The EPR paradox and Bell’s inequalities

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1 Do we really need quantum mechanics?

[see Asher Peres: Quantum Theory — Concepts and Methods]

There is a persistent uneasiness in our perception of quantum mechanics. It appears to be at odds with very fundamental elements of our concept of the world. This was brought to a point by the famous paper by Einstein, Podolsky, and Rosen, Phys. Rev. 47, 777 (1934).

Based on very plausible elements of our world concept, Bell [J.S. Bell, Physics 1, 195 (1964)] has derived his famous inequalities that put limits on the correlations of completely separated systems. The inequalities are at odds with quantum mechanics. The inequalities are also at odds with experiments — at least one of the “plausible elements of our world concept” is wrong.

1.1 The EPR paradox

Quantum mechanics claims that a particle does not “have” simultaneously a momentum and a position, the complete information is in the wave function.
EPR construct a quantum mechanical state that supposedly shows that a system must “have” position and momentum simultaneously, even if they may not be accessible to direct measurement.

There are two essential ingredients for setting up this paradox:

(1) **Locality**: the idea that large spatial separation can ensure independence of two systems.

(2) **Realism**: an operational concept of “physical reality” which should allow us to talk about which properties a system “has”.

Although (2) appears much more fuzzy, it seems that physicists’ suspicion is also directed against (1).

We take QM at face value in the sense that two subsystems are represented by any state in the tensor product space of the two spaces characterizing each of the subsystems:

\[
\Psi^{(a,b)} \in \mathcal{H}_a \otimes \mathcal{H}_b = L^2(dx_a dx_b, \mathbb{R}^3 \times \mathbb{R}^3)
\]

while

\[
\Psi^{(a)} \in \mathcal{H}_a, \quad \text{and} \quad \Psi^{(b)} \in \mathcal{H}_b.
\]

Suppose at some instant in time, you have a system of two particles in a peculiar wave packet state:

\[
\Psi = d(x_1 - x_2 - L)d(p_1 + p_2)
\]

where \(d\) is a function very well localized near 0 (approximating the \(\delta\) function). You might ask whether this is a state in the two particle Hilbert space.

It is: we can just change coordinates \((x_1, x_2) \rightarrow (x_1 - x_2, x_1 + x_2)\) and Fourier transform with respect to \(x_1 +\)
\[ x_2 \rightarrow p_1 + p_2 \] and then safely set up our wavepacket as above. This is a very formal argument with the purpose to show that the wave function is a legitimate one within the formal framework of quantum mechanics. In its original intention it describes two particles about which we only know (1) they are separated by \( L \) and (2) they move at equal momenta in opposite directions. Quantum mechanics tells us that this contains the complete information about the system that is accessible to us. EPR introduce the idea of “physical reality” to reason that even if this may be all that is accessible to us, there “is” more “reality” in such a system.

Element of physical reality: “If, without in any way disturbing a system, we can predict with certainty ... the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity” (Quote from EPR).

A theory can only be legitimately called “complete”, if it includes all “elements of physical reality”.

The position of particle 2 is an “element of physical reality”: we can determine it by measuring the position of particle 1. As the particles are arbitrarily far separated, by assumption (1) we do this measurement “without in any way disturbing” particle 2. By the same argument, we could just as well measure \( p_1 \), and therefore also momentum \( p_2 \) is an element of physical reality. Therefore, EPR reason, quantum mechanics is not “complete”: The word “physical reality” implies that somehow particle 2 “has” a momentum \( p_2 \) and position \( x_2 \), which by quantum theory would be a meaningless statement.

The construction of the EPR paradox was criticized
by Bohr on the basis of the notion of “elements of physical reality”, as there is no measurement that would provide us with both, $x_1$ and $p_1$. This would deprive us of any predictions for $x_2$ and $p_2$ simultaneously: so in which sense could both quantities independently considered “real”? This may or may not be a justified argument. If it were justified, the EPR paradox would be reduced to the problem of particle-wave dualism, which I personally consider rather disquieting in the first place. I am inclined to this point of view.

1.2 Bell’s inequalities

Bell has taken a more general point of view. Instead of discussing the specific form of quantum theory, he set up his famous inequalities based on pretty much the two assumptions underlying the EPR criticism of quantum theory. He then shows that for theories based on these assumptions some inequalities hold that are violated by quantum mechanics. It appears that they also are violated by experiments.

