Fourier Transform

László Erdős

Dec 2, 2004

The purpose of this note is to outline the material we covered about Fourier series and transform.

1 Fourier series

In this Section we work on $[0, 2\pi]$ where 0 and 2π are periodically identified. You can think of if as the unit circle S^1. All the L^p and C^k spaces refer to the corresponding function space on S^1. The periodicity condition does not require any extra for L^p functions ($f(0) = f(2\pi)$ makes no sense), but it requires $f(0) = f(2\pi)$ for continuous functions and $f^{(j)}(0) = f^{(j)}(2\pi)$, $0 \le j \le k$ for C^k functions.

1.1 L^2 Theory of Fourier series

Define

$$e_n(x) := \frac{e^{inx}}{\sqrt{2\pi}} \qquad n \in \mathbf{Z}$$

which is clearly an orthonormal set.

Let $f \in L^2$, then $f \in L^1$ and we can define

$$c_n = (e_n, f) = \int_0^{2\pi} \frac{e^{-inx}}{\sqrt{2\pi}} f(x)dx$$

The **formal** series $\sum c_n e_n(x)$ is called the **Fourier series** of f. The basic question is its convergence (in which sense?) and its relation to f.

If we knew that $\{e_n\}$ were complete, i.e. it would form an ONB, then by the general basis-representation theorem in abstract Hilbert spaces we would know that $\sum c_n e_n$ converges to f in the norm of L^2. In particular, the finite approximate sums,

$$S_N(f)(x) := \sum_{n=-N}^{N} c_n e_n(x)$$

would converge to $f(x)$ in L^2.

Note that $S_N(f)$ is the projection of f onto the finite dimensional space spanned by $\{e_n : |n| \leq N\}$. This follows from the following lemma in abstract Hilbert spaces:

Lemma 1.1 *Let $M \subset H$ be a closed subspace of a Hilbert space. Let $\{e_n\}$ be an ONB in M. Then for any $x \in H$ the projection of x onto M is given by*

$$P_M(x) = \sum (e_n, x) e_n$$

Proof. Since $M \oplus M^\perp = H$, we can uniquely write $x = m + m'$ with $m \in M$, $m' \in M^\perp$. However it is easy to check that

$$x = \Big(\sum (e_n, x) e_n \Big) + \Big(x - \sum (e_n, x) e_n \Big)$$

exactly decomposes x into an element of M plus an element of $M^\perp$ (the first term is in M because M is closed, so the limit of the finite $\sum_{n \leq N} (e_n, x) e_n \in M$ sums lies in M. The second term is in $M^\perp$, since for any e_m clearly $e_m \perp x - \sum (e_n, x) e_n$ by orthogonality). By the uniqueness of the $x = m + m'$ decomposition hence we know that the component of x in M must be $m = \sum (e_n, x) e_n$. $\square$

1.2 Pointwise Theory of Fourier series

So far we have seen that the L^2 theory of F-series is fairly trivial (modulo the missing fact, that e_n is complete). However, here we talk about functions and not just abstract elements of abstract Hilbert spaces. There are several other questions one can ask about the relation between F-series and the original function. For example: In which other sense does the convergence hold? Can one differentiate/integrate Fourier series term-by-term? Etc.

The good analogy is the Taylor series, expcept that there essentially everything is trivial. Recall that a Taylor series converges absolutely inside the interval of convergence (maybe not at the endpoints), it converges uniformly on any compact subset away from the endpoints, it can be integrated and differentiated arbitrary many times inside the interval of convergence etc. The similar questions for F-series are much harder and the answers show greater variety.

I will show the most important things.

Theorem 1.2 *Let $f \in C(S^1)$ be continuous (you can think of it $f \in C[0, 2\pi]$ with periodic boundary condition, $f(0) = f(2\pi)$). Then the Fourier series is uniformly Cesaro summable. It means that*

$$\sup_x |\Sigma_N(f)(x) - f(x)| \to 0 \qquad as \quad N \to \infty$$

where

$$\Sigma_N(f) := \frac{1}{N+1}\Big(S_0(f) + S_1(f) + \ldots + S_N(f)\Big)$$

is the so-called Cesaro sum of the series $\sum c_n e_n$.

Proof. Easy computation (summing up geometric series) gives

$$S_N(f)(x) = \sum_{n=-N}^{N} \frac{e^{inx}}{\sqrt{2\pi}} \int \frac{e^{-iny}}{\sqrt{2\pi}} f(y)dy = \frac{1}{2\pi} \int \sum_{n=-N}^{N} e^{in(x-y)} f(y)dy$$

$$= \int D_N(x-y)f(y)dy = \int D_N(y)f(x-y)dy$$

where

$$D_N(z) := \frac{1}{2\pi} \frac{\sin\Big[\Big(N+\frac{1}{2}\Big)z\Big]}{\sin\frac{z}{2}}$$

is the so-called Dirichlet kernel.

