Bulk Universality for Random Matrices via the local relaxation flow

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Disentis, Jul 26–30, 2010

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INTRODUCTION

Basic question [Wigner]: Consider a large matrix whose elements are random variables with a given probability law. What can be said about the statistical properties of the eigenvalues? Do some universal patterns emerge and what determines them?

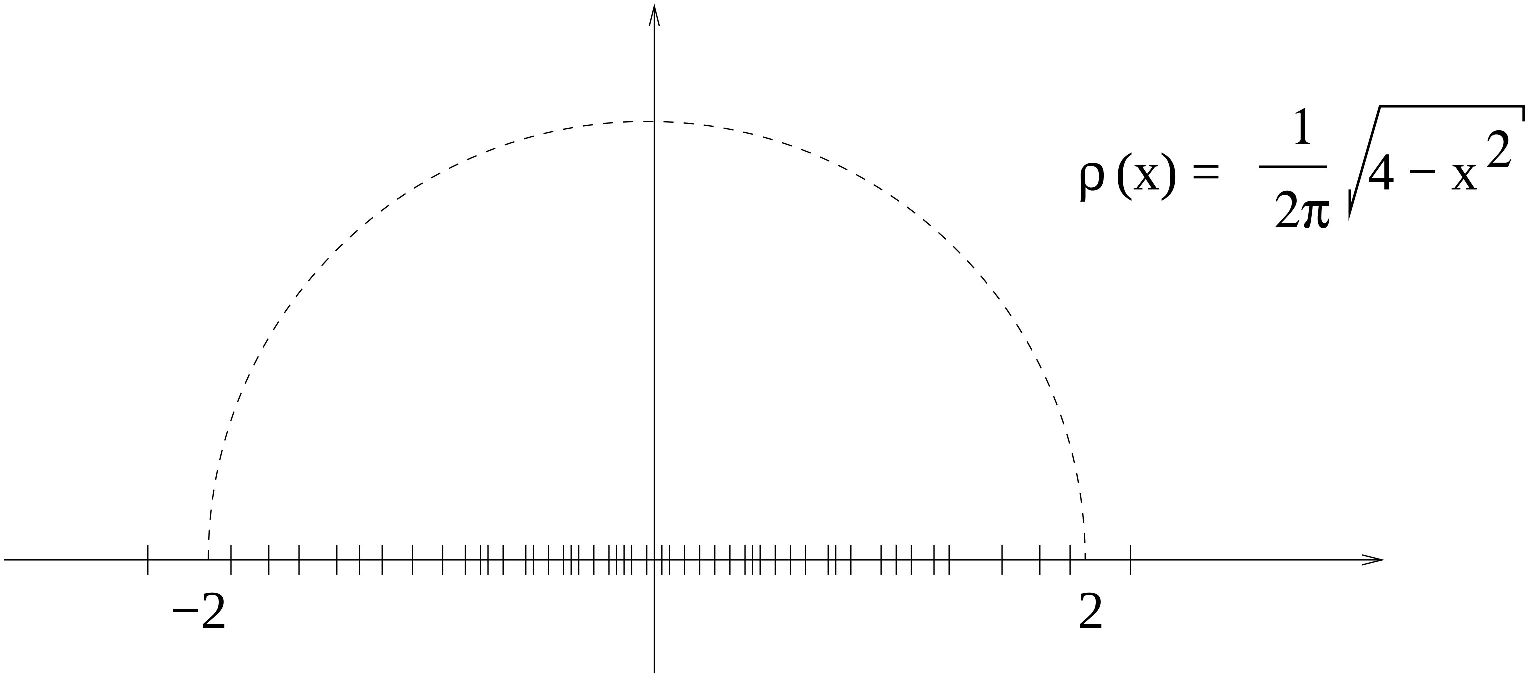
Gaussian Unitary Ensemble (GUE):

$H = (h_{jk})_{1 \leq j,k \leq N}$ hermitian $N \times N$ matrix with

$$h_{jk} = \overline{h}_{kj} = \frac{1}{\sqrt{N}}(x_{jk} + iy_{jk}) \quad \text{and} \quad h_{kk} = \frac{2}{\sqrt{N}} x_{kk}$$

where x_{jk}, y_{jk} (for $j < k$) and x_{kk} are independent, centered Gaussian random variables with variance 1/2.