Except for locality, which seems to be a rather clear cut concept, the essence of Bell’s realism is that it is meaningful to speak of a system to “have” a set of properties irrespective of whether we can measure them simultaneously or not, similar to Einstein’s “element of physical reality”.

Let us assume that we have two particles that are well separated such that manipulations (or measurement) on one particle cannot influence measurements on the other (“locality”, requires space-like separation of the two mea-
surement events in the sense of special relativity). Let us further assume that each particle “has” an internal state that completely determines the outcome of any measurement made on that particle (“realism” or “determinism”). A “particle” here is local by definition (different from a “wave”, that is defined by its variation over space). It does not matter whether we can in principle measure the complete information of that internal state or not.

For an example we imagine the two particles to be photons originating from a common source. We measure passage of the photo through a polarizer with two possible outcomes: 1 for pass, -1 for do not pass. We do a series of measurements \( j = 1, 2, 3, \ldots \), on these photons. We assume in the \( j \)th measurement the particles have the internal states \( \lambda_j \) and \( \mu_j \) which uniquely determine the outcome of any possible measurement. Each \( \lambda_j \) is a sufficiently large set of numbers to fully characterize the internal state of the first particle, likewise \( \mu_j \) for the second particle. The internal states are also called “hidden variables”. As by assumption each particle has its own \( \lambda_j \) and \( \mu_j \) we call them local hidden variables. In particular, the internal states would determine which result, +1 or −1, we would find if we measure polarization in arbitrary directions \( \vec{\alpha}, \vec{\beta}, \text{ or } \vec{\gamma} \). Denote the outcome of measurements in the corresponding directions on the first particle by the functions \( a(\lambda_j), b(\lambda_j), c(\lambda_j) \), and for the second particle \( a(\mu_j), b(\mu_j), c(\mu_j) \). With the restriction to values \( \pm 1 \) this looks like we are extracting a very small part of the internal information, but maybe that
is just what we are doing in our measurement.

Now assume that we generated the two particles in a correlated fashion, such that we know that for any measurement $c(\lambda_j) = c(\mu_j)$. This can be achieved e.g. with photons that for symmetry reasons must have parallel polarization.

For the first particle, we use two directions of the polarizer $\bar{\alpha}$ and $\bar{\gamma}$, for the second particle we use $\bar{\beta}$ and $\bar{\gamma}$. We call the measurement results $a(\lambda_j), b(\mu_j), c(\lambda_j) = c(\mu_j)$ for polarizer angles $\alpha, \beta, \gamma$, respectively. The possible outcomes for all measurements are $\pm 1$ in quantum mechanics; in our general model think of a digital switch that can only show these two results. We could measure $\gamma$ on either particle, and we know for sure, because of symmetry, that the outcome would be the same for either particle. (If you do not like this reasoning, there is a slightly more complex inequality by the name CHSH that does not use this, but rather relies on 4 different measurement angles.) The potential measurement results for each photon pair with internal states $\lambda_j$ and $\mu_j$ fulfill

$$1 - b(\mu_j)c(\lambda_j) = + a(\lambda_j)[b(\mu_j) - c(\mu_j)] \text{ or } = - a(\lambda_j)[b(\mu_j) - c(\mu_j)]$$

which can be easily verified by inserting the values $\pm 1$ for $b$ and $c$.

Note that we cannot actually measure the $b(\mu_j) - c(\mu_j)$ unless we assume that we do not disturb $\mu_j$ by our measurement of $b(\mu_j)$. However, with a polarizer we do disturb the measured system. This is sometimes called “counterfactual reasoning”. It assumes that a property is somehow “there”, even if we cannot give a prescription how to determine it. It reasons that if we were able to
do that measurement, we would get the inequality for each $j$. This is similar to the EPR concept of “physical reality”: it imagines something is there even if nobody can tell how to measure it.

Now we take the average value of these functions over many measurements $j = 1, 2, 3, ...$, i.e. sum up all potential results and divide by the number of measurements. As the left hand side is $\geq 0$, while the right hand side changes sign, we find

$$1 - \langle bc \rangle \geq \langle a[b - c] \rangle \text{ and } [1 - \langle bc \rangle] \geq -\langle a[b - c] \rangle$$

or

$$|\langle ab \rangle - \langle ac \rangle| \leq 1 - \langle bc \rangle.$$  

This is Bell’s inequality.