We would like to see that it behaves like an approximate delta-function, i.e. its main weight is supported around $z \approx 0$, so in the integral $\int D_N(y)f(x-y)dy$ the main contribution would come from the regime $y \sim x$ and it is then $f(x)$ by continuity of f. It is clearly true that

$$\int D_N(z)dz = \frac{1}{2\pi}\int \sum_{n=-N}^{N} e^{inz}dz = 1$$

(only the $n = 0$ contributes), but D_N is not positive, and actually it is highly oscillatory and not concentrated sufficiently strongly around the origin.

The Cesaro sum takes care of this. Again, a simple calculation (summing up geometric series) shows that

$$\Sigma_N(f)(x) = \frac{1}{N+1}\frac{1}{2\pi}\int \left(\sum_{n=0}^{N} \frac{\sin\Big[\Big(N+\frac{1}{2}\Big)z\Big]}{\sin\frac{z}{2}}\right) f(x-y)dy = \int K_N(y)f(x-y)dy$$

with

$$K_N(z) := \frac{1}{2\pi(N+1)}\Big(\frac{\sin\frac{N+1}{2}z}{\sin\frac{z}{2}}\Big)^2$$

which is called the Fejér-kernel. Note that $K_N \geq 0$ (it is a "miracle"), and it enjoys the following properties:

(i) $\int K_N = 1$

(ii) For any $\delta > 0$, $K_N(z) \to 0$ uniformly on $[\delta, 2\pi - \delta]$.

The first one is trivial, since K_N is just the average of D_N's,

$$K_N = \frac{1}{N+1}(D_0 + D_1 + \ldots + D_N)$$

and $\int D_k = 1$ for any k.

The second one follows from the fact that if $z \in [\delta, 2\pi - \delta]$, then $|\sin(z/2)| \geq \delta/4$ (draw the graph), so for such z's

$$K_N(z) \leq \frac{16}{\delta^2(N+1)} \to 0$$

as $N \to \infty$.

Now we can show that $\Sigma_N(f)$ converges uniformly to f for continuous f. Since f is continuous on a compact set (S^1), it is uniformly continuous, so for any $\varepsilon > 0$, $\exists\delta$ s.t. $|f(z) - f(x)| < \varepsilon$ if $|z - x| < \delta$. Then we have

$$|\Sigma_N(f)(x) - f(x)| = \Big|\int K_N(x-z)f(z)dz - f(x)\Big| = \Big|\int K_N(x-z)(f(z) - f(x))dz\Big|$$

(notice how $f(x)$ was smuggled in the integral using $\int K_N = 1$)

$$\leq \int K_N(x-z)|f(z) - f(x)|dz = \int_{|z-x|\geq\delta}(\ldots)dz + \int_{|z-x|<\delta}(\ldots)dz$$

$$\leq 2\|f\|_\infty \int_{|z-x|\geq\delta} K_N(z-x)dz + \varepsilon\int K_N$$

$$\leq 2\|f\|_\infty 2\pi\frac{16}{\delta^2(N+1)} + \varepsilon$$

In the first term we estimated $|f(z) - f(x)| \leq 2\|f\|_\infty$, in the second we used $|f(z) - f(x)| < \varepsilon$ if $|z - x| < \delta$. Note that K_N is positive, so no need to take absolute value.

Now this can be made arbitrarily small for large N, uniformly in x. First you choose the ε sufficiently small, then fix δ and choose N sufficiently large. [THINK THIS OVER, it is very important!] $\square$

Corollary 1.3 *For any* $f \in L^2[0, 2\pi]$ *we have* $S_N(f) \to f$ *in* L^2. *Moreover,* $\{e_n\}_{n\in\mathbf{Z}}$ *is complete*

Proof. Clearly

$$\|f - S_N(f)\|_2 \le \|f - \Sigma_N(f)\|_2 \tag{1.1}$$

because $S_N(f)$ is the projection of f onto the space spanned by e_n, $|n| \le N$, and clearly $\Sigma_N(f)$ is also in this space. Since the projection gives the closest element, this inequality is trivial.

Now let $f \in L^2$. We choose a continuous function g such that $\|f - g\| < \varepsilon$, this is possible, since C is dense in L^2. We write

$$\|f - S_N(f)\|_2 \le \|f - g\|_2 + \|g - S_N(g)\|_2 + \|S_N(g) - S_N(f)\|_2 \le 2\varepsilon + \|g - S_N(g)\|_2$$

using that $\|S_N(g) - S_N(f)\|_2 = \|S_N(g - f)\|_2 \le \|g - f\|_2$ by the fact that any projection has norm at most 1 and S_N is a projection.

Since $g \in C$, using (1.1) for g and Theorem 1.2, we see that $\|g - S_N(g)\|_2 \le \varepsilon$ if choosing N big enough, so the whole thing is smaller than 3ε.

