text{length of } y \text{ often}}$

Induction

$$f : \mathsf{W} \mapsto \mathsf{W} \wedge A[\epsilon] \wedge (\forall x \in \mathsf{W})(A[\mathsf{p_W} x] \to A[x]) \to (\forall x \in \mathsf{W})A[x] \quad (\mathcal{C}\text{-}\mathsf{I_W})$$

where $A[x]$ belongs to the formula class $\mathcal{C}$

Definition ($\Sigma^{\mathsf{b}}_{\mathsf{W}}/\Sigma^{\mathsf{b}-}_{\mathsf{W}}$)

A formula $A[f, x]$ belongs to $\Sigma^{\mathsf{b}}_{\mathsf{W}}$ ($\Sigma^{\mathsf{b}-}_{\mathsf{W}}$) if it is of the form $(\exists y \leq fx)B[f, x, y]$ where $B[f, x, y]$ positive and W-free (and not containing $\forall$).

Theories PT and PT$^-$

$\mathsf{PT} \;\; := \mathsf{B} + (\Sigma^{\mathsf{b}}_{\mathsf{W}}\text{-}\mathsf{I_W})$
$\mathsf{PT}^- := \mathsf{B} + (\Sigma^{\mathsf{b}-}_{\mathsf{W}}\text{-}\mathsf{I_W})$

Important Properties of PT^- (and PT)

Lemma (λ-Abstraction, Fixpoint)

In B*, we have λ-abstraction for any term t and a term* rec *serving as fixed point operator.*

Theorem

The provably total functions of PT^- *coincide with the functions terminating in polynomial time.*

Outline

Language

"Theory of Types and Names"

The Language $\mathcal{L}_\mathsf{T}$ is $\mathcal{L}$ extended with type variables $X, Y, \ldots,$ binary relation symbols $\in, \Re$ (naming), constants $\mathsf{w}, \mathsf{id}, \mathsf{dom}, \mathsf{un}, \mathsf{int}, \mathsf{inv}$

Additional Shortcuts

$$\Re(a) := \exists X(\Re(a, X))$$
$$a \dot{\in} b := \exists X(\Re(b, X) \wedge a \in X)$$

Axioms of PET

Axioms are the axioms of B plus the following:

$$\exists x \Re(x, X) \tag{Expl1}$$

$$\Re(a, X) \wedge \Re(a, Y) \rightarrow X = Y \tag{Expl2}$$

$$\forall z (z \in X \leftrightarrow z \in Y) \rightarrow X = Y \tag{Expl3}$$

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$$a \in \mathsf{W} \rightarrow \Re(\mathsf{w}(a)) \wedge \forall x (x \,\dot{\in}\, \mathsf{w}(a) \leftrightarrow \mathsf{W}_a(x)) \tag{$\mathbf{w}_a$}$$

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$$\Re(a) \rightarrow \Re(\mathsf{inv}(f, a)) \wedge \forall x(x \mathrel{\dot\in} \mathsf{inv}(f, a) \leftrightarrow fx \mathrel{\dot\in} a) \tag{\textbf{inv}}$$

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$$\Re(a) \rightarrow \Re(\mathsf{dom}(a)) \wedge \forall x(x \mathrel{\dot\in} \mathsf{dom}(a) \leftrightarrow \exists y((x, y) \mathrel{\dot\in} a)) \tag{$\mathbf{dom}$}$$

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$$\epsilon \in X \wedge (\forall x \in \mathsf{W})(\mathsf{p_W} x \in X \rightarrow x \in X) \rightarrow (\forall x \in \mathsf{W})(x \in X) \tag{T-$\mathsf{I_W}$}$$

Comprehension: Preparations

Definition (Class of $\Sigma^{\mathrm{b}}_{\mathrm{T}}$ formulas and set of variables $FV_{\mathrm{W}}(A)$)

$A \equiv (s = t)$, $s\downarrow$ or $(s \in X)$ A is a $\Sigma^{\mathrm{b}}_{\mathrm{T}}$ formula and $FV_{\mathrm{W}}(A) := \emptyset$.

$A \equiv \mathrm{W}_a(t)$ A is a $\Sigma^{\mathrm{b}}_{\mathrm{T}}$ formula and $FV_{\mathrm{W}}(A) := \{a\}$ if $a \notin FV_I(t)$.

