Π_3-reflection in Kripke-Platek set theory and nonmonotone inductive definitions from the class $[\Pi^0_1, \ldots, \Pi^0_1]$.

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Impredicative and metapredicative theories

Impredicative	Metapredicative
ID_1: P^A is the **least** fixed points	$\widehat{\mathsf{ID}}_1$: P^A is just **some** fixed points
Inductive definitions: $\mathsf{P}^A_\alpha := \mathsf{P}^A_{<\alpha} \cup F^A(\mathsf{P}^A_{<\alpha})$ **full** ordinal induction ordinals are **well-founded**	Inductive definitions: $\mathsf{P}^A_\alpha := \mathsf{P}^A_{<\alpha} \cup F^A(\mathsf{P}^A_{<\alpha})$ **restricted** ordinal induction ordinals are just **linearly ordered**
KPm, recursive Mahlo **full** $\in$-induction	KPm^0, metapredicative Mahlo **no** $\in$-induction

Impredicative and metapredicative theories

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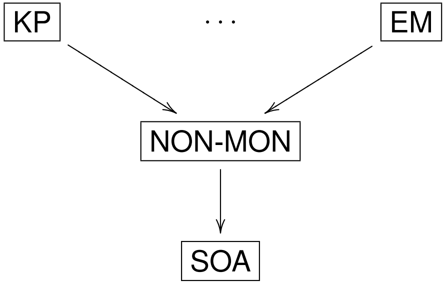
	KPm^0, metapredicative Mahlo **no** $\in$-induction

Non-monotone inductive definitions – a tool to embed admissible set theory into second order arithmetic



Remark

Ordinal analysis is simplest in subsystems of second order arithmetic

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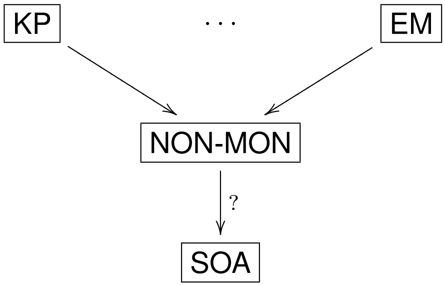

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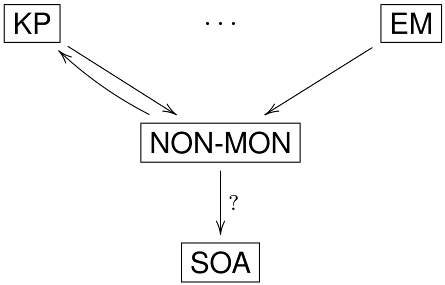

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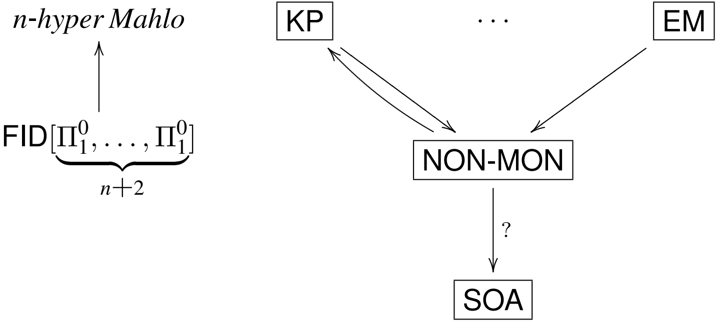

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FID($\mathcal{K}$): A theory of first order inductive definitions

FID($\mathcal{K}$) is formulated in a language extending L_1 by

- ordinal variables $\alpha, \beta, \gamma, \ldots$,
- a binary relation symbol $\alpha < \beta$,
- for each $\mathsf{L}_1(\mathsf{P})$ formula $A(\mathsf{P}, u) \in \mathcal{K}$, a binary relation symbol $\mathsf{P}^A(\alpha, x) =: \mathsf{P}^A_\alpha(x)$.