The completeness of $\{e_n\}_{n\in\mathbf{Z}}$ follows immediately; if $\overline{\text{Span}\{e_n\}}$ were not the whole H, then it would be a proper closed subspace of it, and then there would be a nonzero element $h \in H$ in the orthogonal complement, in particular $h \perp e_n$. Therefore all Fourier coefficients were zero, $S_N(h) \equiv 0$ for any N, and hence $\lim S_N(h) = 0$. On the other hand, $\lim S_N(h) = h$ for any $h \in H$, contradiction. $\square$

The completeness also implies that the general statement about unitary equivalence with ℓ^2 holds. The map

$$F : L^2([0, 2\pi]) \mapsto \ell^2(\mathbf{Z})$$

$$f \mapsto \{(e_n, f)\}_{n\in\mathbf{Z}}$$

is isometrically isomorphic, (in other words: unitary) between the two spaces. It is called the **Fourier transform**.

Note that for convenience we work with $\ell^2(\mathbf{Z})$ instead of the usual $\ell^2 = \ell^2(\mathbf{N})$, but of course these two spaces can be easily mapped into each other by relabelling all integers by natural numbers.

Having completed the L^2 theory, the next question is in which other senses does the Fourier series converge. Pointwise convergence for a general L^2 function does not make sense. Unfortunately, pointwise convergence does not hold even for arbitrary continuous functions, where it could make sense:

Example. There exists $f \in C(S^1)$ such that its F-series does not converge pointwise.

The construction is not easy, and later we'll give a proof of the existence of such function without really constructing it, but it will require an extra theoretical material (Banach-Steinhaus theorem).

The following is a major theorem despite its innocent look:

Theorem 1.4 (Carleson) *If $f \in L^2[0, 2\pi]$, then $S_N(f) \to f$ almost everywhere pointwise.*

1.3 Regularity and Fourier series

If one assumes extra regularity on the function, then one gets better convergence:

Theorem 1.5 *If $f \in C^1(S^1) = C^1_{per}([0, 2\pi])$ (periodicity in the sense that $f(0) = f(2\pi)$ and $f'(0) = f'(2\pi)$), then $S_N(f) \to f$ uniformly*

Since $f' \in C \subset L^2$, it has a F-series, $f' = \sum b_n e_n$ that converges in L^2. Let $f = \sum c_n e_n$ be the F-series of f (that is also L^2). We have

$$b_n = \int \frac{e^{-inx}}{\sqrt{2\pi}} f'(x)dx = in \int \frac{e^{-inx}}{\sqrt{2\pi}} f(x) = inc_n$$

by integration by parts. This shows that the F-series of a C^1 function can be differentiated termwise (clearly, the termwise derivative of $\sum c_n e_n$ is $\sum inc_n e_n$.) Since $\sum n^2 |c_n|^2 = \sum |b_n|^2 < \infty$, we get

$$\sum |c_n| = \sum |c_n| n \cdot \frac{1}{n} \leq \Big(\sum |c_n|^2 n^2\Big)^{1/2} \Big(\sum n^{-2}\Big)^{1/2} < \infty$$

Therefore the series $\sum c_n e_n$ converges uniformly since $\sum |c_n| \|e_n\|_\infty = \sum |c_n|$ is summable. This is exactly the uniform convergence of $S_N(f) \to f$.

In particular, we have seen that if $f \in C^1$, then $\sum |c_n|^2 n^2 < \infty$.

It is natural to ask if the converse is true, i.e. is it true that $\sum |c_n|^2 n^2 < \infty$ implies differentiability. The answer is no, but it is true under a stronger condition

Theorem 1.6 *Suppose that $\sum |c_n| n < \infty$ holds for the Fourier series of an $f \in L^2$ function. Then $f \in C^1$.*

Remark. Note that the condition $\sum |c_n|n < \infty$ is stronger than $\sum |c_n|^2 n^2 < \infty$. If $\sum |c_n|n < \infty$, then in particular $|c_n|n \leq K$ for some constant (every convergent series have bounded terms), therefore $\sum |c_n|^2 n^2 \leq K \sum |c_n|n$, so if the latter is finite, so is the former.

Proof. Recall a standard theorem in analysis: If a sequence of functions f_N converges to f pointwise (actually convergence at one point is enough) and f_N' converges to some function h uniformly, then f is differentiable and $f' = h$. Use this theorem for $f_N := S_N(f)$ and check that $f_N' = \sum_{-N}^{N} inc_n e_n$ converges uniformly. But this is clear from the condition. □

Similar theorem is true for higher derivatives (The proof is by induction on k):

Theorem 1.7 *If $\sum |c_n|n^k < \infty$, then $f \in C^k$.*

1.4 Diagonalization of derivatives

The F-transform diagonalizes the differentiation operator $D := \frac{d}{dx}$ with e_n being the eigenvectors:

$$De_n = ine_n$$

(compare it with $Av = \lambda v$ eigenvalue equation; view D as a linear transformation, i.e. as an "infinite dimensional" matrix, and then e_n's are the eigenvectors with eigenvalues n).