$A \equiv (B \wedge C)$ or $(B \vee C)$ with B and C in $\Sigma^{\mathrm{b}}_{\mathrm{T}}$ and if no conflict arises between FV_I and FV_{W}, then A is a $\Sigma^{\mathrm{b}}_{\mathrm{T}}$ formula and $FV_{\mathrm{W}}(A) := FV_{\mathrm{W}}(B) \cup FV_{\mathrm{W}}(C)$.

$A \equiv \exists x B$ with $B \in \Sigma^{\mathrm{b}}_{\mathrm{T}}$ and $x \notin FV_{\mathrm{W}}(B)$, then A is a $\Sigma^{\mathrm{b}}_{\mathrm{T}}$ formula and $FV_{\mathrm{W}}(A) := FV_{\mathrm{W}}(B)$.

Comprehension: Preparations

Definition ($\rho_A(B,x)$)

For a $\Sigma^{\mathrm{b}}_{\mathrm{T}}$ formula A, we define a term $\rho_A(B,x)$ by induction on the complexity of B in $\Sigma^{\mathrm{b}}_{\mathrm{T}}$, where $x \notin FV_{\mathsf{W}}(B)$ and x not bound in B:

$$
\begin{aligned}
\rho_A(s=t,x) &:= \mathsf{inv}(\lambda x.(s,t),\mathsf{id}),\\
\rho_A(s\downarrow,x) &:= \mathsf{inv}(\lambda x.(s,s),\mathsf{id}),\\
\rho_A(s\in \mathsf{W}_a,x) &:= \mathsf{inv}(\lambda x.s,\mathsf{w}(a)),\\
\rho_A(s\in X,x) &:= \mathsf{inv}(\lambda x.s,\mu_A(X)),\\
\rho_A(C\wedge D,x) &:= \mathsf{int}(\rho_A(C,x),\rho_A(D,x))\\
\rho_A(C\vee D,x) &:= \mathsf{un}(\rho_A(C,x),\rho_A(D,x))\\
\rho_A(\exists y C,x) &:= \mathsf{dom}(\rho_A(C[(x)_0/x,(x)_1/y],x)).
\end{aligned}
$$

where $\mu_A(X)$ assigns an individual variable not occurring in A to the free type variable X.

Restricted Elementary Comprehension

Theorem (Restricted elementary comprehension in PET)

For A a $\Sigma^{\mathrm{b}}_{\mathrm{T}}$ formula with $FV_T(A) = \{X_1, \ldots, X_n\}$ and $FV_{\mathrm{W}}(A) = \{w_1, \ldots, w_m\}$. Let $z_i := \mu_A(X_i)(1 \leq i \leq n)$ and $\rho_{A,x} := \rho_A(A, x)$, then we have:

1. $FV_I(\rho_{A,x}) = (FV_I(A) \setminus \{x\}) \cup \{z_1, \ldots, z_n\}$,
2. $\mathrm{PET} \vdash \mathrm{W}(\vec{w}) \wedge \Re(\vec{z}, \vec{X}) \ \rightarrow \ \Re(\rho_{A,x})$,
3. $\mathrm{PET} \vdash \mathrm{W}(\vec{w}) \wedge \Re(\vec{z}, \vec{X}) \ \rightarrow \ (\forall x)(x \mathrel{\dot{\in}} \rho_{A,x} \leftrightarrow A)$.

Remark

As a consequence of comprehension and type induction, induction is available for $\Sigma^{\mathrm{b}}_{\mathrm{T}}$ formulae.

Outline

Lower Bounds: Preparatory Work

Lemma (Properties of the subword relation)

The following statements are provable in PET*:*

1. $x \in \mathsf{W} \wedge z \in \mathsf{W} \wedge x \subseteq \mathsf{p_W} z \rightarrow x \subseteq z$,
2. $x \in \mathsf{W} \wedge y \in \mathsf{W} \wedge z \in \mathsf{W} \wedge x \subseteq y \wedge y \subseteq z \rightarrow x \subseteq z$ *(Transitivity)*,
3. $x \in \mathsf{W} \wedge y \in \mathsf{W} \wedge x \subseteq y \rightarrow x \leq y$.

Lower Bounds: "Bounded Induction"

$$a \in \mathsf{W} \wedge \epsilon \in X \wedge (\forall x \subseteq a)(\mathsf{p_W} x \in X \rightarrow x \in X) \rightarrow a \in X \qquad (\mathrm{T}\text{-}\mathsf{I}_\mathsf{W}^\mathsf{b})$$

Lemma

We have that $(\mathrm{T}\text{-}\mathsf{I}_\mathsf{W})$ *and* $(\mathrm{T}\text{-}\mathsf{I}_\mathsf{W}^\mathsf{b})$ *are provably equivalent in* PET *without* $(\mathrm{T}\text{-}\mathsf{I}_\mathsf{W})$.

