Existence of the algebraic basis

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Definition A subset $\{v_\alpha\}_{\alpha \in A} \subset V$ of a vector space V over the field F is called **spanning set** if

$$\text{Span}\{v_\alpha\} = V$$

where

$$\text{Span}\{v_\alpha\} = \Big\{ \sum_\alpha c_\alpha v_\alpha \;:\; c_\alpha \in F \text{ and } c_\alpha = 0 \text{ with finitely many exceptions} \Big\}$$

Definition A subset $\{v_\alpha\}_{\alpha \in A} \subset V$ of a vector space V over the field F is called **linearly independent** if whenever

$$\sum_\alpha c_\alpha v_\alpha = 0$$

holds for some coefficients c_α, that are all zero with at most finitely many exceptions, then all coefficients must be zero.

The index set A is an abstract set to label different elements. The Greek letter $\alpha \in A$ instead of i, n or k indicates that the index set is not necessarily the set of natural numbers in particular it may be uncountable.

However, note that the summation in the definition of linear independence and spanning property is always finite.

Definition A subset $\{v_\alpha\}_{\alpha \in A} \subset V$ of a vector space V is called **basis** if it is linearly independent and spanning.

Remark: This basis is also called *algebraic* or *Hamel* basis referring to the fact that only algebraic operations are used in its definition. Later we will see that this is NOT the useful notion of a basis. We will have to allow infinite sums in order to obtain a practically useful basis. The price is that we will have to define infinite sums (via convergence of its finite truncations) i.e. we have to introduce analysis (norm, convergence etc.) in our vectorspace.

Theorem: Every vectorspace V has a basis.

Proof. Consider the set S of all linearly independent subsets of V, i.e.

$$S := \Big\{ \{v_1, v_2, \dots v_k\} \ : \ \text{lin indep.} \Big\}$$

Introduce a natural ordering $\prec$ on S according to the containment; for any $s, s' \in S$ we have $s \prec s'$ whenever $s \subset s'$ (we allow $s = s'$ as well, i.e. $\subset$ is not strict containment). Clearly $(S, \prec)$ is a partially ordered set.

We check that every totally ordered subset $S' \subset S$ has an upper bound in S. Consider

$$\tau := \bigcup_{\sigma \in S'} \sigma$$

i.e. the union of all elements of S'. Clearly $\sigma \prec \tau$, so τ is an upper bound, but we also have to show that $\tau \in S$. Let $\tau = \{v_\alpha\}_{\alpha \in A}$, we have to check linear independence. Consider a finite linear combination

$$\sum_{j=1}^{k} c_{\alpha_j} v_{\alpha_j} \qquad (*)$$

that is zero. Let $\sigma_j \in S'$ such that $v_{\alpha_j} \in \sigma_j$. Since S' is totally ordered, we can assume that $\sigma_1 \prec \sigma_2 \prec \dots \prec \sigma_k$ (otherwise relabel these k elements), in particular σ_k contains all other subsets. Therefore $v_1, v_2, \dots v_k \in \sigma_k$. Viewing (*) as a linear combination of elements of σ_k, that is a linearly independent set, we conclude that all c_{α_j} are zero.

Note that the finiteness of the sum was absolutely essential, otherwise we may not have a "biggest" subset σ_k.

We have seen that the condition of Zorn lemma is satisfied, therefore S has a maximal element, call it σ, we claim that it is a basis. This is a subset of linearly independent elements of V, we need to show that it spans.

Suppose that it does not span, i.e. there exists a vector $v \in V$ that cannot be written as a linear combination of elements from σ. Construct

$$\sigma' = \sigma \cup \{v\}$$

Clearly $\sigma \prec \sigma'$ and $\sigma \neq \sigma'$. Since σ was maximal in S, σ' cannot belong to S, i.e. σ is not linearly independent. I.e. there is a finite, nontrivial linear combination (not all coeff's are zero) that gives zero. This linear combination must contain v with a nonzero coefficient, otherwise we would have a linear combination containing only elements from σ (with nonzero coefficients), which is impossible by the linear independence of σ. Therefore

$$cv + \sum_{j=1}^{k} c_{\alpha_j} v_{\alpha_j} = 0$$

where $v_{\alpha_1}, \ldots, v_{\alpha_k}$ are elements of σ and the coefficient c of v is not zero. Therefore we can express v as

$$v = \sum_{j=1}^{k} \Big(-\frac{c_{\alpha_j}}{c} \Big) v_{\alpha_j}$$

a linear combination of elements of σ, contradicting the claim, that v is not in the span of σ. This contradiction proves that σ does span. □

Let me emphasize that the above proof is only an existence proof that relies on a non-constructive step, the Zorn lemma. The basis obtained in this way has no practical use whatsoever. It has some theoretical interest, but it cannot be used for any calculation etc. because the basis elements are not in our hand.

Some proofs in mathematics are somewhat simpler if one assumes Zorn Lemma, so we will do that. However, this is not a real restriction. In all practical cases, the usage of Zorn lemma can be avoided with a bit more effort and with a result that is of more restricted validity than the one with Zorn Lemma, however, it always covers the practically important cases.