Instead of the operator D it is better to look at $-iD$, the formulas are nicer. Note that using integration by parts, $-iD$ is a symmetric operator

$$(f, (-iD)g) = -i \int \bar{f}(x) g'(x) dx = i \int \bar{f}'(x) g(x) dx = \int \overline{-if'(x)} g(x) dx = ((-iD)f, g)$$

while D itself would be antisymmetric. This is the main reason we like $-iD$ better than D.

If you consider the differentiation map under F-transform, then

$$F(-iDf) = \{nc_n\}$$

if $F(f) = \{c_n\}$. I.e. the action of the $-iD$ in the Fourier representation is simply multiplication by n.

Again, higher derivatives are similar, for example $-\Delta = (-iD)^2$ in the Fourier picture is just multiplication by n^2.

1.5 Sobolev spaces

We have seen at Theorem 1.6 and in the Remark afterwards that differentiable functions cannot be nicely characterized by F-transform. However, we can use the F-transform to extend the concept of differentiability. The following definition is fundamental:

Definition 1.8 (Sobolev spaces) *We define*

$$H^k := \{f \in L^2[0, 2\pi] \ : \ \sum n^{2k}|c_n|^2 < \infty\}$$

to be the Sobolev space of order k. For $f \in H^k$, the function $\sum (in)^k c_n e_n$ (that makes sense as an L^2 function) is called the **weak k-th order derivative of f**. *If $f \in C^k$, then this notion coincides with the usual k-th order derivative.*

We have seen that if $\sum |c_n||n|^k < \infty$, then $f \in C^k$. The condition $\sum |c_n|^2 |n|^{2k} < \infty$ is weaker, nevertheless we could define the k-th derivative of f, as an L^2 function. We did **not** define it as the limit of the difference quotient since that does not exist. We defined it via Fourier transform, but one can check that every properties of the derivatives hold and it is indeed an extension of the usual derivative.

The space H^k has a major advantage over C^k, it is a (complete) Hilbert space with the scalar product

$$(f, g)_{H^k} := \sum (1 + |n|^{2k}) \overline{c_n} d_n$$

where $f = \sum c_n e_n$ and $g = \sum d_n e_n$. The norm is called the k-th order Sobolev norm:

$$\|f\|_{H^k} := \Big(\sum (1 + |n|^{2k}) |c_n|^2 \Big)^{1/2}$$

This concept perfectly makes sense for any $k \geq 0$, not just for integers. It is clearly a Hilbert space, because it is an L^2 space on $\mathbf{Z}$ with the weigthed counting measure

$$\mu(\{n\}) = 1 + |n|^{2k}$$

There are several alternative definitions of the Sobolev norm(s). The following norm

$$\|f\|'_{H^k} := \Big(\sum (1 + n^2)^k |c_n|^2 \Big)^{1/2}$$

is not the same as $\|f\|_{H^k}$, but it is equivalent to it in the following sense: there exist positive constants, K_1, K_2, depending only on k, such that

$$K_1 \|f\|_{H^k} \leq \|f\|'_{H^k} \leq K_2 \|f\|_{H^k}$$

In particular the two norms define the same topology (i.e. the concept of convergence is the same), so for every practical purpose it does not matter which one you use. (Most books actually prefer $\|f\|'_{H^k}$).

Note that

$$C^k \subset H^k \subset L^2$$

The set C^k is dense in L^2 in the usual L^2-norm, in other words, if you close C^k in the $\|\cdot\|_2$, then you get L^2. But if you close C^k in the stronger $\|\cdot\|_{H^k}$ norm, then the closure is H^k.

The major advantage is that one can interchange derivatives and limits under a control given by a scalar product. We can interchange derivatives and limits if uniform control is given. But typically it is hard to check if f_n converges uniformly. It is much easier to check convergence in $\|\cdot\|_{H^k}$ norms because you can compute on the Fourier side.

For example, suppose that $f_n \in C^1$ and f_n is Cauchy in $\|\cdot\|_{H^1}$. This is a checkable condition in many cases. Then f_n converges to f in $\|\cdot\|_{H^1}$, and f will have weak derivative, namely $f_n' \to f'$ in L^2 sense. It may be that $f \notin C^1$, but still we can talk about its derivative, and having introduced this concept, the limit and differentiation can be interchanged.

Another interesting point is the following. We have seen that $C^1 \subset H^1 \subset L^2$, and it is also easy to check that $H^k \subset H^\ell$ whenever $k \geq \ell$.

We also know that $C^1 \subset C \subset L^2$. Question: where is C "nested" in the Sobolev space hierarchy. These questions are answered by various *Sobolev-embedding* theorems and are nontrivial for general spaces. On the $[0, 2\pi]$ the answer is the following:

Theorem 1.9 (Sobolev) *For any $\varepsilon > 0$*

$$H^{1/2+\varepsilon}[0, 2\pi] \subset C[0, 2\pi]$$

but $H^{1/2}[0, 2\pi] \not\subset C[0, 2\pi]$.

We'll give the proof of the first statement in the Exercise session.

1.6 Solution to the heat equation on S^1

We show the power of the F-transform by solving the simplest PDE (partial diff. equation).