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Further abbreviations:

- $\mathsf{P}^A(s) := \exists\beta \mathsf{P}^A_\beta(s)$,
- $\mathsf{P}^A_{<\alpha}(s) := (\exists\beta < \alpha)\mathsf{P}^A_\beta(s)$,
- $s \in F^A(\mathsf{P}^A) := A(\mathsf{P}^A, s)$,

The axioms of FID$(\mathcal{K})$

- The axioms of PA without induction.
- $<$ is **just a linear** ordering on the ordinals with least element 0.
- $\mathsf{P}^A_\alpha = \mathsf{P}^A_{<\alpha} \cup F^A(\mathsf{P}^A_{<\alpha})$.
- $s \in F^A(\mathsf{P}^A) \to \mathsf{P}^A(s)$.
- Induction along N for all formulas.
- **restricted** ordinal induction:

$$\forall\alpha[(\forall\beta < \alpha)(B(\beta,\delta) \to B(\alpha,\delta)] \to \forall\alpha B(\alpha,\delta),$$

for each Δ_0^{ON} formula of the form $\delta \leq \gamma \to B'(\gamma,\delta)$ with only the displayed ordinal variables free.

An important observation

Each element $s \in \mathsf{P}^A$ enters P^A for a reason:

$$\mathsf{P}^A_\alpha(s) \wedge \mathsf{P}^A_{<\alpha}(s) \rightarrow (\exists\beta < \alpha)\mathsf{P}^A_\beta(s) \wedge \neg\mathsf{P}^A_{<\beta}(s).$$

Operator forms form $[\Pi^0_1, \ldots, \Pi^0_1]$

Definition

$A(P, u)$ is in $[\Pi^0_1, \Pi^0_1]$, if $A_1(P, u)$, $A_2(P, u)$ are Π^0_1, and $A(P, u)$ is as follows:

$$[F^{A_1}(P) \not\subseteq P \wedge A_1(P, u)] \vee [F^{A_1}(P) \subseteq P \wedge A_2(P, u)].$$

Definition

$A(P, u)$ is in $\underbrace{[\Pi^0_1, \ldots, \Pi^0_1]}_{n}$, if $A_1(P, u) \ldots, A_n(P, u)$ are Π^0_1, and $A(P, u)$ is a follows:

$$\bigvee_{1 \le i \le n} [\bigwedge_{j<i} (F^{A_j}(P) \subseteq P) \wedge F^{A_i}(P) \not\subseteq P \wedge A_i(P, u)].$$

KPu^0 and KPm^0: KPu and KPm without foundation

KPu^0 is formulated in $\mathsf{L}_1(\in, \mathsf{N})$. It formalizes a universe of sets above the natural numbers N, which are the urelements.

- For all axioms $A(\vec{a})$ of PA except induction, $\vec{a} \in \mathsf{N} \to A^{\mathsf{N}}(\vec{a})$ is an axiom of KPu^0.
- Kripke-Platek axioms: Pair, union, Δ_0-separation, Δ_0-collection.
- Complete induction on the natural numbers for **sets**.

An admissible set is a transitive model of KPu^0. KPm^0 is an extension of KPu^0 formulated in $\mathsf{L}_1(\in, \mathsf{N}, \mathsf{Ad})$. It formalizes the existence of

- arbitrary large admissible sets that are **linearly ordered by** $\in$.
- Π_2-reflection on admissibles.

n-hyper Mahlo

Let $\pi_2(e,x)$ be a universal Π_2 formula. A^a is the formula obtained from A by replacing each quantifier $\mathcal{Q}x$ in A by $(\mathcal{Q}x \in a)$. Occasionally, we write $a \models A$ for A^a. and $a \in \mathsf{Ad}$ for $\mathsf{Ad}(a)$.

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Definition

- $\mathsf{Ad}_0(a) := \mathsf{Ad}(a) \wedge \forall x \in a)(\exists b \in a)(x \in b \wedge \mathsf{Ad}(b))$, and $\mathsf{Ad}_{n+1}(a)$ is

  $$a \in \mathsf{Ad} \wedge (\forall e \in \mathsf{N})[a \models \pi_2(x,e) \rightarrow (\exists b \in \mathsf{Ad}_n \cap a)(x \in b \wedge b \models \pi_2(x,e))].$$

  If $\mathsf{Ad}_0(a)$, then we say that a is 1-inaccessible,
  if $\mathsf{Ad}_{n+1}(a)$, then we say that a is n-hyper Mahlo.