Classical ensembles: GUE, GOE, GSE, Wishart ($H = A^* A$)



Eigenvalues: $x_1 \leq x_2 \leq \ldots\ldots \leq x_N$

Typical eigenvalue spacing is $x_{i+1} - x_i \sim \frac{1}{N}$ (in the bulk)

Observations: i) Semicircle density. ii) Level repulsion.

Holds for GUE, GOE, GSE. For Wishart: Marchenko-Pastur law

SINE KERNEL

Probability density of the eigenvalues (wrt. Lebesgue)

$$p(x_1, x_2, \dots, x_N)$$

The k-point marginal density ($\Longrightarrow$ correlation) is given by

$$p_N^{(k)}(x_1, x_2, \dots, x_k) = \int_{\mathbb{R}^{N-k}} p(x_1, \dots x_k, x_{k+1}, \dots, x_N) \mathrm{d}x_{k+1} \dots \mathrm{d}x_N$$

Special case: $k = 1$ (density)

$$\varrho_N(x) := p_N^{(1)}(x) = \int_{\mathbb{R}^{N-1}} p(x, x_2, \dots, x_N) \mathrm{d}x_2 \dots \mathrm{d}x_N$$

It allows to compute expectation of observables with one eigenvalue:

$$\mathbb{E} \frac{1}{N} \sum_{i=1}^{N} O(x_i) = \int O(x) \varrho_N(x) \mathrm{d}x \to \frac{1}{2\pi} \int O(x) \sqrt{4 - x^2} \mathrm{d}x$$

Higher k is similar, for example, $k = 2$ gives

$$\mathbb{E}\frac{1}{N(N-1)}\sum_{i\neq j}^{N} O(x_i, x_j) = \int O(x,y) p_N^{(2)}(x,y)\,\mathrm{d}x\mathrm{d}y$$

Correlation functions answer to the questions like:

"How many eigenvalues are in an interval I?"

"What is the prob. that there are 5 eigenvalues in I and 3 in J?"

They do not (directly) answer to questions like;

"What is the distribution of the 2010-th eigenvalue?"

"What is the distribution of the gap between neighboring evalues?"

Nevertheless, the second question can be answered by exclusion-inclusion formulas.

Local level correlation statistics [Wigner, Dyson, Mehta]

$$\lim_{N\to\infty} [\rho_N(E)]^{-2} p_N^{(2)}\left(E + \frac{x_1}{N\rho_N(E)}, E + \frac{x_2}{N\rho_N(E)}\right)$$

$$= \det\left\{S(x_i - x_j)\right\}_{i,j=1}^{2}, \quad S(x) = \frac{\sin \pi x}{\pi x}, \quad |E| < 2$$

$$= 1 - \left(\frac{\sin \pi(x_1 - x_2)}{\pi(x_1 - x_2)}\right)^2 \quad \Longrightarrow \quad \text{Level repulsion}$$

k-point correlation functions are given by $k \times k$ determinants:

$$\lim_{N\to\infty} [\rho_N(E)]^{-k} p_N^{(k)}\left(E + \frac{x_1}{N\rho_N(E)}, E + \frac{x_2}{N\rho_N(E)}, \dots, E + \frac{x_k}{N\rho_N(E)}\right)$$

$$= \det\left\{S(x_i - x_j)\right\}_{i,j=1}^{k}$$

Note that the limit is independent of E as long as E is in the bulk spectrum, i.e. $|E| < 2$.