Let $f(x,t)$ be periodic function (imagine it is the temperature at point $x \in S^1$ at time t), satisfying

$$\partial_t f = \Delta_x f$$

where $\Delta_x = d^2/dx^2$. we set the initial condition $f(x,0) = f_0(x)$, where $f_0 \in L^2(S^1)$ is given.

Suppose that $f_t = f(\cdot, t) \in C^2$ for any $t > 0$ (we'll prove it later) and let its F-transform be

$$f_t = \sum c_n(t) e_n$$

(note that the F-coefficients depend on t). Then

$$\Delta_x f = \sum(-n^2)c_n(t)e_n$$

and

$$\partial_t f_t = \sum c_n'(t)e_n$$

(we'll see in a moment, than $c_n'(t)$ decays sufficiently fast as $n \to \infty$ so that you can interchange the infinite sum and the t-derivative).

Equating the F-coefficients, we have

$$c_n'(t) = -n^2 c_n(t)$$

and the initial condition is $c_n(0) = c_n$, where $f_0 = \sum c_n e_n$. Therefore

$$c_n(t) = e^{-tn^2} c_n$$

So we can summarize

Theorem 1.10 *Let $f_0 \in L^2(S^1)$, $f_0 = \sum c_n e_n$. The the function*

$$f(t,x) = \sum e^{-tn^2} c_n e_n(x)$$

is smooth (C^∞), for any $t > 0$ it solves

$$\partial_t f = \Delta_x f$$

and as $t \to 0$

$$\lim_{t \to 0} f(t, \cdot) = f_0$$

where the limit is in L^2

Note that the time direction is important: the solution works for $t > 0$. If $t < 0$, then the F-series of the presumed solution would blow up (unless only finitely many c_n's are nonzero).

Proof. We just have to check the conditions we imposed in the calculation above the theorem. If $t > 0$, then

$$\sum n^k \cdot e^{-tn^2} |c_n| \leq \left(\sum |c_n|^2\right)^{1/2} \left(\sum n^{2k} e^{-2tn^2}\right)^{1/2} < \infty$$

since the Gaussian function decays much faster than any polynomial. Using Theorem 1.7, we obtain that $f_t \in C^k$ for any k.

Similarly one can check that ∂_t can be brought inside the summation, because the derivative series uniformly converges by $\sum |c_n'(t)| = \sum n^2 e^{-tn^2} < \infty$.

Finally we can check that

$$\|f(t,\cdot) - f_0\|^2 = \sum (e^{-2tn^2} - 1)|c_n|^2 \to 0$$

as $t \to 0$ by monotone (or dominated) convergence.

Note that the formal solution to the heat equation is

$$f_t = e^{t\Delta} f_0$$

so the question is how to define $e^{t\Delta}$. More general, one can consider a function $G : \mathbf{R} \to \mathbf{R}$ and ask how to define $G(\Delta)$. The answer is simply multiplication on the Fourier side, i.e.

$$[G(\Delta)f](x) := \sum_n G(-n^2) c_n e_n(x)$$

if $f(x) = \sum_n c_n e_n(x)$ and $f \in L^2$. But one has to make sure that the RHS makes sense, i.e. you can define $G(\Delta)$ on the function f only if

$$\sum_n |G(-n^2)|^2 |c_n|^2 < \infty$$

The fractional Sobolev spaces (i.e. when the order k is not integer) give rise to the possibility to define fractional derivatives. The natural definition would be

$$(-iD)^k f = \sum_n n^k c_n e_n(x)$$

for functions where the RHS is well defined, but this has a major flaw for noninteger k. When k is not integer and n is a negative number, then the definition of n^k is not unique (think of $k = 1/2$, $n = -1$, then $\sqrt{-1} = \pm i$). To avoid the problem of choosing the "right" version, one defines only $(-\Delta)^{k/2}$ instead of $(-iD)^k$, i.e.

$$(-\Delta)^{k/2} f := \sum_n |n|^k c_n e_n(x)$$

Note that the absolute value removes the ambiguity, but this operator is not the same as $(-iD)^k$ (although the identity $[(-iD)^2]^{k/2} = (-\Delta)^{k/2}$ looks tempting, one should remember that taking fractional powers even for complex numbers is a multivalued operation. Without keeping this mind, one would run into the

$$i = (-1)^{1/2} = [(-1)^2]^{1/4} = 1^{1/4} = 1$$

nonsense).

It is also clear for integer k's, e.g.

$$(-\Delta)^{1/2} f := \sum_n |n|^k c_n e_n(x) \neq \sum_n n^k c_n e_n(x) = (-iD)f$$

despite the fact that the square of both operators are the same:

$$(-\Delta)^{1/2}(-\Delta)^{1/2} = -\Delta = (-iD)(-iD)$$

But this should not surprise us, since $(-1)(-1) = 1 \cdot 1$ but $-1 \neq 1$. As usual, taking the absolute value (which is the nonnegative square root of the square: $|n| = \sqrt{n^2}$) loses information; in this way both $(-iD)^k$ and $(iD)^k$ are translated into the same operator.