- 0-hyper Mahlo is KPm^0 and $n+1$-hyper Mahlo is KPu^0 plus Π_2-reflection on admissibles that are n-hyper Mahlo:

  $$(\forall e \in \mathsf{N})[\pi_2(x,e) \rightarrow (\exists b \in \mathsf{Ad}_n)(x \in b \wedge b \models \pi_2(x,e))].$$

Parameter-free Δ-induction along $(a \cap \mathsf{Ad}, \in)$

Lemma

Assume that $A(u, v)$ is a Δ formula with only the displayed variables free. If

$$\emptyset \neq \mathcal{C} := \{x \in \mathsf{Ad} : A(x, d)\} \text{ and } (\forall x \in \mathcal{C})(d \in x),$$

then $\mathcal{C}$ has an $\in$-least element.

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Assume that $\mathcal{C}$ has no $\in$-least element.

- For each $a \in \mathcal{C}$, there is an $b \in \mathcal{C}$ with $b \in a$. Further, $\bigcap \mathcal{C} = \bigcap (\mathcal{C} \cap b) \in a$.
- Hence $c := \bigcap \mathcal{C}$ is a set, and $c \in c$.
- c is an intersection of admissible and thus satisfies Δ-separation.

  Therefore, $r := \{x \in c : x \notin x\} \in c$, and so $r \in r \Leftrightarrow r \notin r$!

A first application – hierarchies along $(a \cap \mathsf{Ad}, \in)$

For each $\mathsf{L}_1(\mathsf{P})$ formula $A(\mathsf{P}, u)$, $\mathsf{Hier}^A(f, a)$ is the conjunction of the following two Δ_0 formulas:

- $\mathsf{Dom}(f) = a \cap \mathsf{Ad}$,
- $(\forall x \in \mathsf{Dom}(f))[f(x) = f_{<x} \cup \{n \in \mathsf{N} : A^{\mathsf{N}}(f_{<x}, n)\}]$,

where $f_{<x} := \bigcup_{y \in x} f(y)$. Otherwise, $\{a \in \mathsf{Ad} : \neg(\exists f \in a^+)\mathsf{Hier}^A(f, a)\}$ had a $\in$-least element!

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Lemma (Provable in KPm^0)

For each $\mathsf{L}_1(\mathsf{P})$ *formula* $A(\mathsf{P}, u)$,

$$(\forall a \in \mathsf{Ad})(\exists! f \in a^+)\mathsf{Hier}^A(f, a),$$

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Embedding FID$[\underbrace{\Pi^0_1, \Pi^0_1}_{n+2}]$ into n-hyper Mahlo

If $A(\mathsf{P}, u)$ is an $\mathsf{L}_1(\mathsf{P})$ formula, then for each admissible a, $\mathsf{P}^A_a := f(a)$ and $\mathsf{P}^A_{<a} := f_{<a}$, where $f \in a^+$ is the unique function satisfying $\mathsf{Hier}^A(f, a^+)$.

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Lemma (Provable in KPm^0) (auxiliary lemma)

Let $C(U^+, V^-, u)$ be a Π^0_1 formula of L_1 and $a \in \mathsf{Ad}$. Then,

$$(\forall b \mathring{\in} a) C^{\mathsf{N}}(\mathsf{P}^A_{<a}, \mathsf{P}^A_b, n) \to C^{\mathsf{N}}(\mathsf{P}^A_{<a}, \mathsf{P}^A_{<a}, n).$$

Let $x \mathring{\in} a := x \mathring{\in} a \cap \mathsf{Ad}$.