Bounding Functions $f : \mathsf{W} \mapsto \mathsf{W}$

Lemma

There is a closed term max *such that* PET *proves:*

1. $f : \mathsf{W} \mapsto \mathsf{W} \to \mathsf{max} f : \mathsf{W} \mapsto \mathsf{W}$,
2. $f : \mathsf{W} \mapsto \mathsf{W} \wedge f^* = \mathsf{max} f \wedge x \in \mathsf{W} \wedge y \in \mathsf{W} \wedge x \subseteq y \to f^* x \leq f^* y)$,
3. $f : \mathsf{W} \mapsto \mathsf{W} \wedge f^* = \mathsf{max} f \wedge x \in \mathsf{W} \to fx \leq f^* x)$,
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Proof (Sketch)

$\mathrm{max} := \lambda f.\lambda x.f(\mathrm{max}_{\mathrm{arg}} fx)$ where $\mathrm{max}_{\mathrm{arg}}$ is a functional detecting the argument maximising the function f up to x.
Proof of (1) by proving $f : \mathrm{W} \mapsto \mathrm{W} \rightarrow \mathrm{max}_{\mathrm{arg}} f : \mathrm{W} \mapsto \mathrm{W}$ by $(\mathrm{T}\text{-}\mathrm{I}_{\mathrm{W}}^{\mathrm{b}})$ on $(\exists y \leq a)((\mathrm{max}_{\mathrm{arg}} f)x = y)$. Proof of (2) and (3) by induction.

Lower Bounds

Theorem

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Proof

Steps of proving $\mathsf{PET} \vdash (\Sigma_{\mathsf{W}}^{b-}\text{-}\mathsf{I}_{\mathsf{W}})$:

1. Take Σ_{W}^{b-} formula $A[x] \equiv (\exists y \leq fx)B[f, x, y]$ and assume $f : \mathsf{W} \to \mathsf{W} \wedge A[\epsilon] \wedge (\forall x \in \mathsf{W})(A[x] \to A[\mathsf{s}_0 x] \wedge A[\mathsf{s}_1 x])$

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2. Prove $(\exists y \leq fx)B[x, y] \leftrightarrow (\exists y \leq f^* c)(y \leq fx \wedge B[x, y])$ for $c \in \mathsf{W}$ and $x \subseteq c$.

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3. With Comprehension we can construct type X such that $(\forall x \subseteq c)(x \in X \leftrightarrow (\exists y \leq f^* c)(y \leq fx \wedge B[x, y]))$ as $f^* c \in \mathrm{W}$

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1. Take Σ^{b-}_{W} formula $A[x] \equiv (\exists y \leq fx)B[f, x, y]$ and assume $f : \mathsf{W} \to \mathsf{W} \wedge A[\epsilon] \wedge (\forall x \in \mathsf{W})(A[x] \to A[\mathsf{s}_0 x] \wedge A[\mathsf{s}_1 x])$
2. Prove $(\exists y \leq fx)B[x, y] \leftrightarrow (\exists y \leq f^* c)(y \leq fx \wedge B[x, y])$ for $c \in \mathsf{W}$ and $x \subseteq c$.
3. With Comprehension we can construct type X such that $(\forall x \subseteq c)(x \in X \leftrightarrow (\exists y \leq f^* c)(y \leq fx \wedge B[x, y]))$ as $f^* c \in \mathsf{W}$
4. $\epsilon \in X \wedge (\forall x \subseteq c)(\mathsf{p}_{\mathsf{W}} x \in X \to x \in X)$ immediate from above.
5. By $(\mathrm{T}\text{-}\mathsf{I}^b_{\mathsf{W}})$: $c \in X$ $\square$

Outline

Structures for PET

Definition ($\mathcal{L}_\mathsf{T}$ structure)

A $\mathcal{L}_\mathsf{T}$-structure $\mathcal{M}^\star$ is a tuple

$$(\mathcal{M}, \mathcal{T}, \mathcal{E}, \mathcal{R}, \mathrm{w}, \mathrm{id}, \mathrm{dom}, \mathrm{un}, \mathrm{int}, \mathrm{inv})$$

where (i) $\mathcal{M}$ is a $\mathcal{L}$-structure, (ii) $\mathcal{T}$ is a non-empty set of subsets of $|\mathcal{M}|$, (iii) $\mathcal{E}$ is the usual $\in$ relation on $|\mathcal{M}| \times \mathcal{T}$, (iv) $\mathcal{R}$ is a non-empty subset of $|\mathcal{M}| \times \mathcal{T}$, and (v) $\mathrm{w}, \mathrm{id}, \mathrm{dom}, \mathrm{un}, \mathrm{int}, \mathrm{inv}$ are elements of $|\mathcal{M}|$.