Finally, I'd like to mention, that we dealt with one-dimensional setup for mere simplicity. Everything is true in higher dimensions. Consider $T^d = [0, 2\pi]^d$ the d-dimensional torus, which is the cube $[0, 2\pi]^d$ with its opposite sides identified. This in particular means that for continuous functions on T^d we require periodicity in all variables:

$$f(x_1, x_2, \ldots, x_{k-1}, 0, x_{k+1}, \ldots) = f(x_1, x_2, \ldots, x_{k-1}, 2\pi, x_{k+1}, \ldots)$$

(for all $x_1, x_2, \ldots$ without x_k). For spaces with higher derivatives we also require the periodicity of the corresponding derivatives.

We define

$$e_{\mathbf{n}}(\mathbf{x}) := \frac{e^{i\mathbf{n}\cdot\mathbf{x}}}{(2\pi)^{d/2}} \qquad \mathbf{x} \in [0, 2\pi]^d$$

for any $\mathbf{n} = (n_1, \ldots n_d) \in \mathbf{Z}^d$. This is an ONB. For any $f \in L^2([0, 2\pi]^d)$ we define

$$c_{\mathbf{n}} := (e_{\mathbf{n}}, f) = \int_{T^d} \frac{e^{-i\mathbf{n}\cdot\mathbf{x}}}{(2\pi)^{d/2}} f(\mathbf{x}) d\mathbf{x}$$

to be the F-coefficient. Then

$$f = \sum_{\mathbf{n} \in \mathbf{Z}^d} c_{\mathbf{n}} e_{\mathbf{n}}$$

in $L^2([0, 2\pi]^d)$-sense, i.e.

$$S_N(f) \to f$$

in L^2 where

$$S_N(f) := \sum_{\substack{\mathbf{n}\in\mathbf{Z}^d \\ |n_i|\le N}} c_{\mathbf{n}} e_{\mathbf{n}}$$

All one-dimensional theorems (but sometimes not the proof) goes through directly.

2 Fourier transform in $\mathbf{R}^n$

The presentation basically Lieb-Loss Chapter 5, but with the convention used in Reed-Simon. All spaces are over $\mathbf{R}^n$.

2.1 L^1 Fourier transform

Let $f \in L^1(\mathbf{R}^n)$, then its F-transform is defined as

$$\hat{f}(k) := \frac{1}{(2\pi)^{n/2}} \int e^{-ikx} f(x)dx$$

Note that the integral makes sense. In the exponent we have $kx = k \cdot x$ scalar product of two n-vectors, but for simplicity we'll just write it as kx.

Sometimes we'll denote the F-transform by $F(f) := \hat{f}$.

Remark. Whenever you do F-transform, there is a constant hassle with the 2π's. Different books use different conventions, sometimes $\hat{f}$ is defined without the $(2\pi)^{-n/2}$ prefactor, sometimes they put it in the exponent: $\int e^{-2\pi ikx} f(x)dx$ etc. I use the convention of Reed-Simon. But whenever you read another book and the 2π's don't come out right, then you should check at the beginning how that book defined the F-transform...

One can view the F-transform as a map: $\hat{}: f \mapsto \hat{f}$ between $L^1 \mapsto L^\infty$. It is clear that $\|\hat{f}\|_\infty \le \|f\|_1$.

Proposition 2.1 (Properties of F-transform) *(i) [Convolution] Let $f, g \in L^1$, then*

$$\widehat{(f \star g)} = (2\pi)^{n/2} \hat{f}\hat{g}$$

i.e. the convolution goes into product under F-transform

(ii) [Translation]

$$\widehat{f(\cdot - h)} = e^{-ikh}\hat{f}(k)$$

(iii) [Scaling]

$$\widehat{f(\cdot/\lambda)} = \lambda^n \hat{f}(\lambda k).$$

Proof. Calculation. The only point is in (i) to use Fubini properly:

$$(f \star g)\hat{}(k) = \frac{1}{(2\pi)^{n/2}} \int e^{-ikx} \int f(x-y)g(y)dydx$$

$$= \frac{1}{(2\pi)^{n/2}} \int e^{-iky} g(y) \Big(\int e^{-ik(x-y)} f(x-y)dx \Big) dy = (2\pi)^{n/2} \hat{f}(k)\hat{g}(k)$$

where the interchange of integration is guaranteed because $f, g \in L^1$, so the integrand is always in $L^1(dxdy)$.

2.2 Fourier transform of Gaussian function.

Let

$$f(x) = \frac{1}{(2\pi)^{n/2}} e^{-x^2/2}$$

then $\int f = 1$. We compute

$$\hat{f}(k) = \frac{1}{(2\pi)^n} \int e^{-ikx} e^{-x^2/2} = \frac{1}{(2\pi)^n} g(k) e^{-k^2/2}$$

with

$$g(k) := \int e^{-\frac{1}{2}(x+ik)^2} dx$$

(integral is convergent). One can easily check by integration by parts, that $\nabla g(k) = 0$, therefore $g(k) = g(0) = (2\pi)^{n/2}$. Hence we obtain

$$\hat{f} = f$$

(Actually the Gaussian is the only function with the property that the F-transform leaves it invariant).