Gap distribution

$$\mathbb{E}\,\frac{\#\text{ of e.v. pairs with gap less than } s \text{ near } E}{\#\text{ of eigenvalues near } E} \to \int_0^s p(\alpha)\mathrm{d}\alpha,$$

$$p(\alpha) := \frac{\mathrm{d}^2}{\mathrm{d}\alpha^2}\det(1-\mathcal{K}_\alpha)\mathrm{d}\alpha$$

$$\mathcal{K}_\alpha(x,y) = \frac{\sin\pi(x-y)}{\pi(x-y)} \text{ as operator on } L^2((0,\alpha))$$

Wigner surmise (not exact but very good approximation)

$$p(\alpha) \approx \frac{\pi\alpha}{2}e^{-\pi\alpha^2/4} \qquad \text{for the symmetric case}$$

$$p(\alpha) \approx 32\pi^{-2}\alpha^2 e^{-4\alpha^2/\pi} \qquad \text{for the hermitian case}$$

More precisely: Fix an energy E with $|E| < 2$ and for any $s > 0$ and for some N-dependent parameter t with $1/N \ll t \ll 1$ let

$$\Lambda(s) := \frac{1}{2Nt\varrho_{sc}(E)}\#\Big\{1 \le j \le N-1 \;:\; \lambda_{j+1}-\lambda_j \le \frac{s}{N\varrho_{sc}(E)},\; |\lambda_j-E| \le t\Big\}$$

i.e., the proportion of rescaled eigenvalue differences below a threshold s in a large but still microscopic vicinity of an energy E.

By the exclusion-inclusion formula, change of variables and the universal limit of the correlation functions,

$$\begin{aligned}\mathbb{E}\Lambda(s) =& \frac{1}{2Nt\varrho}\sum_{m=2}^{\infty}(-1)^m\binom{N}{m}\int_{-t}^{t}\mathrm{d}v_1\ldots\int_{-t}^{t}\mathrm{d}v_m \mathbf{1}\Big\{\max|v_i-v_j| \le \frac{s}{N\varrho}\Big\}\\ &\times p_N^{(m)}(E+v_1, E+v_2, \ldots, E+v_m)\\ \to& \sum_{m=2}^{\infty}\frac{(-1)^m}{(m-1)!}\int_0^s \mathrm{d}a_2\ldots\int_0^s \mathrm{d}a_m \det\left(\frac{\sin\pi(a_i-a_j)}{\pi(a_i-a_j)}\right)_{i,j=1}^m,\end{aligned}$$

where $\varrho = \varrho_{sc}(E)$.

History, Applications and Conjectures

- **Nuclear Physics:** the excitation spectra of heavy nuclei are expected to have the same local statistical properties as the eigenvalues of GOE (Wigner, 1955).

- **Quantum Chaos Conjecture** (vague)

classical dynamics with potential V	e.v. gap of $-\Delta + V$
chaotic	GOE statistics
integrable	Poisson statistics

Geodesic flow: Bohigas-Giannoni-Schmit (1984), Berry-Tabor (1977)

• **Riemann ζ-function:** Gap distribution of zeros of ζ function is given by GUE (Montgomery, 1973).

• **Anderson Model (1958):** V_ω random potential on $\mathbb{R}^d$ or $\mathbb{Z}^d$

random Schrödinger operator: $\quad H = -\Delta + \lambda V_\omega$

the eigenvalue distribution of H in the **delocalization regime** (λ small) is given by GOE. In the localization regime (λ large or near the spectral edge) Poisson statistics is proven.

Universality conjecture (vague): Statistics of eigenvalues depends on symmetry but is otherwise independent of the details of the ensemble.