Lemma (Provable in KPm^0) (Π_2-reflection on stages))

Let $A(u,v)$ be Δ such that $x,y,z \in \mathsf{Ad} \wedge y \in z \wedge A(x,y) \rightarrow A(x,z)$ and a 1-inaccessible. Then,

$$(\forall x \mathring{\in} a)(\exists y \mathring{\in} a)A(x,y) \rightarrow (\exists b \mathring{\in} a)(\forall x \mathring{\in} b)(\exists y \mathring{\in} b)A(x,y).$$

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- Assuming the premise, let $f(0) := \emptyset^+$ and, if $f(n) = c \in \mathsf{Ad}$, then $f(n{+}1)$ is the $\in$-least admissible of the class

$$\{z \in \mathsf{Ad} : c \in z \wedge (\forall x \in c)(\exists y \in z)A(x,y)\}.$$

- f is in a. Further, we can view $f(n)$ as a Σ-function symbol. Now let b be the $\in$-least element of the class

$$\{z \in \mathsf{Ad} : (\forall n \in \mathsf{N})(f(n) \in z)\}.$$

Lemma (Provable in KPm^0) (base case))

Let $A(P,u)$, $B(P,u)$ *be* $\mathsf{L}_1(\mathsf{P})$ *formulas with* B Π^0_1. *Suppose that* a *is* 1*-inaccessible. Then*

$$B^{\mathsf{N}}(\mathsf{P}^A_{<a}, n) \rightarrow (\exists b \mathring{\in} a) B^{\mathsf{N}}(\mathsf{P}^A_{<b}, n).$$

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$$B^{\mathsf{N}}(\mathsf{P}^A_{<a}, n) \to (\exists b \mathring{\in} a) B^{\mathsf{N}}(\mathsf{P}^A_{<b}, n).$$

- Let $C(U^+, V^-, u)$ s. t. $B(X,x) \leftrightarrow C(X,X,x)$. Assume $A^{\mathsf{N}}(\mathsf{P}^A_{<a}, n)$.
- $(\forall b \mathring{\in} a) C^{\mathsf{N}}(\mathsf{P}^A_{<a}, \mathsf{P}^A_b, n)$ (V^-)
- $(\forall b \mathring{\in} a)(\exists c \mathring{\in} a) C^{\mathsf{N}}(\mathsf{P}^A_c, \mathsf{P}^A_b, n)$ (Σ reflection in a).
- $(\forall b \mathring{\in} d)(\exists c \mathring{\in} d) C^{\mathsf{N}}(\mathsf{P}^A_c, \mathsf{P}^A_b, n)$ for some $d \mathring{\in} a$. (Π_2-refl. Lemma)
- $(\forall b \mathring{\in} d) C^{\mathsf{N}}(\mathsf{P}^A_{<d}, \mathsf{P}^A_b, n)$ (U^+)
- By the aux. Lemma: $C^{\mathsf{N}}(\mathsf{P}^A_{<d}, \mathsf{P}^A_{<d}, n)$, i.e. $B^{\mathsf{N}}(\mathsf{P}^A_{<d}, n)$.

Lemma (Provable in KPm^0)

Let $A(P, u)$ be an operator form from $[\Pi^0_1, \ldots, \Pi^0_1]$ with components $A_0, \ldots, A_n$. For all $k \leq n$, if $a \in \mathsf{Ad}_k$, then for all $i \leq k$,

$$A_i^{\mathsf{N}}(\mathsf{P}^A_{<a}, n) \rightarrow n \in \mathsf{P}^A_{<a}.$$

Lemma (Provable in KPm^0)

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$$A_i^{\mathsf{N}}(\mathsf{P}^A_{<a},n)\to n\in\mathsf{P}^A_{<a}.$$