2.3 Fourier transform of L^2 functions

Theorem 2.2 *Let $f \in L^1 \cap L^2$. Then*

(i) $\hat{f} \in L^2$ and $\|f\|_2 = \|\hat{f}\|_2$ (also called Plancherel formula, similarly to Parseval identity for F-series).

(ii) The F-transform as a map $f \to \hat{f}$ can be extended as a bounded map on L^2 from $f \in L^1 \cap L^2$ to $f \in L^2$.

(iii) If $f, g \in L^2$, then $(f, g) = (\hat{f}, \hat{g})$. I.e. the Fourier transform is isometry on L^2.

Remark: We do not know yet that the F-transform is bijection, isometry is weaker. It is, but this will come later.

Proof. (i) Since $\hat{f} \in L^\infty$, for any $\varepsilon > 0$ we have

$$\int |\hat{f}(k)|^2 e^{-\varepsilon k^2} dk < \infty$$

We compute it

$$\int |\hat{f}(k)|^2 e^{-\varepsilon k^2} dk = \frac{1}{(2\pi)^n} \int \Big(\int \overline{f(x)} e^{ikx} dx \Big) \Big(\int f(y) e^{-iky} dy \Big) e^{-\varepsilon k^2} dk$$

$$= \frac{1}{(2\pi)^n} \int \overline{f(x)} f(y) \int e^{ik(x-y)} e^{-\varepsilon k^2} dk dx dy$$

The integrals can be interchanged since $f(x)f(y)e^{-\varepsilon k^2}$ is in the L^1 space of $dxdydk$. The last Gaussian integral can be explicitly computed, we obtain

$$= \int \overline{f(x)} f(y) j_\varepsilon(x-y) dx dy$$

where

$$j_\varepsilon(x-y) := \frac{1}{(4\varepsilon\pi)^{n/2}} e^{-\frac{(x-y)^2}{4\varepsilon}}$$

therefore, in summary

$$\int |\hat{f}(k)|^2 e^{-\varepsilon k^2} dk = (f, f \star j_\varepsilon) \tag{2.2}$$

Recall an old theorem, that we proved (Theorem 2.16 in [Lieb-Loss])

Theorem 2.3 *Let $j(x)$ be such that $\int j = 1$ and $\int |j| < \infty$. Let $j_\delta(x) := \delta^{-n} j(x/\delta)$. Then for any $f \in L^p$ $(p < \infty)$ we have $f \star j_\delta \to f$ in L^p as $\delta \to 0$.*

Using this theorem, we recognize that the right hand side of (2.2) goes to $\|f\|^2$ as $\varepsilon \to 0$. In particular the right hand side is bounded, uniformly in ε. Therefore by monotone convergence, the LHS converges to $\|\hat{f}\|_2$, so we get (i).

(ii) The extension is a standard argument. Let $f \in L^2$, take an approximating sequence $f_n \in L^1 \cap L^2$ such that $f_n \to f$ in L^2 (e.g. $f_n(x) := f(x)\chi(|x| \le n)$. Then

$$\|\hat{f}_n - \hat{f}_m\|_2 = \|f_n - f_m\|_2$$

by part (i), but this is a Cauchy sequence since $f_n \to f$. Therefore $\hat{f}_n$ is also Cauchy and its limit exists. This limit is defined to be the F-transform of f, again denoted by $\hat{f}$. One has to check that this definition does not depend on the approximating sequence. But if $\hat{f}_n \to F$ and $\hat{g}_n \to G$, where $f_n \to f$ and $g_n \to f$, then clearly $\|\hat{f}_n - \hat{g}_n\| = \|f_n - g_n\| \leq \|f_n - f\| + \|g_n - f\| \to 0$, so $F = G$. Again, by limiting argument we see that the Parseval identity, $\|f\| = \|\hat{f}\|$ holds for $f \in L^2$ as well.

(iii) From $\|f\| = \|\hat{f}\|$ we easily conclude that $(f, g) = (\hat{f}, \hat{g})$ because in any complex H-space the following, so called polarization identity holds:

$$(x, y) = \frac{1}{4}\Big[\|x+y\|^2 - \|x-y\|^2 - i\|x+iy\|^2 + i\|x-iy\|^2\Big]$$

(it is amusing to check it). This ensures that the norm determines the scalar product (not just the other way around).

2.4 Inversion formula

Theorem 2.4 *For any $g \in L^2(\mathbf{R}^n)$ we define*

$$\check{g}(x) := \hat{g}(-x) = \frac{1}{(2\pi)^{n/2}} \int e^{ikx} g(k) dk$$

Then

$$(\hat{f})\check{} = f$$

In other words, the Fourier transform is invertible, the inverse transform is almost the same, but note that the sign in the exponent is different. In particular, we have

Corollary 2.5 *The F-transform $f \to \hat{f}$ on L^2 is unitary.*

This directly follows from the isometry and the invertibility.