KNOWN RESULTS FOR UNITARY ENSEMBLES

Unitary ensemble: Hermitian matrices with density

$$\mathcal{P}(H)\mathrm{d}H \sim e^{-\mathrm{Tr}\,V(H)}\mathrm{d}H$$

Invariant under $H \to UHU^{-1}$ for any unitary U (GUE)
Joint density function of the eigenvalues is explicitly known

$$p(\lambda_1, \ldots, \lambda_N) = \text{const.} \prod_{i<j} (\lambda_i - \lambda_j)^{\beta} e^{-\sum_j V(\lambda_j)}$$

classical ensembles $\beta = 1, 2, 4$ (orthogonal, unitary, symplectic symmetry classes; GOU, GUE, GSE for Gaussian case)

Correlation functions can be explicit computed via orthogonal polynomials due to the Vandermonde determinant structure.

large N asymptotic of orthogonal polynomials $\Longrightarrow$ local eigenvalue statistics indep of V. But density of e.v. depends on V.

Many previous results:

Dyson (1962-76), Mehta (1960-) classical Gaussian ensembles via Hermite polynomials

General case by Deift etc. (1999), Pastur-Schcherbina (2008), Bleher-Its (1999), Deift etc (2000-, GOE and GSE), Lubinsky (2008) . . .

All results are limited to invariant ensembles.

PREVIOUS METHODS FOR RANDOM MATRICES

- **Moment method:** compute $\mathbb{E}\mathrm{Tr}H^k$.

  Semicircle law on macro scale [Wigner, 1955].

  Suitable on the spectral edges [Sinai, Soshnikov 1999] for the Tracy-Widom law.

- **Green Function Method:** compute $\mathbb{E}\mathrm{Tr}\frac{1}{z-H}$.

  Resolvent is unstable inside the spectrum, randomness stabilizes.

- **Explicit formulas + Asymptotic analysis.**

  E.g: orthogonal polynomial methods or Brezin-Hikami and Johansson's formulae for hermitian case.

  Problems: Need explicit formulas. Universality ???

(HERMITIAN) WIGNER ENSEMBLE

$H = H^* = (h_{jk})_{1 \le j,k \le N}$ is a hermitian $N \times N$ matrix, $N \gg 1$.

$$h_{kj} = \frac{1}{\sqrt{N}}\Big(x_{kj} + iy_{kj}\Big) \qquad\qquad h_{kk} = \frac{2}{\sqrt{N}} x_{kk}$$

where x_{kj}, y_{kj} $(1 \le k < j \le N)$ and x_{kk} $(1 \le k \le N)$ are i.i.d. with

$$\mathbb{E}\, x_{jk} = 0 \qquad \text{and} \qquad \mathbb{E}\, x_{jk}^2 = \frac{1}{2} \qquad \left(\Rightarrow \quad \mathbb{E}\,|h_{jk}|^2 = \frac{1}{N}\right)$$

No explicit formula for the eigenvalue distribution.

Not unitary invariant, unless $V(x) = x^2$ Gaussian.

Symmetric Wigner ensemble is defined similarly.

Our final result: Universality for (generalized) Wigner matrices

SEMICIRCLE LAW (WIGNER 1955)

$$\mathcal{N}[I] := \text{number of eigenvalues in interval } I$$

$$\varrho_{sc}(E) := \frac{1}{2\pi}\sqrt{(4-E^2)_+}$$

For any $\delta > 0$ and $|E| \leq 2$

$$\lim_{\eta\to 0}\lim_{N\to\infty}\mathbb{P}\Big(\Big|\frac{\mathcal{N}[E-\frac{\eta}{2}, E+\frac{\eta}{2}]}{N\eta} - \varrho_{sc}(E)\Big| \geq \delta\Big) = 0$$

Remark 1: Semicircle is independent of the distribution of entries

Remark 2: Wigner's result concerns the macroscopic density, i.e. density in intervals containing order N eigenvalues.

