(UN)DECIDABLITY

Undecidable:

predicate calculus, Peano arithmetic (Church)

Decidable:

- Presburger arithmetic (Presburger)
- Elementary theory of the ordered field $\mathbb{R}$ (Tarski)
("Tarski Principle": completeness of the axiom system for real closed fields)
- Elementary theory of the field $\mathbb{C}$
("Poor Man's Lefschetz Principle": completeness of the axiom system for algebraically closed fields of fixed characteristic)
- Elementary theory of non-trivial divisible ordered abelian groups
- Elementary theory of algebraically closed non-trivially valued fields of fixed characteristic (Robinson).

VALUATIONS

A) Rational function fields

For a given rational function:

multiplicity of
a zero: positive
a pole: negative

Example: K any field, a, b, c, d distinct elements of K.
Take the rational function

$$r(X) \;=\; \frac{(X-a)^3}{(X-b)(X-c)^5}$$

and consider the following valuations on $K(X)$:

$$\begin{aligned}
v_{X-a}\,(r(X)) &= 3 \\
v_{X-b}\,(r(X)) &= -1 \\
v_{X-c}\,(r(X)) &= -5 \\
v_{X-d}\,(r(X)) &= 0
\end{aligned}$$

Evaluation homomorphisms and places

The evaluation of polynomials at an element $a \in K$

$$K[X] \ni f(X) \mapsto f(a) \in K$$

is a ring homomorphism. It remains a ring homomorphism on the valuation ring

$$\mathcal{O}_{X-a} = \left\{ \frac{f(X)}{g(X)} \mid f, g \in K[X],\ g(a) \neq 0 \right\}$$

$$\mathcal{O}_{X-a} \ni \frac{f(X)}{g(X)} \mapsto \frac{f(a)}{g(a)} \in K\ .$$

Extend this homomorphism to a **place** P_{X-a} by setting

$$r(X)P_{X-a} = \begin{cases} r(a) & \text{if } r(X) \in \mathcal{O}_{X-a} \\ \infty & \text{otherwise.} \end{cases}$$

For

$$r(X) = \frac{(X-a)^3}{(X-b)(X-c)^5}$$

we have:

$$\begin{aligned} r(X)P_{X-a} &= 0 \\ r(X)P_{X-b} &= \infty \\ r(X)P_{X-c} &= \infty \\ r(X)P_{X-d} &= \frac{(d-a)^3}{(d-b)(d-c)^5} \in K \end{aligned}$$

B) p-adic valuations

We can do the same for rational numbers that we did for rational functions. Choose a prime number p. Take $m, n \in \mathbb{Z} \setminus \{0\}$ and write

$$\frac{m}{n} \;=\; p^{\nu} \cdot \frac{m'}{n'}$$

with $\nu \in \mathbb{Z}$ and $m', n' \in \mathbb{Z} \setminus \{0\}$ such that p does not divide m' and n'. Then set

$$v_p\left(\frac{m}{n}\right) \;=\; \nu \;.$$

The canonical epimorphism

$$\mathbb{Z} \ni m \;\mapsto\; \overline{m} \in \mathbb{Z}/p\mathbb{Z}$$

extends to a homomorphism on the valuation ring

$$\mathcal{O}_p \;=\; \{\frac{m}{n} \mid m, n \in \mathbb{Z},\ (p, n) = 1\}$$

$$\mathcal{O}_p \ni \frac{m}{n} \;\mapsto\; \frac{m}{n}P_p := \overline{m} \cdot \overline{n}^{-1} \in \mathbb{Z}/p\mathbb{Z}$$

because $\mathbb{F}_p := \mathbb{Z}/p\mathbb{Z}$ is a field, and we set

$$\frac{m}{n}P_p \;:=\; \infty \quad \text{if} \quad \frac{m}{n} \notin \mathcal{O}_p \;.$$

The field $\mathbb{Q}_p$ of p-adic numbers is the completion of $\mathbb{Q}$ under the p-adic metric $|x - y|_p = p^{-v_p(x-y)}$.

C) Fields of formal Laurent series

Take any field K and define

$$K((t)) \;:=\; \{\sum_{i=N}^{\infty} c_i t^i \mid N \in \mathbb{Z},\, c_i \in K\}$$

This is a field. The t-adic valuation is given by

$$v_t\left(\sum_{i=N}^{\infty} c_i t^i\right) \;:=\; N \quad \text{ if } c_N \neq 0$$

and the t-adic place is defined through

$$tP_t \;:=\; 0\,.$$

The valuation ring is

$$\mathcal{O}_t \;=\; K[[t]] \;=\; \{\sum_{i\geq 0} c_i t^i \mid c_i \in K\}\,.$$

We have

$$\left(\sum_{i\geq 0} c_i t^i\right) P_t \;=\; c_0 \in K\,,$$

and the elements outside of the valuation ring are sent to ∞.

Value groups and residue fields

In all of our examples so far, the values of the non-zero elements were in the ordered abelian group $\mathbb{Z}$, and the finite images under the place were elements of the field K ($= \mathbb{F}_p$ in the p-adic case).