Before the rigorous proof, let me show you how a physicist think about it. Let me do $n = 1$ dimension. We would like to check that

$$\frac{1}{2\pi} \int e^{ikx} \int e^{-iky} f(y) dy dk = f(x)$$

If you disregard the question of validity of interchanging integrals, then we have to show that

$$\frac{1}{2\pi} \int e^{ik(x-y)} dk = \delta(x-y) \tag{2.3}$$

where $\delta(z)$ is the delta function at the origin.

There are some confusion about the delta function, it looks like nonrigorous mathematics. The way to think about it is that it is not a function but really a measure that is written like $\delta(z)dz$. This is the Lebesgue-Stieltjes measure associated to the absolutely honest monotonic function $H(x) = \chi(x \geq 1)$ (which is by the way called the Heaviside function). The measure assigns the value 1 to any set that contains the origin and 0 otherwise. The integration with respect to this measure is

$$\int f(z)\delta(z)dz = f(0)$$

So the real meaning of (2.3) is exactly

$$\int \Big(\frac{1}{2\pi}\int e^{ik(x-y)}dk\Big) f(y)dy = f(x)$$

The rigorous proofs have to do some regularization of the divergent integral in (2.3). This is somewhat delicate and one can get the wrong result if one is not careful. The physicists have an enormous experience and instinct to do it always (mostly) right without going through rigorous steps.

Rigorous proof of the Inversion formula. We need the following

Lemma 2.6 *Let $f \in L^2$, $\lambda > 0$ then*

$$\frac{1}{(2\pi\lambda)^{n/2}}\int e^{-\frac{(x-y)^2}{2\lambda}} f(y)dy = \frac{1}{(2\pi)^{n/2}}\int e^{-\frac{\lambda k^2}{2}} e^{ikx}\hat{f}(k)dk \tag{2.4}$$

Proof: First we check if for $f \in L^1$. Then the RHS is

$$RHS = \frac{1}{(2\pi)^n}\int e^{-\frac{\lambda k^2}{2}} e^{ikx} e^{-iky} f(y)dkdy$$

and the integrals can be freely interchanged since we have L^1 control in both k and y. The Gaussian dk integral can be computed and one gets the LHS.

If $f \in L^2$, then we approximate it by $f_n \in L^1$, $f_n \to f$ (in L^2). By Plancherel (extended to L^2) we obtain $\hat{f}_n \to \hat{f}$, and plugging these limits into (2.4), checking that the Gaussian factors on both sides give the necessary domination, the claim follows from dominated convergence. □

Once (2.4) is established, we take $\lambda \to 0$ limit. The LHS goes to $f(x)$ (in L^2) by Theorem (2.3) (one has to check that the Gaussian on the LHS with the given normalization has integral 1). On the RHS we have $e^{-\lambda k^2/2}\hat{f}(k) \to \hat{f}(k)$ pointwise convergence (as $\lambda \to 0$) and as $\hat{f} \in L^2$

we have also L^2 convergence (dominated convergence). Using Plancherel for the inverse F-transform (which is clearly valid since the F-transform and its inverse differ only by a reflection $x \mapsto -x$) we get that the RHS of (2.4), that is the inverse F-transform of $e^{-\lambda k^2/2}\hat{f}(k)$ also converges to the inverse F-transform of $\hat{f}$ and this was to be proved. $\square$

2.5 Heat equation on $\mathbf{R}^n$

Let $f \in C^1 \cap L^2$, then a simple integration by parts shows that

$$\widehat{\nabla f}(k) = ik\hat{f}(k)$$

(note that k is a vector and ∇ is also a vector). Similarly for higher derivatives, for example $\widehat{\Delta f}(k) = k^2\hat{f}(k)$.

Consider the heat equation

$$\partial_t f_t = \Delta f_t$$

for the unknown function $f_t(x) = f(x,t)$ with initial data $f(x,0) = f_0$

Suppose that $f_t \in C^2$, then we can take the F-transform of the equation:

$$\partial \hat{f}_t = -k^2 \hat{f}_t(k)$$

so the solution is

$$\hat{f}_t(k) = e^{-tk^2}\hat{f}_0(k)$$

Back to x-space

$$f_t(x) = \left(e^{-tk^2}\hat{f}_0\right)^{\check{}} = \frac{1}{(2\pi)^{n/2}}\left(e^{-tk^2}\right)^{\check{}} \star f_0 = \frac{1}{(4\pi t)^{n/2}}\int e^{-\frac{(x-y)^2}{4t}} f_0(y)dy$$

where we used how the F-transform (and its inverse) behaves under convolution. Once this representation is given, it is easy to check that indeed $f_t \in C^2$ (even more: $f_t \in C^\infty$) for $t > 0$. Finally one checks that $f_t \to f_0$ in L^2 as $t \to 0+0$ exactly as in the case of S^1.