We cannot measure $\langle ab - ac \rangle$, but we may measure the “correlation functions” $\langle ab \rangle$ and $\langle ac \rangle$ separately. If the distribution of $\lambda_j$ and $\mu_j$ is statistical, random, we can split our measurement indices $j = 1, 2, 3, ...$ into three subsets $J$ and $K$ and $L$, compute the average values for each subset $\langle ab \rangle_J$, $\langle ac \rangle_K$ and $\langle bc \rangle_L$ and assume $\langle ab \rangle_J \sim \langle ab \rangle$ and likewise for $\langle bc \rangle$ and $\langle ac \rangle$. Randomness of the “internal states” $\lambda_j$ and $\mu_j$ may be ensured by randomly selecting the subsets $J$, $K$ and $L$. This hypothesis may be experimentally corroborated by just extending the measurement series to ever larger numbers. Bell’s inequality puts a rigorous bound on the correlation functions of different correlation functions of the same observable.

In view of this statistical argument in is difficult to see what could be wrong with counterfactual reasoning; but
of course, we are here at the very limits of our imagination and logics, far form the terrain that is secured by everyday experience. Therefore we need to proof statements, not ask, why they should be wrong. Thus, it remains a sore point in this whole chain of reasoning. It may limit the validity of the arguments to models, where in principle we can measure \( b(\mu_j) - c(\mu_j) \). After all, what is the point of talking of a property, if there is no effect which can be identify as an unambiguous consequence of this property? In the end, connecting an effect to a property is what we call a measurement. A property that does not lead to any effect ever, isn’t a property. Permission for counterfactual reasoning after all is somehow subsumed in the “reality” of the “hidden” variables.

1.3 The correlation of polarization measurements

The crucial tests of the assumptions of “physical reality” and locality of nature to date were all performed with light, i.e. with photons. We therefore briefly discuss the quantum mechanical polarization measurements of photons.

Polarization state of a photon: we can measure the polarization of a photon by inserting a polarizer into its path of propagation, say, along direction \( z \). Then the photon can have polarization directions in the \( xy \)-plane. A polarizer lets the photon pass, if the polarizer’s direction is parallel to the polarization direction of the photon, it does not let it pass, if the polarization direction is perpendicular to the direction of the polarizer. After the polarizer we know the polarization direction of the photon.
ton to be the same as the polarizer’s: measuring a photon behind a polarizer means to project the wave function on the polarization state in direction of the projector.

If \(|x\rangle\) and \(|y\rangle\) designate polarization states in the respective directions, a polarization measurement with a polarizer in direction \(\vec{\alpha} = (\cos \alpha, \sin \alpha, 0)\) is represented by the operator

\[
P_{\alpha} = (|x\rangle \cos \alpha + |y\rangle \sin \alpha)(\cos \alpha \langle x| + \sin \alpha \langle y|)
\]

This is manifestly a projector with eigenvalues 0 and 1. For convenience, we will work with the derived operators

\[
\sigma_{\alpha} = 2P_{\alpha} - 1 = (|x\langle x| - |y\rangle \langle y|) \cos 2\alpha + (|x\langle y| + |y\rangle \langle x|) \sin 2\alpha
\]

with eigenvalues \(\pm 1\). This can also be written in matrix form

\[
\sigma_{\alpha} = \begin{pmatrix} |x\rangle \\ |y\rangle \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cos 2\alpha + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sin 2\alpha \begin{pmatrix} \langle x| \\ \langle y| \end{pmatrix}
\]

With respect to the basis \(|x\rangle, |y\rangle\) these operators are represented by the Pauli matrices \(\sigma_{\alpha} = \cos 2\alpha \sigma_z + \sin 2\alpha \sigma_x\) and we see that polarization measurement can be mathematically mapped onto measurements of spin directions in the \(xz\)-plane.

**Problem 1.1:** Verify the mathematical form of the observable for polarization measurement used for discussing the violation of Bell’s inequalities:

\[
P_{\alpha} = (|x\rangle \cos \alpha + |y\rangle \sin \alpha)(\cos \alpha \langle x| + \sin \alpha \langle y|)
\]

\[
\sigma_{\alpha} = 2P_{\alpha} - 1 = (|x\langle x| - |y\rangle \langle y|) \cos 2\alpha + (|x\langle y| + |y\rangle \langle x|) \sin 2\alpha
\]
and finally
\[ \sigma_\alpha = \left( \begin{array}{c} |x\rangle \\ |y\rangle \end{array} \right) \cdot \left[ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cos 2\alpha + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sin 2\alpha \right] \left( \begin{array}{c} \langle x \rangle \\ \langle y \rangle \end{array} \right) \]

