

Extended Crash course on Topology

László Erdős

Jan 25, 2004

Topology is a very basic field of mathematics; in a systematic introduction into mathematics it precedes analysis (but usually one starts with analysis for pedagogical reasons).

The idea is that we would like to generalize the concept of open/closed sets, convergence and continuity for the most general situation. So far we learned these concept for metric spaces which were already generalizations of the usual Euclidean spaces, where you have seen these concepts first.

Metric spaces are still special cases of those, so-called **topological** spaces, where all these concepts can be defined. Metric space naturally comes with a quantitative bonus: one can tell not just that a sequence converges, but also how fast it does so, because it does make sense to measure the distance from an element x_n of a sequence to its limit. However, in principle, the concept of convergence does not require distance, it really requires a collection of sets about each point, called “neighborhoods” that play the role of the small open balls around that point. When such a collection of neighborhoods (with certain natural properties) are specified, we can talk about continuity and convergence. We say that a **topology** is given on the base set X .

There is another way to specify the topology on any base set X and we follow this route, since it is somewhat faster.

Let X be an arbitrary set and let $\mathcal{O} \subset P(X)$ be a family of its subsets ($P(X)$ is the set of all subsets of X) that satisfy the following three properties:

- (i) \mathcal{O} is closed under finite intersection (this means that if $O_1, \dots, O_n \in \mathcal{O}$, then $\bigcap_i O_i \in \mathcal{O}$).
- (ii) \mathcal{O} is closed under union of arbitrary number of elements.
- (iii) $\emptyset, X \in \mathcal{O}$.

Note that these properties are satisfied by the collection of the open sets in any metric space. Now we *require* them, and given \mathcal{O} , we *declare* these sets to be open. With this declaration, we defined a topology on X .

Once the open sets are known, closed sets (complements of opens) and compact sets (any open cover thereof contains a finite cover) are defined naturally.

A set $N \subset X$ is called the neighborhood of a point x if there is an open set O such that $x \in O \subset N$.

A function between two topological spaces, $f : (X, \mathcal{O}_X) \mapsto (Y, \mathcal{O}_Y)$ is said to be continuous if the preimage of any open set is open, i.e. if $f^{-1}(A) \subset \mathcal{O}_X$ for any $A \in \mathcal{O}_Y$. A sequence x_n converges to x if for any open set $O \in \mathcal{O}_X$ that contains x , there exists a threshold N such that $x_n \in O$ whenever $n \geq N$.

You can read more about it in Werner Appendix B.2. or in Sec IV.1 of Reed-Simon.

It is an unpleasant feature of general topological spaces that one cannot prove continuity by sequences. If a function f is continuous at x and $x_n \rightarrow x$, then it still holds that $f(x_n) \rightarrow f(x)$ (check!), but this is not sufficient to prove that a function is continuous. This is due to the fact that a point can have so many different neighborhoods that a countable sequence cannot “explore” all of them.

There is a generalized concept of sequences (called nets) which replace the role of sequences, but this leads too far (see: Reed-Simon IV.2 or Werner p. 480).

So far we have seen that if somebody gives us a topology (collection of open sets) we can decide whether a function is continuous. We can reverse this argument. Suppose we have a collection \mathcal{F} of functions (for simplicity, scalar valued) defined on X . We would like to define a topology such that all $f \in \mathcal{F}$ be continuous. This is not hard, if we choose $\mathcal{O} = P(X)$, then we are done, but this is a useless definition. So we aim at the weakest (=least number of open sets) topology so that all $f \in \mathcal{F}$ are still continuous. To construct such a topology is not straightforward (of course every preimage $f^{-1}(a, b)$ of every open intervals under any $f \in \mathcal{F}$ must be included in \mathcal{O} , but we need many more sets to satisfy (i)–(iii), so we can keep on adding finite intersections and arbitrary unions of elements of \mathcal{O} to \mathcal{O}). But it is very easy to see that such topology exists. Simply take all families \mathcal{O} that satisfy (i)–(iii) and that contain all sets of the form $f^{-1}(a, b)$, $a, b \in \mathbf{R}$, $f \in \mathcal{F}$. There is at least such family, namely $\mathcal{O} = P(X)$. Now take the intersection of all these families. This is a family, that still satisfies (i)–(iii) (think over) and still contains all $f^{-1}(a, b)$, and it is the smallest for these properties.

The upshot is that for any given collection of functions, there is the weakest topology so that all these functions are continuous. Of course it does not mean that only these functions are continuous, clearly the sums, products etc. of such functions are still continuous.

The best example to see this new concept is $L^\infty(X)$ (without measure, i.e. this is the set of everywhere defined bounded functions). With the usual supremum norm it is a normed space, hence metric, in particular continuity can be checked via sequences.

We can define a topology different from the norm topology on this set $L^\infty(X)$ by requiring

that all functions $x : f \mapsto f(x)$ be continuous (and consider the weakest such topology). Don't get misled by the letters, here x denotes the function (on the space of functions f) and f is its variable. For example the following sets are open

$$S_{x,c,\varepsilon} := \{f \in L^\infty(X) : |f(x) - c| < \varepsilon\}$$

and by (i)–(iii) all finite intersections and arbitrary unions of such sets are also open.

When restricted to the unit ball, $[-1, 1]^X$, you can see that this is weakest topology on the direct product that makes all projections continuous, because the map

$$x : f \in [-1, 1]^X \mapsto f(x)$$

is exactly the projection map onto the x -component of the direct product.

This new topology is not the norm topology. This is not easy to see directly. One argument is that in the norm topology the unit ball is not compact, while in the new topology we have Tychonoff theorem:

Theorem [Tychonoff] Let K_α , $\alpha \in A$ be a family of compact topological spaces. Then their direct product, $\prod_{\alpha \in A} K_\alpha$ is compact in the product topology, i.e. in the weakest topology that makes all projections continuous.

This theorem is proved in Reed-Simon (Thm IV.5 and at the end: Supplement to IV.3) or in Werner Thm. B. 2.10, but it is not short. For countable direct product a Cantor diagonalization trick works (EXERCISE).

Notice that even the definition of the direct product with uncountable many elements is nontrivial and requires the Axiom of choice. The direct product $\prod_{\alpha \in A} X_\alpha$ of sets X_α is by definition the set of all functions $f : A \rightarrow \bigcup_{\alpha \in A} X_\alpha$ such that $f(\alpha) \in X_\alpha$. It is not obvious that this set is nonempty, actually it does not follow (in case of uncountable A) from the standard axioms of set theory. One can include it as an independent axiom. It is equivalent to Zorn lemma. (Zorn lemma is a theorem that one can prove from the Axiom of choice; or you can set Zorn lemma as an axiom and prove the “theorem of choice” about the nonemptiness of the direct product). We remark that such transfinite methods are not needed for countable many products.

Finally, if X is a Banach space, then the weak* topology on its dual, X^* is the weakest topology that makes all functionals of the form

$$x : \ell \mapsto \ell(x)$$

from X^* to the scalars, continuous. This is the topology that appears in the generalized version of Banach-Alaouglu (claiming that the unit ball of the dual of a Banach space is weak* compact).

We remark that the proof of Banach-Alaouglu is fairly easy from Tychonoff (see Thm IV.21 in Reed-Simon), but it uses the concept of nets which we skipped.