UNIVERSALITY IN THE BULK

Theorem [E-Yau-Yin, 2010] Suppose that H is a hermitian or symmetric Wigner matrix with $\mathbb{E}h_{ij} = 0$, and $\sigma_{ij}^2 = \mathbb{E}|h_{ij}|^2$, with

$$(cN^{-1} \leq) \quad \sigma_{ij}^2 \leq CN^{-1}$$

and normalization

$$\sum_j \sigma_{ij}^2 = 1, \qquad \forall i$$

and subexponential decay. Then the corr. functions of the local statistics are given by the GUE/GOE in the bulk, i.e. for $|E| < 2$, $\forall k$ for any nice test fn $O : \mathbb{R}^k \to \mathbb{R}$ we have (# is GUE or GOE):

$$\lim_{b\to 0} \lim_{N\to\infty} \frac{1}{2b} \int_{E-b}^{E+b} \mathrm{d}E' \int_{\mathbb{R}^k} \mathrm{d}\alpha_1 \ldots \mathrm{d}\alpha_k \, O(\alpha_1, \ldots, \alpha_k)$$
$$\times \frac{1}{\varrho_{sc}(E)^k} \Big(p_N^{(k)} - p_{\#,N}^{(k)} \Big) \Big(E' + \frac{\alpha_1}{N\varrho_{sc}(E)}, \ldots, E' + \frac{\alpha_k}{N\varrho_{sc}(E)} \Big) = 0.$$

[Earlier result for Wigner with $\sigma_{ij}^2 = 1/N$: E-Schlein-Yau, 2009]

DYSON BROWNIAN MOTION

Gaussian convolution matrix (after rescaling) $H_t = H_0 + \sqrt{t}V$ interpolates (Johansson [2001] for $t > 0$, hermitian case). It can also be obtained by evolving the matrix elements with an OU process:

$$\partial_t u_t = L u_t, \qquad L = \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \frac{x}{2}\frac{\mathrm{d}}{\mathrm{d}x}, \qquad H_t = e^{-t/2}H_0 + (1-e^{-t})^{1/2}V.$$

Denote the prob density of the eigenvalues by $f_t(\mathbf{x})\mu(\mathrm{d}\mathbf{x})$.

$$\text{GUE}: \ \mu = Ce^{-N\mathcal{H}(\mathbf{x})}\mathrm{d}\mathbf{x}, \quad \mathcal{H}(\mathbf{x}) = \sum_{i=1}^N \frac{x_i^2}{2} - \frac{1}{N}\sum_{i \neq j} \log|x_j - x_i|$$

$$(PDE) \qquad \partial_t f_t = \mathcal{L} f_t, \qquad -\int f \mathcal{L} f d\mu = \frac{1}{2N}\int |\nabla f|^2 d\mu, =: D_\mu(f)$$

$$\mathcal{L} = \frac{1}{2N}\sum_{i=1}^N \partial_i^2 + \sum_{i=1}^N \Big(-\frac{1}{2}x_i + \frac{1}{N}\sum_{j\neq i}\frac{1}{x_i - x_j}\Big)\partial_i,$$

$$(SDE) \qquad \mathrm{d}x_i = \frac{1}{\sqrt{N}}\mathrm{d}B_i + \Big(-\frac{1}{2}x_i + \frac{1}{N}\sum_{j\neq i}\frac{1}{x_i - x_j}\Big)\mathrm{d}t$$

Derivation of DBM (for the BM case, $h_{ij} = N^{-1/2}B_{ij}$ for simplicity):

$$\frac{\partial \lambda_\alpha}{\partial h_{ik}} = \bar{u}_\alpha(i)u_\alpha(k), \qquad \frac{\partial u_\alpha(i)}{\partial h_{\ell k}} = \sum_{\beta\neq\alpha}\frac{1}{\lambda_\alpha - \lambda_\beta}\bar{u}_\beta(\ell)u_\alpha(j)u_\beta(i)$$

$$\frac{\partial^2 \lambda_\alpha}{\partial h_{ik}\partial h_{\ell j}} = \sum_{\beta\neq\alpha}\frac{1}{\lambda_\alpha - \lambda_\beta}u_\beta(\ell)\bar{u}_\alpha(j)\bar{u}_\beta(i)u_\alpha(k) + c.c.$$