Model Construction

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- Take model $\mathcal{M}$ of PT^-.
- Choose decent interpretation for constants.
- Inductively define
  $\mathcal{T}_k := \{ext(m) : m \in R_k\}, \mathcal{R}_k := \{(m, ext(m)) : m \in R_k\}$,
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  $k > 0$ $\; R_k := \{\mathrm{un}(a, b), \mathrm{int}(a, b) : a, b \in R_{k-1}\}$
  $\cup\{\mathrm{inv}(f, a), \mathrm{dom} a : a \in R_{k-1}\}$ and e.g.
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- Set $\mathcal{T} := \bigcup_{k\in\mathbb{N}} \mathcal{T}_k$ and $\mathcal{R} := \bigcup_{k\in\mathbb{N}} \mathcal{R}_k$.

Upper Bounds

Theorem (Model extension)

Any model $\mathcal{M}^\star$ constructed as described above from a model $\mathcal{M}$ of PT^- *satisfies the following conditions:*

1. $\mathcal{M} \models A \iff \mathcal{M}^\star \models A$ *for any $\mathcal{L}$ sentence A,*
2. $\mathcal{M}^\star \models \mathrm{T}\text{-}\mathsf{I}_{\mathsf{W}}$,
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Proof of 2

We show that every type $X \in \mathcal{T}$ is weakly $\Sigma^{\mathsf{b}-}_{\mathsf{W}}$ definable, i.e. that $X = \{m \in |\mathcal{M}| : \mathcal{M} \models A[m]\}$ for A $\Sigma^{\mathsf{b}-}_{\mathsf{W}}$ formula with a fixed bound. This is proved by induction on the level k when X is added to R_k.

Outline

Uniformity

Cantini showed that the Uniformity Principle **UP** can be added to PT without strengthening the theory.

$$(\forall x)(\exists y \in \mathsf{W})A(x,y) \rightarrow (\exists y \in \mathsf{W})(\forall x)A(x,y) \text{ for } A \text{ positive} \qquad (\mathbf{UP})$$

UP entails the following bounded uniformity axiom:

$$\forall x(\exists y \leq t)A[x,y] \rightarrow (\exists y \leq t)(\forall x)A[x,y] \text{ for } A \text{ positive} \qquad (\mathbf{UP}\text{'})$$

In the presence of $\mathbf{UP}'$ we can add an universal type to PET:

$$\Re(a) \rightarrow \Re(\mathsf{all}\, a) \wedge \forall x(x \,\dot{\in}\, \mathsf{all}\, a \leftrightarrow \forall y(\langle x,y\rangle \,\dot{\in}\, a)) \qquad (\mathbf{all})$$

Lemma

PT + (**UP**) *is contained in* PET + (**all**) *and* PET + (**all**) *is a conservative extension of* PT + (**UP**) *for closed* $\mathcal{L}$ *formulae.*

Other Extensions

Choice $(\forall x \in \mathsf{W})(\exists y \in \mathsf{W})A(x, y) \rightarrow$
$(\exists f : \mathsf{W} \mapsto \mathsf{W})(\forall x \in \mathsf{W})A(x, fx)$
for A positive and containing type variables only in the form $t \in X$

Totality $\forall x \forall y(xy\downarrow)$

Extensionality $\forall f \forall g(\forall x(fx \simeq gx) \rightarrow f = g)$

Theorem

The provably total functions of PET *augmented by any combination of the principles* (**all**), Choice, Totality, *and* Extensionality *coincide with the polynomial time computable functions.*

Further Work

We are currently studying the addition of disjoint join:

Join: $\Re(a) \land f : a \mapsto \Re \rightarrow \Re(\mathsf{j}(a, f) \land$
$\forall x(x \mathrel{\dot{\in}} \mathsf{j}(a, f) \leftrightarrow (x)_0 \mathrel{\dot{\in}} a \land (x)_1 \mathrel{\dot{\in}} f(x)_0)$

Furthermore, we plan to study weak theories of partial (self referential) truth.