Finite trees as ordinals

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University of Oslo

Honouring Wilfried
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Typical trees

The natural numbers:

$$0 = \; .$$

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Typical trees

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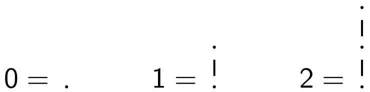

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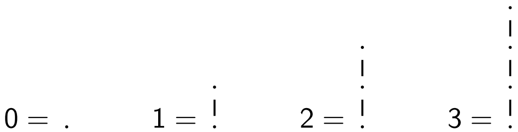

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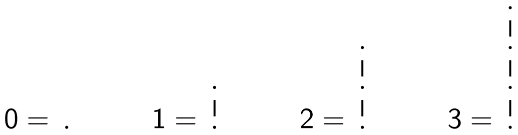

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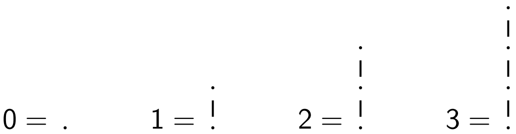

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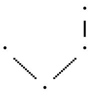
$\omega =$  $\omega^{\omega} =$ 

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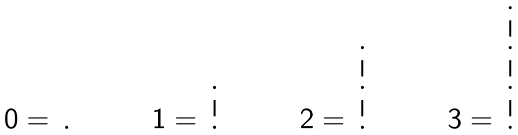

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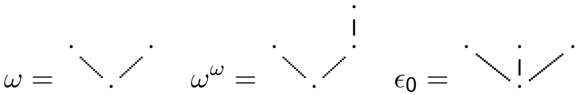

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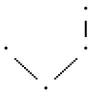
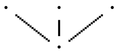
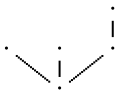
$\omega =$  $\omega^\omega =$  $\epsilon_0 =$  $\Gamma_0 =$ 

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  - Length of sequences
  - Rightmost element where they differ

Elementary properties

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- Decidable
- Transitive
- Linear
- Equality is the usual tree equality

Some ordinal functions

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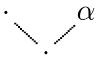

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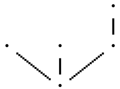
where $\sim$ means we jump over fix points.
In general we get the fix point free *n*-ary Veblen functions.

Approximating from below 1

$\Gamma_0 =$ 

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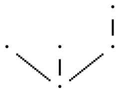
$\Gamma_0 =$ 

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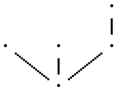
$\Gamma_0 =$ 

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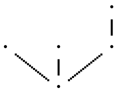
$\Gamma_0 =$ 

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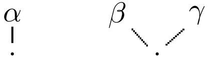

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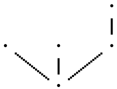
$\Gamma_0 =$ 

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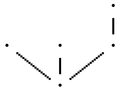
$\Gamma_0 =$ 

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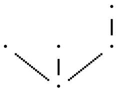
$\Gamma_0 =$ 

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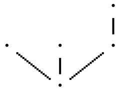
$\Gamma_0 =$ 

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This gives all trees less than Γ_0.

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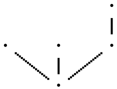
$\Gamma_0 =$ 

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Both arguments are straightforward.

Further work

Linear extensions of embeddings

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$$\left| \begin{array}{ccc} A & & B \\ & \diagdown \; \diagup & \\ & \cdot & \end{array} \right| \leq \begin{array}{ccc} |A| & & |B| \\ & \diagdown \; \diagup & \\ & \cdot & \end{array} \oplus \begin{array}{ccc} |B| & & |A| \\ & \diagdown \; \diagup & \\ & \cdot & \end{array}$$

This gives Higmans lemma. Further work gives Kruskals theorem.

Further work

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