**Problem 1.2:** Show that the expectation value for a simultaneous measurement of polarization for a state \( |e\rangle := \left( |x\rangle \otimes |x\rangle + |y\rangle \otimes |y\rangle \right)/\sqrt{2} \) is
\[ \langle e|\sigma_\alpha \otimes \sigma_\beta|e\rangle = \cos 2(\alpha - \beta). \]

Assume you have a source of light that emits two photons at a time and that is rotationally invariant. Such a source could be an atom in an excited \( s \)-state (\( L = 0 \)) that decays to its \( L = 0 \) ground state by emitting two photons. As the atomic initial state is rotationally invariant, the total system after decay must also be rotationally invariant. And as the final atomic state is \( L = 0 \) also the state of the photons must be \( L = 0 \). This is a requirement of symmetry only.

Assume we measure only photons emitted in a well-defined direction (call it the \( z \)-direction) from the atom at two far separated locations \( A \) and \( B \). A complete basis for the polarization states of the two photons is \( |x\rangle \otimes |x\rangle, |x\rangle \otimes |y\rangle, |y\rangle \otimes |x\rangle, |y\rangle \otimes |y\rangle \). As the total state is rotationally invariant, it is in particular invariant under rotations around the \( z \)-axis, which leave only the “entangled” two-photon polarization states
\[ |x\rangle \otimes |x\rangle + |y\rangle \otimes |y\rangle \text{ and } |x\rangle \otimes |y\rangle - |y\rangle \otimes |x\rangle \]
where the latter has odd particle exchange symmetry. That means that in two photons emitted from a rotationally invariant process have parallel polarizations. If
we measure the polarization of one we can infer the polarization of the other. This discrete quantity now replaces what was momentum in the original formulation of the EPR paradox.

1.4 Experimental test of Bell’s inequalities

It is easy to see that the expectation value of the photon-pair state for two polarizers at the angles $\alpha$ and $\beta$

$$[\langle x \rangle \otimes \langle x \rangle + \langle y \rangle \otimes \langle y \rangle ] \sigma_\alpha \otimes \sigma_\beta [\langle x \rangle \otimes \langle x \rangle + \langle y \rangle \otimes \langle y \rangle ] = \cos 2(\alpha - \beta)$$

Be that as it may, Bell’s inequality relates expectation values of measurements to each other that can be computed by quantum mechanics. When we choose e.g. angles $\alpha = 0^\circ$, $\beta = 30^\circ$ and $\gamma = 60^\circ$ we violate Bell’s inequality

$$|\cos(-60^\circ) - \cos(-120^\circ)| + \cos(-60^\circ) = |1/2 + 1/2| + 1/2 = 3/2 > 1$$

This would be bad for quantum mechanics, if experiments had not found the same kind of violation of Bell’s inequality. So, it is bad for our preferred, intuitive, and only known way of thinking about reality.

The actual experiment [Aspect et al., Phys. Rev. Lett. 49, 1804 (1982)] uses four different angles $\alpha, \beta, \gamma, \delta$ and the inequality

$$|\langle ab \rangle + \langle bc \rangle + \langle cd \rangle - \langle da \rangle| \leq 2$$

which was found to be violated by 5 standard deviations, but in perfect agreement with the QM prediction. It appears that “local realism” is not a property of the world. The experiment has been confirmed many times since then, and with even more striking error margins.
This is where we stand today. We do not know whether “realism”, the idea that a system somehow “has” all properties that we can measure, and has them simultaneously, or the concept that far separated systems are independent of each other, or both are wrong. Certain explicit formulations of non-local theories have been ruled out by an experiment 2007, but this is not a universal statement for all non-local theories. Currently there seems to be popular inclination to think of the world as being inherently non-local. Thinking of the possible implications of this for our ability to understand and predict events makes me shudder. The alternative of it being not “real”, i.e. the properties of things not being only “their” properties but rather a joint product of “us” and “them”, is not much of a consolation. I dare say that all of mankind’s thinking is based on the concept of objects “out there” which we can perceive and about which we can think, but which have their “nature” or “reality” independently of us. Philosophers have always known that this may be an untenable position because the idea is very difficult to make precise. However, they have not offered useful alternatives. Now we have measurements that seem to tell us that the idea is wrong. Quantum mechanics may be right. But who understands it? So how shall we form a correct image of reality?