$$\mathrm{d}\lambda_\alpha = \frac{1}{\sqrt{N}}\sum_{ik}\bar{u}_\alpha(i)u_\alpha(k)\mathrm{d}B_{ik} + \frac{1}{N}\sum_{ik}\sum_{\beta\neq\alpha}\frac{1}{\lambda_\alpha - \lambda_\beta}u_\beta(i)\bar{u}_\alpha(k)\bar{u}_\beta(i)u_\alpha(k)\mathrm{d}t$$

$$= \frac{1}{\sqrt{N}}\mathrm{d}B_\alpha + \frac{1}{N}\sum_{\beta\neq\alpha}\frac{1}{\lambda_\alpha - \lambda_\beta}\mathrm{d}t$$

where one can check that $\sum_{ik}\bar{u}_\alpha(i)u_\alpha(k)\mathrm{d}B_{ik}$ are independent white noises for different α's.

$$(SDE) \qquad \mathrm{d}x_i = \frac{1}{\sqrt{N}}\mathrm{d}B_i + \Big(-\frac{1}{2}x_i + \frac{1}{N}\sum_{j\neq i}\frac{1}{x_i - x_j}\Big)\mathrm{d}t$$

Idea: Equilibrium is the invariant ensemble (GUE, etc.) with known local statistics. For local statistics, only local equilibrium needs to be achieved which is faster.

Bakry-Emery method: Speed of convergence to a measure $\mu = e^{-\mathcal{H}}/Z$ is given by the lower bound on the Hessian matrix $\mathcal{H}''$:

$$\mathcal{H}'' \geq K \quad \Longrightarrow \quad S_\mu(f_t) \lesssim e^{-tK}, \quad D_\mu(\sqrt{f_t}) \lesssim e^{-tK}$$

and LSI holds: $KS_\mu(f_t) \leq D_\mu(\sqrt{f_t})$.

In our case, $K = O(1)$, so time to global equilibrium is $O(1)$.

[Morally, Johansson's work is $t \sim O(1)$, but with different method and for different reason]

KEY STEPS

Step 1. Good local semicircle law including a control near the edge.

Method: System of self-consistent equations for the Green function, control the error by large deviation methods.

Step 2. Universality for Gaussian divisible matrices with a small ($\sim N^{-\varepsilon}$) Gaussian component.

Method: Modify DBM to speed up its local relaxation, then show that the modification is irrelevant for statistics involving differences of eigenvalues.

Step 3. Universality for arbitrary Wigner matrices.

Method: Remove the small Gaussian component in Step 2 by resolvent perturbation theory and moment matching.

DYSON CONJECTURE

Theorem [Strong local ergodicity of DBM] [E-Yau-Yin, 2010] Suppose that H is a hermitian or symmetric Wigner matrix with $\mathbb{E}h_{ij} = 0$, and $\sigma_{ij}^2 = \mathbb{E}|h_{ij}|^2$, with

$$(cN^{-1} \le) \quad \sigma_{ij}^2 \le CN^{-1}, \qquad \sum_j \sigma_{ij}^2 = 1, \qquad \forall i$$

and subexponential decay. Let $E \in [2-c, 2+c]$ with some $c > 0$. Then for any $\varepsilon' > 0$ and $0 < b < c/2$, any integer $n \ge 1$ and for any nice test function $O : \mathbb{R}^n \to \mathbb{R}$ we have

$$\lim_{N\to\infty} \sup_{t \ge N^{-1+\varepsilon'}} \Big| \int_{E-b}^{E+b} \frac{\mathrm{d}E'}{2b} \int_{\mathbb{R}^n} \mathrm{d}\alpha_1 \ldots \mathrm{d}\alpha_n \, O(\alpha_1, \ldots, \alpha_n) \frac{1}{\varrho(E)^n}$$
$$\times \Big(p_{t,N}^{(n)} - p_{\mu,N}^{(n)} \Big) \Big(E' + \frac{\alpha_1}{N\varrho(E)}, \ldots, E' + \frac{\alpha_n}{N\varrho(E)} \Big) \Big| = 0$$

where $p_{t,N}^{(n)}$ are the corr. functions of the evalues of the DBM flow.

