Basis in Hilbert spaces

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The purpose of this note is to outline the concept of the basis in a Hilbert space in more details than Reed-Simon does.

Recall that in finite dimensional spaces a basis could be defined as a maximal linearly independent set or as a minimal spanning set. The words maximal and minimal refer to inclusion, i.e. a set S is maximal for a certain property (like lin. independence) if for any $S' \supset S$, $S' \neq S$ the property does not hold.

Recall also that both the definition of linear independence and span used finite sums.

In infinite dimensions one would like to extend the concept of span. It turns out that if one looks for a set $\{x_\alpha\}_{\alpha\in A}$ such that every element x in the Hilbert space $\mathcal{H}$ could be uniquely written as

$$x = \sum c_\alpha x_\alpha \qquad (*)$$

where the sum contains *finitely many* nonzero terms, then the concept of "basis" becomes too complicated: the necessary set $\{x_\alpha\}_{\alpha\in A}$ has uncountably many elements and it is non-construtive, hence totally useless for practical purposes.

So we'll relax the "spanning" condition to require that finite sums of the form (*) be only dense in $\mathcal{H}$, in other words, any element in $\mathcal{H}$ could be aribitrarily approximated by sums of the form (*) (Recall the way how the simple functions are dense in L^p)

Lemma 0.1 *Let X be a Banach space. Let $x_1, x_2, \dots$ be vectors in X such that $\sum \|x_n\| < \infty$. Then $\sum x_n$ exists (i.e. $y_N := \sum_{n=1}^N x_n$ has a limit as $N \to \infty$) and is independent of the order of the summands. This means that if $\sigma : \mathbf{N} \to \mathbf{N}$ is a bijection, then*

$$\sum_{n=1}^{\infty} x_n = \sum_{n=1}^{\infty} x_{\sigma(n)}$$

Proof. For the limit, show that

$$y_N := \sum_{n=1}^{N} x_n$$

is a Cauchy sequence, using that

$$\|y_N - y_M\| \le \Big| \sum_{n=N+1}^{M} x_n \Big| \le \sum_{n=N+1}^{M} \|x_n\|$$

which goes to zero as $N, M \to \infty$, since it is the tail of a convergent series.

For the rearrangement, let

$$y_N := \sum_{n=1}^{N} x_n, \qquad \hat{y}_N := \sum_{n=1}^{N} x_{\sigma(n)}$$

and we need to check that $\lim y_N = \lim \hat{y}_N$. Consider

$$\|y_N - \hat{y}_N\| = \Big\| \sum_{n=1}^{N} x_n - \sum_{n=1}^{N} x_{\sigma(n)} \Big\|$$

Note that if N is big enough (bigger than $\sigma^{-1}(1)$), then x_1 cancels out. If N is even bigger, namely also bigger than $\sigma^{-1}(2)$, then x_2 also cancels out. And so on... For every K one can find $N(K)$ (namely $\max\{\sigma^{-1}(1), \sigma^{-1}(2), \dots \sigma^{-1}(K)\}$) such that if $N \ge N(K)$, then the terms $x_1, x_2, \dots x_K$ cancel. Therefore the rest is estimated by at most twice of all the remaining elements:

$$\|y_N - \hat{y}_N\| \le 2 \sum_{n=K+1}^{\infty} \|x_n\|$$

for $N \ge N(K)$. This shows that $\lim y_N = \lim \hat{y}_N$. $\square$

With a very similar argument one can prove the following:

Lemma 0.2 *Let X be a Hilbert space. Let $x_1, x_2, \dots$ be orthogonal vectors in X such that $\sum \|x_n\|^2 < \infty$. Then $\sum x_n$ exists (i.e. $y_N := \sum_{n=1}^{N} x_n$ has a limit as $N \to \infty$) and is independent of the order of the summands. This means that if $\sigma : \mathbf{N} \to \mathbf{N}$ is a bijection, then*

$$\sum_{n=1}^{\infty} x_n = \sum_{n=1}^{\infty} x_{\sigma(n)}$$

Proof. Follow the same proof as above, just at the last formula you use Parseval identity (generalized Pythagoras) instead of triangle inequality, that says that if $\{x_n\}$ are orthogonal, then

$$\Big\| \sum_{1}^{N} x_n \Big\|^2 = \sum_{1}^{n} \|x_n\|^2$$

For example, one has

$$\|y_N - \hat{y}_N\|^2 \le 2 \sum_{n=K+1}^{\infty} \|x_n\|^2$$

for $N \ge N(K)$. This again goes to zero since it is the tail of a convergent series.

