Π^0_2 *Conservation in Proof Theory*

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- Recursive proof trees can be encoded by natural numbers.
- E.g. Pohlers: *Proof-theoretical analysis of ID_ν by the method of local predicativity.* In: W. Buchholz, S. Feferman, W. Pohlers, W. Sieg: *Iterated inductive definitions and subsystems of analysis: Recent proof-theoretical studies* (1981)

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- Provides a reduction of a theory **T** (e.g. $\mathbf{T} = \Delta^1_2$-**CA** $+$ **BI**) to **PA** $+ \bigcup_{\alpha<\lambda}$ **TI**(α) where λ is a sufficiently large ordinal (representation system).

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- **Drawback:** Has probably never been sufficiently formalized. Glossing over detail. A lot of handwaving.

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- It works for $\mathbf{PA} + \bigcup_{\alpha<\lambda} \mathbf{TI}(\alpha)$. Would it work for $\mathbf{KPi} = \Delta^1_2\text{-}\mathbf{CA} + \mathbf{BI}$?
- Does this machinery work for elementary in place of primitive recursive functions?

Complexity of Ordinal Representation Systems

- Rick Sommer has investigated the question of complexity of ordinal representation systems at great length. His case studies revealed that with regard to complexity measures considered in complexity theory the complexity of ordinal representation systems involved in ordinal analyses is rather low. It appears that computations on ordinals in actual proof-theoretic ordinal analyses can be handled in the theory

$$I\Delta_0 + \Omega_1$$

where Ω_1 is the assertion that the function $x \mapsto x^{\log_2(x)}$ is total.

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- Characterizes them as the λ**-descent recursive functions**.

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- Characterizes them as the **λ-descent recursive functions**.
- Treatment is semi-rigorous.

The Descent Computable Functions

Theorem (Friedman, Sheard)

The provably computable functions of

$$\mathbf{PA} + \bigcup_{\alpha<\lambda} \mathbf{TI}(\alpha)$$

are all functions f of the form

$$f(\vec{m}) = g(\vec{m}, \text{least } n.h(\vec{m}, n) \trianglelefteq h(\vec{m}, n+1)) \tag{1}$$

where g and h are elementary and, for some $\alpha \in A$,

$$\mathbf{EA} \vdash \forall \vec{x} y\, h(\vec{x}, y) \in A_\alpha.$$

The above class of functions is called the **descent computable functions over A_λ.**

Main Step

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- Ordinal analysis reduces a theory **T** which is a subsystem of second order arithmetic or set theory to

$$\mathbf{PA} + \bigcup_{\alpha<\lambda} \mathbf{TI}(\alpha),$$

where λ is the proof-theoretic ordinal of **T**.

Friedman's question:

1. Proof theory (almost?) always is establishing a conservative extension result that says that a given formal system **T** is a conservative extension of a system of arithmetic transfinite induction on a notation system, for Σ^0_1 sentences, or even Π^0_2 sentences.

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2. The conservative extension statement itself is a Π^0_2 sentence. All I really need is that this Π^0_2 sentence has a reasonable Skolem function. E.g., a "reasonable" primitive recursive function will do. By "reasonable" I mean, e.g., that its presentation in the primitive recursion calculus of Kleene uses at most, say, 2^{1000} symbols.

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3. From this, I can apply Friedman/Sheard with the old classical theory of infinitary derivations, linking to elementary recursive descent recursion, and no longer need your expertise.

Friedman's question:

1. $2^{1000} \simeq 10^{300} > \sharp$ atoms in the visible universe

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- The **strongly positive formulas** are built up from formulas $P(t)$ and atomic formulas of **HA** by means of $\wedge, \vee, \forall, \exists$.
- $\widehat{\mathbf{ID}}^{i}_{1}$ is obtained from **HA** by adding for each **strongly positive** operator form $\Phi(P, x)$ a new predicate symbol I_Φ and the axiom

$$(\mathrm{FP}_\Phi) \qquad \forall x\, [\Phi(I_\Phi, x) \leftrightarrow I_\Phi(x)].$$

Moreover, the induction schema is extended to the new language.

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$$\forall x \exists y\, B(x, y) \rightarrow \exists e \forall x\, B(x, \{e\}(x)).$$

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Theorem 1 For each strongly positive operator form Φ there is an arithmetical formula $A_\Phi(x)$ such that

$$\mathbf{HA} + \mathrm{CT}_0 \vdash \forall x[\Phi(A_\Phi, x) \leftrightarrow A_\Phi(x)].$$

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3. $\mathbf{HA} \vdash \exists e\,(e\mathfrak{r}C) \rightarrow C$ for almost negative C.

$\widehat{\mathbf{ID}}_1^i + \bigcup_{\alpha<\lambda} \mathbf{TI}(\alpha)$ *as a metatheory for ordinal analysis*

- A semi-formal system à la Schütte is given by a derivability predicate $\mathcal{D}(\alpha, \rho, \Gamma)$ meaning 'Γ is derivable with order α and cut-rank ρ' defined by transfinite recursion on α as follows:

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$(*)$ $\mathcal{D}(\alpha, \rho, \Gamma) \Leftrightarrow$

$\alpha < \lambda$, and either Γ contains an axiom or Γ is the conclusion of an inference with premisses $(\Gamma_i)_{i \in I}$ such that for every $i \in I$ there exists $\beta_i < \alpha$ with $\mathcal{D}(\beta_i, \rho, \Gamma_i)$, and if the inference is a cut it has rank $< \rho$.