- Let $C_i(U^+,V^-,u)$ s. t. $A_i(X,x)\leftrightarrow C_i(X,X,x)$. Assume $A_i(\mathsf{P}^A_{<a},n)$.
- $k=0$: if $A_0(\mathsf{P}^A_{<a},n)$, then $A_0(\mathsf{P}^A_{<b},n)$ for some $b\mathring{\in} a$, and $n\in\mathsf{P}^A_b$ whether A_0 is active at stage b or not.
- $k\to k{+}1$: As before, $(\forall b\mathring{\in} a)(\exists c\mathring{\in} a)C_i^{\mathsf{N}}(\mathsf{P}^A_c,\mathsf{P}^A_b,n)$. $(i\le k{+}1)$
- Since $a\in\mathsf{Ad}_{k+1}$, there is an $a'\in\mathsf{Ad}_k$ with $(\forall b\mathring{\in} a')(\exists c\mathring{\in} a')C_i^{\mathsf{N}}(\mathsf{P}^A_c,\mathsf{P}^A_b,n)$.
- $i\le k$: $n\in\mathsf{P}^A_{<a'}$ by the I.H.
  $i=k{+}1$: A_{k+1} is active at stage a' thus $n\in\mathsf{P}^A_{a'}$.

Remark

- $\mathsf{KPu}^0 + \exists a \mathsf{Ad}_{n+1}(a)$ *is* KPu^0 *above a model of* n*-hyper Mahlo.*
- $$|\mathsf{KPu}^0 + \exists a \mathsf{Ad}_{n+1}(a)| = |n\text{-hyper Mahlo} + \text{fml ind. along } \mathsf{N}|.$$

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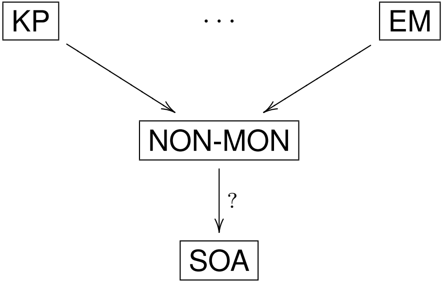
$$|\mathsf{KPu}^0 + \exists a \mathsf{Ad}_{n+1}(a)| = |n\text{-}\textit{hyper Mahlo} + \textit{fml ind. along } \mathsf{N}|.$$

Theorem

$|\mathsf{FID}[\underbrace{\Pi^0_1, \ldots, \Pi^0_1}_{n+2}]| \leq |n$*-hyper Mahlo + fml ind*$|$, *and*

$|\bigcup_n \mathsf{FID}[\underbrace{\Pi^0_1, \ldots, \Pi^0_1}_{n}]| \leq |\mathsf{KPu}^0 + \Pi^1_3\text{-}\mathsf{Refl}_{\mathsf{Ad}}|$.

Non-monotone inductive definitions – a tool to embed set theory into SOA?



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A family of theories $\mathsf{T}^{\vec{\alpha}}$ build by two operations

Let $C(U)$ be an L_2 formula and $\pi^1_2(U,u)$ a universal Π^1_2 formula. Then

- Limit: $l(C) := \forall X \exists Y [X \dot{\in} Y \wedge Y \models C(X)]$.
- Π^1_2-reflection: $p(C)$ is the universal closure of

$$\pi^1_2(X,e) \to \exists Y [X \dot{\in} Y \wedge Y \models C(X) \wedge Y \models \pi^1_2(X,e)].$$

Next, for $\alpha, \beta, \gamma, \ldots$ below Φ_0, we assign the theories as follows:

The theories of the form $\mathsf{T}^{\alpha,\beta,\gamma}$ and $\mathsf{T}^{\alpha,\Phi_0,0}$

- $\mathsf{T}^0(\emptyset) := \exists Y [Y \models \exists X (X = \{x : \pi^0_1(\emptyset, x, e)\}]$,
- $\mathsf{T}^{\alpha,\beta,\gamma} := l^\gamma \circ (l \circ p^1)^\beta \circ (l \circ p^2)^\alpha \ (\mathsf{T}^0)$.
- $\mathsf{T}^{\alpha,\Phi_0,0} := (p^2) \circ (l \circ p^2)^\alpha \ (\mathsf{T}^0)$.

Theorem

Let $0 < \Phi_0$ *be the least ordinal closed under all* n*-ary Veblen functions* φ^n*. Then we have for all* $\alpha_1, \ldots, \alpha_{k-1} < \Phi_0$*,*

- $|\mathsf{T}^{\alpha_{k-1},\ldots,\alpha_1}| = \varphi^k \alpha_{k-1} \ldots \alpha_1 0$.
- $|\mathsf{T}^{\alpha_{k-1},\ldots,\alpha_i,\Phi_0,\vec{0}}| = \varphi^k \alpha_{k-1} \ldots \alpha_i \omega \vec{0} 0$.