Definition 0.3 *A collection of vectors $\{x_\alpha\}_{\alpha \in A}$ in a Hilbert space is called orthonormal system (ONS), if $\|x_\alpha\| = 1$, and $(x_\alpha, x_\beta) = 0$ for $\alpha \ne \beta$.*

Definition 0.4 *A collection of vectors $\{x_\alpha\}_{\alpha \in A}$ in a Hilbert space is called linearly independent, if whenever*

$$\sum_\alpha c_\alpha x_\alpha = 0$$

for a FINITE sum (i.e. all but finitely many c_α's are zero), then all $c_\alpha = 0$.

It is easy to check that

$$(ONS) \Longrightarrow \text{lin. indep}$$

(simply multiply $\sum_\alpha c_\alpha x_\alpha = 0$ by x_β to get $c_\beta = 0$ for any β).

Definition 0.5 *A collection of vectors $\{x_\alpha\}_{\alpha \in A}$ in a HIlbert space is called orthonormal basis (ONB) if it is a maximal ONS.*

Note: we did not define basis in general, only ONB.

Lemma 0.6 *An ONS is maximal iff its finite linear combinations are dense.*

Proof. Suppose the closure $N := \overline{\{\sum c_\alpha x_\alpha\}}$ is not the whole H-space (here the sum has only finitely many nonzero elements). This is closed, so by Riesz Lemma, there exists $x_0 \in N^\perp$ and you can check that $\{x_0, x_\alpha, \alpha \in A\}$ is a bigger ONS, that is a contradiction.

To prove the other direction: Suppose that the ONS is not maximal, i.e. there exists an x_0 normalized vector orthogonal to every other x_α. Since the finite lin. combinations are dense, one can find (finitely many nonzero) coefficients such that

$$\left\| x_0 - \sum c_\alpha x_\alpha \right\| \le 1/2$$

But then

$$1 = |(x_0, x_0 - \sum c_\alpha x_\alpha)| \le \|x_0\| \left\| x_0 - \sum c_\alpha x_\alpha \right\| \le 1/2$$

contradiction. □

From here the rest is the same as in RS Sec II.3. Let me point out two things that are somewhat hidden.

Remark 1. In the proof of Theorem II.6 one needs to know that

$$\sum_{\alpha \in A} |(x_\alpha, y)|^2 < \infty \qquad (*)$$

implies that at most countably many (x_α, y)'s are nonzero. Note that there is no condition at all on a cardinality of A in general.

This follows from the fact (CHECK!) that for any $n \in \mathbf{N}$, the number of $|(x_\alpha, y)|$ that are bigger than $1/n$ is finite.

Remark 2. It follows from Theorem II.6 that the representation in a given basis is unique. By linearity it is enough to check that

$$0 = \sum c_\alpha x_\alpha$$

for some $\sum_\alpha |c_\alpha|^2 < \infty$ implies that $c_\alpha = 0$. Note that this does not directly follow from the fact that ONS is linearly independent, since now we talk about infinite sum. So we need the following statement (which is the precise way of stating the converse of Thm II.6):

Lemma 0.7 *Suppose that c_α's are such that $\sum_\alpha |c_\alpha|^2 < \infty$. Then $y = \sum_\alpha c_\alpha x_\alpha$ is well defined and $\|y\|^2 = \sum_\alpha |c_\alpha|^2$.*

Proof. We can choose $\alpha_1, \alpha_2, \ldots$ to list the nonzero c_α's (there are countably many, see above). Let $y_n = \sum_{j=1}^n c_{\alpha_j} x_j$, check that y_n Cauchy, hence $y_n \to y$. The result is independent of the order by Lemma 0.2. Therefore

$$0 = \lim_{n\to\infty} \|y - \sum_{j=1}^n c_{\alpha_j} x_j\|^2 = \lim_{n\to\infty} \Big(\|y\|^2 - \sum_{j=1}^n |c_{\alpha_j}|^2 \Big)$$

by Pythagoras, that shows the second part of the claim.