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- $(*)$ can be viewed as a fixed-point axiom which together with $\bigcup_{\alpha<\lambda} \mathbf{TI}(\alpha)$ defines $\mathcal{D}$ implicitly, whence the metatheory $\widehat{\mathbf{ID}}_1^i + \bigcup_{\alpha<\lambda} \mathbf{TI}(\alpha)$ suffices.

The strength of Kruskal's theorem

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Theorem 2. For every n, $\Pi^1_2 - \mathbf{BI}_0^-$ proves $\mathbf{KT}_n$, i.e. Kruskal's theorem for finite at most n branching trees.

$\Pi^1_2 - \mathbf{BI}_0 \nvdash \forall x\, \mathbf{KT}_x$.

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Theorem 3. $\mathbf{ACA}_0$ proves that Kruskal's theorem is equivalent to the uniform Π^1_1-reflection for $\Pi^1_2 - \mathbf{BI}_0$.

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$$(\forall\text{-Inst}) \qquad \forall X\, A(X) \to A(F)$$

where $A(X)$ is an arithmetic formula and $F(u)$ is an arbitrary formula of second order arithmetic. $A(F)$ results from $A(X)$ by replacing every subformula $t \in X$ by $F(t)$.

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2. How to transform a proof of $\forall X\, A(X)$ into a proof of $A(F)$?
3. Easy if the proof of $\forall X\, A(X)$ is cut-free: **substitution**.

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  3. **$T^* \mathrel{\genfrac{}{}{0pt}{}{|\beta}{|0}} \Xi, \forall X F(X)$ implies $T^* \mathrel{\genfrac{}{}{0pt}{}{|f(\beta)}{|\varrho}} \Xi, \Gamma$ for every set of weak formulas Ξ and $\beta < \Omega$.**

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Then $T^* \mid\frac{\alpha}{\varrho}\, \Gamma$ holds.

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4. If $T^* \overset{\alpha_0}{\underset{\varrho}{\vdash}} \Gamma, F(U)$ holds for some $\alpha_0 \lhd \alpha$ and a non-arithmetic formula $F(U)$, then $T^* \overset{\alpha}{\underset{\varrho}{\vdash}} \Gamma, \exists X F(X)$.

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- $\widehat{\mathbf{ID}}^i_n$(*strong*) can be interpreted in intuitionistic analysis $\mathbf{EL} + \mathbf{AC}$-$\mathbf{NF}$ basically by the same proof as the classical second recursion theorem. $\mathbf{EL} + \mathbf{AC}$-$\mathbf{NF}$ is conservative over **HA** by **Goodman's theorem**.

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(2) $\widehat{\mathbf{ID}}^{i}_{n}$ is formulated for **strongly positive** operator forms. But the iterated inductive definition of $T^* \overset{\alpha}{\underset{\varrho}{\vdash}} \Gamma$ seems to require a **strictly positive** iterated inductive definition.

Metatheory for the Ω-rule cont'd

Two Drawbacks:

(1) The employment of Goodman's theorem makes the proof less explicit. Hard to say how long a fully formalized proof would be.

(2) $\widehat{\mathbf{ID}}^i_n$ is formulated for **strongly positive** operator forms. But the iterated inductive definition of $T^* \mathrel{\substack{\alpha \\ \vdash \\ \varrho}} \Gamma$ seems to require a **strictly positive** iterated inductive definition.

- The strictly positive (with respect to P) formulas of $\mathcal{L}_1(P, Q)$ formulas are closed under the following clause:

  If $\mathbf{A}$ is an $\mathcal{L}_1(Q)$ formula and B is **strictly positive**, then $\mathbf{A} \rightarrow B$ is **strictly positive**.

Metatheory for the Ω-rule cont'd

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- Proof can be easily made fully formal.
- An **accessibility operator form** $\mathcal{A}(P, Q, x, y)$ is of the form $A(x, y) \wedge \forall z[B(x, y, z) \rightarrow P(z)]$, where A, B belong to $\mathcal{L}_1(Q)$.

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Friedman's impractical matters

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- **Section 4: Practical Matters**
- Let $\mathcal{Q}$ be the well-quasi ordering of all finite trees with 6 labels under label preserving homoemorphisms.
- Let **SWQ**($\mathcal{Q}$) be the statement that for any c there exists a number k which is so large that, for any sequence $\langle T_0, \ldots, T_k \rangle$ of trees in $\mathcal{Q}$ with $|T_i| \leq c \cdot (i+1)$ for all $i \leq k$, there exist indices $i < j \leq k$ such that T_i is homoemorphically embeddable into T_j.
- Let $\Psi_{\mathcal{Q}}(c)$ be the smallest such k.

Impractical matters cont'd

Theorem. **SWQ**($\mathcal{Q}$) is provable in Π^1_2-**BI** but **not** in Π^1_2-$\mathbf{BI}_0$.

Impractical matters cont'd

Theorem. $\mathbf{SWQ}(\mathcal{Q})$ is provable in Π^1_2-$\mathbf{BI}$ but **not** in Π^1_2-$\mathbf{BI}_0$.

- When c is specialized we obtain a Σ^0_1 statement $\mathbf{SWQ}_c(\mathcal{Q})$.

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Why?

- For a theory T define

$$\chi_T(n)$$

to be the least integer k such that if $\exists x\, A(x)$ is any Σ^0_1 statement provable in T using $\leq n$ symbols, then $A(m)$ is true for some $m \leq k$.

Impractical matters cont'd

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- Let
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1. **Theorem.** The proof in [RW] of the 1-consistency of $\mathbf{T} := \Pi^1_2\text{-}\mathbf{BI}_0$ in $\mathbf{PA} + \mathbf{TI}(\theta\Omega^\omega 0)$ can be carried out by a proof shorter than 2^{1000} symbols.

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Thank you!