Example

- The theory $\mathsf{T}^{1,0}$ is the limit of $p(\mathsf{T}^0) \simeq \Sigma^1_1$-$\mathsf{DC}_0$.
  $|\mathsf{ATR}_0| = \varphi^3 100 = \Gamma_0$.
- T^{1,Φ_0} formalizes Π^1_2-reflection on models of ATR_0.
  $|\mathsf{ATR}_0 + (\Sigma^1_1\text{-}\mathsf{DC})| = \varphi^3 1 \omega 0$.
- $\mathsf{T}^{\Phi_0,0}$ is $p^2(\mathsf{T}^0) = p(\mathsf{T}^{\Phi_0})$.
  $|\Pi^1_2\text{-}\mathsf{Refl}_{(\Sigma^1_1\text{-}\mathsf{DC})}| = \varphi^3 \omega 00$.

Π^1_n-reflection

Definition

Let C be some L_2 sentence. Then

$$\Pi^1_n\text{-Refl}_C(W,e) := \pi^1_n(W,e) \to \exists M[W \dot{\in} M \wedge M \models C \wedge M \models \pi^1_n(W,e)],$$

is an instance of Π^1_n reflection on models of C.

Further, $\Pi^1_n\text{-Refl}_C := \forall X, x \Pi^1_n\text{-Refl}_C(X,x)$, which is a Π^1_{n+1} sentence.

Remark (The contraposition of $\Pi^1_n\text{-Refl}_C(W,e)$)

Suppose that $A(U)$ is Σ^1_n. Then

$$\forall M[W \dot{\in} M \wedge M \models C \to A^M(W)] \to A(W), \text{ i.e.}$$

$$W \dot{\notin} M, M \not\models C, A^M(W), A(W).$$

Iterated Π^1_n-reflection exhausts Π^1_{n+1}-reflection

Definition (Iterated Π^1_n-reflection)

- $\mathsf{S}^0_n := (\mathsf{ACA}) := \forall X, e \exists Y[Y = \{x : \pi^0_1(X, e, x)\}]$,
- $\mathsf{S}^{k+1}_n := \Pi^1_n\text{-}\mathsf{Refl}_{S^k_n}$.

Lemma

If Γ is a finite set of Σ^1_n formulas, then

$$\mathsf{ACA}_0 + \Pi^1_{n+1}\text{-}\mathsf{Refl}_{(\mathsf{ACA})} \,\big|\!\frac{k}{*}\; \Gamma \Longrightarrow \mathsf{S}^k_n \vdash \Gamma.$$

- $\mathsf{T} \vdash^{k+1}_{*} \Gamma$ is obtained by a cut with $[\neg]\Pi^1_{n+1}$-$\mathsf{Refl}_{(\mathsf{ACA})}$.

  Assume that $\pi^1_{n+1}(U,s) = \forall X B(X,U)$.

  $\wedge$- and $\forall$-inversion and the I.H. yield

$$\begin{array}{lcl} \mathsf{S}^k_n & \vdash & \Gamma, \exists Y B(Y,U), \\ \mathsf{S}^k_n & \vdash & \Gamma, U \dot{\notin} M, M \not\models (\mathsf{ACA}), M \not\models \forall X B(X,U). \end{array}$$

- In S^{k+1}_n we have models of S^k_n above arbitrary sets:

$$\begin{array}{lcl} \mathsf{S}^{k+1}_n & \vdash & \vec{W}, U \dot{\notin} M, M \not\models \mathsf{S}^k_n, \Gamma^M, M \models \forall X B(X,U), \\ \mathsf{S}^{k+1}_n & \vdash & \Gamma, U \dot{\notin} M, M \not\models \mathsf{S}^k_n, M \not\models \forall X B(X,U). \end{array}$$

- A cut yields $\mathsf{S}^{k+1}_n \vdash \vec{W}, U \dot{\notin} M, \Gamma, \Gamma^M, M \not\models \mathsf{S}^k_n$.
- By contraposition of Π^1_n-$\mathsf{Refl}_{\mathsf{S}^k_n}$ we get $\mathsf{S}^{k+1}_n \vdash \Gamma$.

