RESOLUTION OF SINGULARITIES

For every algebraic variety V we would like to find a non-singular variety V' which is birationally equivalent to V.

Hironaka (1965): This is possible in characteristic 0 in all dimensions.

Abhyankar, and others: This is possible in positive characteristic 0 for dimensions ≤ 3.

OPEN QUESTION: Does resolution of singularities hold in positive characteristic for all dimensions?

If we can't solve a problem globally, we try to solve it locally. Can we at least get rid of a single singularity? Given a variety V with a singular point, can we find a birationally equivalent variety V' on which the corresponding point is non-singular? This local version of resolution of singularities is called **local uniformization**.

Zariski proved local uniformization in characteristic 0 in 1940 and used it to prove resolution of singularities in characteristic 0 for dimension 2.

OPEN QUESTION: Does local uniformization hold in positive characteristic for all dimensions?

What is the correspondance that associates to a point on the variety V a point on the new variety V'?

Every variety V defined over a field K has a coordinate ring $K[V]$ and a function field $K(V)$, the quotient field of $K[V]$.

V and V' are birationally equivalent if and only if

$$K(V) = K(V') \ .$$

Every point on V is given by a homomorphism from $K[V]$ into some extension field of K. What is the corresponding homomorphism on $K[V']$?

Solution: extend the homomorphism to a place of $K(V)$, then restrict this place to $K[V']$. If $K[V']$ lies in the valuation ring of the place, then this restriction is a homomorphism on $K[V']$.

That the corresponding point on V' be non-singular is a condition on the place, namely that the Implicit Function Theorem holds. This in turn is essentially the condition in the multidimensional version of Hensel's Lemma, and so local uniformization is a valuation theoretical property.

Connection between the local uniformization problem and the decidability problem

Singularity = failure of the Implicit Function Theorem = Non-trivial valuation theoretical ramification.

• Local uniformization = elimination of ramification from the generation of the algebraic function field.

In order to prove completeness or decidability of algebraic theories, we pass from logic to algebra by use of sufficiently saturated models (reduction to embedding lemmas). By Robinson's Test and saturation, we reduce to finitely generated extensions. In the field case, these are just the algebraic function fields.

• In order to prove embedding lemmas for algebraic function fields, we again have to eliminate ramification!

We have to eliminate ramification.

In characteristic 0, there is only tame ramification.

In positive characteristic, we also have to deal with wild ramification, which is much harder because of the presence of the **defect**.

Classification problems

a) Classification up to isomorphism

– simple groups

– finitely generated non-trivial torsion-free abelian groups

b) Classification up to elementary equivalence

Often works when there is no hope of a classification up to isomorphism.

Classification of valued fields by their value groups and residue fields: the Ax–Kochen–Ershov Principle

$$vK \equiv vL \wedge Kv \equiv Lv \implies (K,v) \equiv (L,v)$$

For which valued fields does this hold?

Presence of the defect destroys the connection between valued fields and their value groups and residue fields!

Try to avoid or tame the defect.

My own contributions

• Ax–Kochen–Ershov Principle for tame valued fields (the largest class of valued fields for which the principle has been proved to date). Decidability of the theory of a tame valued field relative to those of its value group and its residue field.

If Γ is a p-divisible ordered abelian group, then $\mathbb{F}_p((t^\Gamma))$ is a tame field.

• Local uniformization for Abhyankar places in arbitrary characteristic and dimension.

• Local uniformization, up to a finite extension of the function field (alteration), in arbitrary characteristic and dimension.

$\mathbb{F}_p((t))$

Unfortunately, $\mathbb{F}_p((t))$ is not a tame field!

So the decidability problem is still open!

[K, 2001]: There are sentences describing the behaviour of additive polynomials on $\mathbb{F}_p((t))$ that are independent from the (adjusted) axiom system taken over from $\mathbb{Q}_p$. So there is much more "going on" in $\mathbb{F}_p((t))$ than in $\mathbb{Q}_p$.

On the positive side:

Denef and Schoutens (2003):
If resolution of singularities holds in characteristic p, then the existential elementary theory of the discrete valuation ring $\mathbb{F}_p[[t]]$ is decidable.

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