Crash course on Complex Analysis

László Erdős

Feb 6, 2003

I assume that the complex numbers and the basic representations (polar coordinates) and algebraic operations are known. If somebody has trouble with that, make up for it IMMEDIATELY!

The note below summarizes a few basic features of complex analysis.

Let $\Omega \subset \mathbf{C}$ be an open set and let $f : \Omega \mapsto \mathbf{C}$ be a function. This function has a limit at $z_0 \in \overline{\Omega}$,

$$c = \lim_{z \to z_0} f(z)$$

if $\forall \varepsilon > 0$, $\exists \delta$ such that $|f(z) - c| < \varepsilon$ whenever $0 < |z - z_0| < \delta$.

The function is called differentiable at a point $z \in \Omega$ if the limit

$$f'(z) := \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

exists. Note that h is a complex number.

This is a much stronger requirement than the differentiability of the same f viewed as a function from $\mathbf{R}^2 \to \mathbf{R}^2$. (WHY? Think it over!) E.g. the function $f(x, y) = (x, 0)$ is differentiable as a real function, but not differentiable if viewed as a complex function $f(z) = Re(z)$ (real part) with the usual identification $z = x + iy$.

Because of this strength, a very surprising deep fact is true:

Theorem 1. If f is differentiable on an open set Ω then f' is also differentiable, hence f is infinitely differentiable. In this case f is called **analytic**, or **holomorphic** on Ω.

Power series can be defined exactly as in real. These are formal expressions of the form

$$\sum_{n \geq 0}^{\infty} a_n (z - z_0)^n$$

where $a_n \in \mathbf{C}$. The concept of convergence and absolut convergence of power series is exactly the same as in real.

A power series always convergence on a disk with radius R (maybe $R = 0$ or $R = \infty$). This means that inside the open disk it converges (even absolutely), outside the closed disk it never converges, and on the boundary of disk both situation may occur.

The radius of convergence is again the same, it is given by

$$R := \frac{1}{\limsup |a_n|^{1/n}}$$

(Hadamard theorem). If $R > 0$, then the power series absolutely converges for any $|z - z_0| < R$ and the convergence is uniform on any compact subset of this open disk.

Power series can be term by term differentiated and integrated on the domain of uniform convergence. All these properties are proven exactly as in real.

If f is analytic on Ω, and $z_0 \in \Omega$, then one can define its Taylor series about z_0 as

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$$

Theorem 2. Under these conditions, the the Taylor series converges inside the biggest disk with center z_0 that fit into Ω and the value of the series is f.

This theorem implies that in complex analysis there is no problem with identifying the limit. The Taylor series of the function converges inside the biggest meaningful domain and it reproduces the function. Conversely, every power series with positive radius of convergence is its own Taylor series.

ZEROS

Let f be analytic on Ω. Suppose that $f(z_n) = 0$ for a sequence of points z_n and $z_n \to z_0$ with $z_0 \in \Omega$, i.e. a sequence of zeros converge inside Ω. Then $f \equiv 0$.

For the proof simply notice that f can be expanded intro Taylor series around z_0:

$$f(z) = \sum_n a_n (z - z_0)^n$$

If f is not identically zero, then there is a nonzero coefficient. Let a_k be the first one. Write

$$f(z) = a_k (z - z_0)^k [1 + g(z)]$$

where clearly $g(z_0) = 0$. By continuity, $1 + g(z)$ has no root in a small neighborhood of z_0, therefore f has no root in a small neighborhood of z_0 outside of z_0. This contradicts the accumulation of zeros.

LINE INTEGRALS. The definition is the same as in $\mathbf{R}^2$: If γ is a (nice) curve in Ω, where f is defined and f is at least continuous, then

$$\int f(z) = \lim \sum_i [f(z_i) - f(z_{i-1})](z_i - z_{i-1})$$

where the limit is over the refinements of finite polygons inscribed in γ (i.e. $z_1, z_2, \ldots$ must be points in this order along γ and $\max |z_i - z_{i-1}| \to 0$). Note that the integral depends on the orientation, if you reverse the curve, the integral changes to its minus.

Theorem 3 If Ω is simply connected ("no holes") and f is analytic on Ω then

$$\int_\gamma f(z) dz = 0$$

along any closed curve $\gamma \subset \Omega$.

Corollary: integration contours can be deformed within the domain of definition. One application:

Theorem 4 [Cauchy formula]: If Ω is simply connected ("no holes") and f is analytic on Ω then

$$f(z) = \frac{1}{2i\pi} \int_\gamma \frac{f(\xi)}{\xi - z} d\xi$$

where the integration contour γ is a simple closed curve going around z once in a counter-clockwise direction.

The proof is trivial: deform γ to a very tiny circle of radius ε around z. The value of the integral does not change, so

$$f(z) = \frac{1}{2i\pi} \int_{|\xi - z| = \varepsilon} \frac{f(\xi)}{\xi - z} d\xi \approx \frac{1}{2i\pi} \int_{|\xi - z| = \varepsilon} \frac{f(z)}{\xi - z} d\xi = f(z) \frac{1}{2i\pi} \int_{|\xi - z| = \varepsilon} \frac{1}{\xi - z} d\xi = f(z)$$

The last integral can be explicitly computed. In the second step we used that f is continuous. The approximation becomes precise as $\varepsilon \to 0$.

LAURENT SERIES

Suppose f is analytic on $\Omega \setminus \{z_0\}$ where $z_0 \in \Omega$. Then f can be expanded into a Laurent series around z_0, i.e. there exists coefficients a_n such that

$$f(z) = g(z) + \sum_{n \geq 1} \frac{a_n}{(z - z_0)^n}$$

where $g(z)$ is analytic on Ω (also at z_0!) and the series converges in the biggest ball with center z_0 that lies in Ω.

We say that $f(z)$ has a **pole** at z_0 if the sum is finite, otherwise $f(z)$ has **essential singularity**. The coefficient a_1 is called the **residue** of f at z_0.

A function f is called **meromorphic** on Ω if there exists a discrete set $A \subset \Omega$ (no limit point within Ω) such that f is analytic on $\Omega \setminus A$ and at every point of A the function has a pole.

Theorem 5. (Liouville) If f is analytic on $\mathbf{C}$ (called **entire** function) and $|f(z)| \leq M$ for some M, i.e. f is bounded, then f is constant.

Proof: Write

$$f(z) = \sum a_n z^n$$

as the power series expansion. This converges everywhere since f is analytic. Write $z = re^{i\varphi}$ in polar form, then

$$f(re^{i\varphi}) = \sum a_n r^n e^{in\varphi} \qquad (*)$$

Fix r and view this function as a function $f : [0, 2\pi] \to \mathbf{C}$, i.e. as a function of the θ variable only. It is clearly in $L^2[0, 2\pi]$, and the right hand side of (*) is just its Fourier transform. By the Parseval identity (unitarity of F-tr.)

$$\int |f(re^{i\varphi})|^2 d\varphi = \sum_n |a_n|^2 r^{2n}$$

If f is bounded by M, then the left hand size is bounded by $2\pi M$. Therefore

$$\sum_n |a_n|^2 r^{2n}$$

is uniformly bounded, then $|a_n|^2 r^{2n}$ is uniformly bounded for any n and r. But this is possible only if $a_n \equiv 0$ for any $n \neq 0$. This exactly means that $f(z) = a_0$, constant.