Read a formula of ID_1 as an L_2 formula by reading $s \in \mathsf{P}^A$ as an abbreviation for $\forall X[F^A(X) \subseteq X \rightarrow s \in X]$.

Lemma

Suppose that ID_1 *proves* Γ *and that (the translation of) all formulas that occur in the proof-tree are at most* Σ^1_n*.*

$$\mathsf{ID}_1 \overset{k}{\underset{*}{\vdash}} \Gamma \Longrightarrow \mathsf{ACA} + \mathsf{S}^k_n \vdash \Gamma.$$

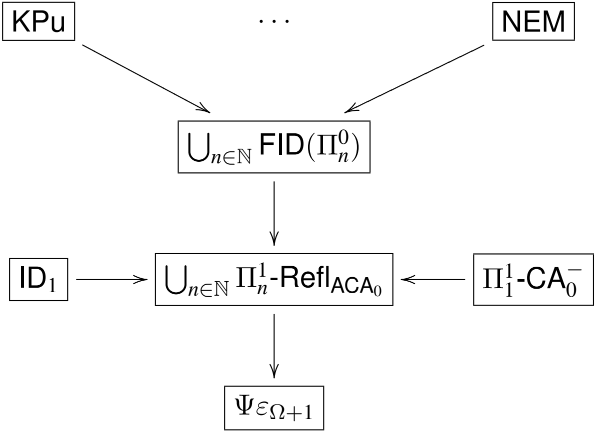
Read a formula of ID_1 as an L_2 formula by reading $s \in \mathsf{P}^A$ as an abbreviation for $\forall X[F^A(X) \subseteq X \rightarrow s \in X]$.

Lemma

Suppose that ID_1 *proves* Γ *and that (the translation of) all formulas that occur in the proof-tree are at most* Σ^1_n*.*

$$\mathsf{ID}_1 \mathrel{\substack{k \\ \vdash \\ *}} \Gamma \Longrightarrow \mathsf{ACA} + \mathsf{S}^k_n \vdash \Gamma.$$

- Have: $\mathsf{ID}_1 \mathrel{\substack{k+1 \\ \vdash \\ *}} \Gamma, \mathsf{P}^A \subseteq F^B(\mathsf{P}^A)$.
- By I.H. $\mathsf{S}^k_n \mathrel{\substack{k \\ \vdash \\ *}} \Gamma, F^A(F^B(\mathsf{P}^A)) \subseteq F^B(\mathsf{P}^A)$.
- $\mathsf{S}^{k+1}_n \vdash M \not\models \mathsf{S}^k_n, \Gamma^M, F^A(F^B(\mathsf{P}^A \restriction M)) \subseteq F^B(\mathsf{P}^A \restriction M)$.
- $\mathsf{P}^A \restriction M$ is a set in S^{k+1}_n: $\mathsf{S}^{k+1}_m \vdash M \not\models \mathsf{S}^k_n, \Gamma^M, s \notin \mathsf{P}^A, s \in F^B(\mathsf{P}^A \restriction M)$.
- Now Π^1_n-$\mathsf{Refl}_{\mathsf{S}^k_n}$ yields $\mathsf{S}^{k+1}_n \vdash \Gamma, s \notin \mathsf{P}^A, s \in F^B(\mathsf{P}^A)$.


KPu
. . .
NEM
$\bigcup_{n\in\mathbb{N}} \mathsf{FID}(\Pi^0_n)$
ID_1
$\bigcup_{n\in\mathbb{N}} \Pi^1_n\text{-}\mathsf{Refl}_{\mathsf{ACA}_0}$
$\Pi^1_1\text{-}\mathsf{CA}^-_0$
$\Psi\varepsilon_{\Omega+1}$