For arbitrary valued fields (L, v), we have the

valuation ring: $\mathcal{O}_v := \{a \in L \mid v(a) \geq 0\}$ with unique maximal ideal $\mathcal{M}_v := \{a \in L \mid v(a) > 0\}$

value group: the ordered abelian group $vL := \{va \mid 0 \neq a \in L\}$

residue field: the field $Lv := \mathcal{O}_v / \mathcal{M}_v$

and the homomorphism part of the place is the canonical epimorphism

$$\mathcal{O}_v \to \mathcal{O}_v / \mathcal{M}_v \ .$$

D) Power series fields

Take any field K and any ordered abelian group Γ and define

$$K((t^\Gamma))$$

to be the set of all power series

$$\sum_{\gamma\in\Gamma} c_\gamma t^\gamma$$

with coefficients $c_\gamma \in K$ for which the support

$$\{\gamma \in \Gamma \mid c_\gamma \neq 0\}$$

is well-ordered. This is a field.

The t-adic valuation is given by

$$v_t\left(\sum_{\gamma\in\Gamma} c_\gamma t^\gamma\right) \;:=\; \min\{\gamma \in \Gamma \mid c_\gamma \neq 0\}$$

and the t-adic place is defined as before.

This valued field has value group

$$v_t K((t^\Gamma)) \;=\; \Gamma\;,$$

and residue field

$$K((t^\Gamma))v_t \;=\; K\;.$$

Power series fields as non-standard models

For every divisible ordered abelian group Γ,

$$\mathbb{R}((t^{\Gamma}))$$

is a (nonstandard) model of the elementary theory of $\mathbb{R}$.

But such power series fields can never be models of the elementary theory of $\mathbb{R}$ with the exponential function [Kuhlmann, Kuhlmann & Shelah].

However, such models can be constructed as unions over ascending chains of such power series fields, each of which carries a non-surjective logarithm which becomes surjective in the union.

Artin's conjecture

Let $i \geq 0$ and $d \geq 1$ be integers. A field K is called $\mathrm{C}_i(d)$ if every form (that is, homogeneous polynomial) of degree d in $n > d^i$ variables has a nontrivial zero. Further, K is called C_i if it is $\mathrm{C}_i(d)$ for every $d \geq 1$.

Artin's conjecture:

$\mathbb{Q}_p$ is a C_2 field, for every prime p.

$\mathbb{F}_p$ is a C_1 field (Chevalley).
$\mathbb{F}_p((t))$ is a C_2 field (Lang).

$\mathbb{Q}_p$ and $\mathbb{F}_p((t))$ are very much alike:

- same value group: $\mathbb{Z}$
- same residue field: $\mathbb{F}_p$
- both are complete under their valuation, whence henselian.

But one has characteristic 0, the other characteristic p.

Is $\mathbb{Q}_p$ a C_2 field like $\mathbb{F}_p((t))$?

No, Artin's conjecture is not true: For $d \geq 4$, not all $\mathbb{Q}_p$ are $C_2(d)$ fields. Terjanian showed that the form

$$f(X_1, X_2, X_3) + f(Y_1, Y_2, Y_3) \\ +f(Z_1, Z_2, Z_3) + 4f(U_1, U_2, U_3) \\ +4f(V_1, V_2, V_3) + 4f(W_1, W_2, W_3)$$

with

$$f(X_1, X_2, X_3) = X_1^4 + X_2^4 + X_3^4 \\ -X_1^2X_2^2 - X_1^2X_3^2 - X_2^2X_3^2 \\ -X_1^2X_2X_3 - X_1X_2^2X_3 - X_1X_2X_3^2$$

does not admit a nontrivial zero in $\mathbb{Q}_2$.

But Ax and Kochen proved in 1965 that Artin's conjecture is "almost true":

Theorem:
For every positive integer d there exists a finite set of primes $A = A(d)$ such that $\mathbb{Q}_p$ is a $C_2(d)$ field, for every prime $p \notin A$.

Proof:

$$\prod_{p \in \mathbb{P}} \mathbb{Q}_p / \mathcal{D} \equiv \prod_{p \in \mathbb{P}} \mathbb{F}_p((t)) / \mathcal{D}$$

as valued fields, where $\mathbb{P}$ is the set of all prime numbers and $\mathcal{D}$ is a non-principal ultrafilter on $\mathbb{P}$.

This is because both ultraproducts are henselian valued fields with the same value group $\prod_{p\in\mathbb{P}} \mathbb{Z}/\mathcal{D}$, and the same residue field $\prod_{p\in\mathbb{P}} \mathbb{F}_p/\mathcal{D}$ of characteristic 0.

Ax–Kochen–Ershov Principle:

If (K,v) and (L,v) are henselian valued fields with Kv of characteristic 0, then

$$vK \equiv vL \;\wedge\; Kv \equiv Lv \quad\Longrightarrow\quad (K,v) \equiv (L,v)$$

Also in 1965, Ax and Kochen, and independently, Ershov proved:

• If (K,v) is a henselian valued field with Kv of characteristic 0, and if $\mathrm{Th}(vK)$ and $\mathrm{Th}(Kv)$ are decidable, then so is $\mathrm{Th}(K,v)$.

• $\mathrm{Th}(\mathbb{Q}_p)$ is decidable.

OPEN QUESTION: Is $\mathrm{Th}(\,\mathbb{F}_p((t))\,)$ decidable?

• Cherlin and others: In a language with a predicate for a cross-section (i.e., for the image of an embedding of the value group), $\mathrm{Th}(\,\mathbb{F}_p((t))\,)$ is undecidable!

• [K, 1989]: If Γ is a p-divisible ordered abelian group and $\mathrm{Th}(\Gamma)$ is decidable, then so is $\mathrm{Th}(\,\mathbb{F}_p((t^\Gamma))\,)$, in the pure language of valued fields.
In particular, $\mathrm{Th}(\,\mathbb{F}_p((t^{\mathbb{Q}}))\,)$ is decidable.