Dyson's conjecture [1962]:
Local equilibrium is already attained after time $t \sim N^{-1+\varepsilon'}$.

HISTORICAL DETOUR

We have been subsequently improving the local semicircle law in several papers focusing on shrinking the window. The following prototype result is optimal away from the edge:

Local Semicircle Law. [E-Schlein-Yau, 2007-08]
Let $\mathcal{N}[I] := \#\{\lambda_n \in I\}$ be no. of evalues in $I \subset \mathbb{R}$. Then

$$\mathbb{P}\left\{ \left| \frac{\mathcal{N}[E - \frac{K}{2N}, E + \frac{K}{2N}]}{K} - \varrho_{sc}(E) \right| \geq \delta \right\} \leq Ce^{-c\delta\sqrt{K}}$$

for all $|E| < 2$ and $K > 0$ uniformly in $N \geq N_0(E, \delta)$ and the eigenvectors are completely delocalized.

Method: Green function + Large deviation

Local semicircle law is always the starting point of any analysis.

Extended states: eigenvector delocalization

No concept of abs. continuous spectrum unlike in random Schr.

Let $\mathbf{v} = (v_1, v_2, \dots v_N) \in \mathbb{C}^N$ be ℓ^2-normalized.

Complete localization: a few large component, e.g.

$$\mathbf{v} = (1, 0, 0, \dots 0) \quad \Longrightarrow \|\mathbf{v}\|_p = 1 \text{ for all } 2 \le p \le \infty$$

Complete delocalization: components have the same size, e.g.

$$\mathbf{v} = \big(N^{-1/2}, N^{-1/2}, \dots\big) \quad \Longrightarrow \|\mathbf{v}\|_p = N^{\frac{1}{p}-\frac{1}{2}} \text{ for all } 2 \le p \le \infty$$

Theorem 2. [E-Schlein-Yau, 2008] Assume $\mathbb{E}\, e^{|x_{ij}|^\varepsilon} < \infty$.
Fix $|E| < 2$, $K > 0$ and $2 < p < \infty$, then for large M

$$\mathbf{P}\left\{ \exists \mathbf{v}, \|\mathbf{v}\|_2 = 1,\ H\mathbf{v} = x\mathbf{v},\ |x - E| \le \frac{K}{N},\ \|\mathbf{v}\|_p \ge MN^{\frac{1}{p}-\frac{1}{2}} \right\} \le Ce^{-c\sqrt{M}}$$

Level repulsion and Wegner estimate

Theorem 3. [E-Schlein-Yau, 2008] Fix $k \geq 1$ and assume that the distribution ν is sufficiently smooth and decays. Let $|E| < 2$, then

$$\mathbb{P}\Big(\mathcal{N}[E - \frac{\varepsilon}{2N}, E + \frac{\varepsilon}{2N}] \geq k\Big) \leq C_k\, \varepsilon^{k^2} \qquad \text{uniformly in } \varepsilon,\ N$$

---

i) **Level repulsion** for $k = 2$. [For Poisson ev's it would be ε^k].

ii) Exponent is **optimal,** it agrees with the GUE case:

$$p(x_1, \dots x_N) \sim \prod_{j<k} (x_j - x_k)^2 e^{-\sum_j x_j^2} \quad \Rightarrow \quad \mathbb{P}(\mathcal{N}_\varepsilon \geq k) \approx \varepsilon^{k^2}$$

(Exponent in **symmetric** case is different, but also optimal)

Method: Green function + local semicircle + integration by parts in probability space

Universality for hermitian Wigner matrices

Johansson [2001], Ben Arous-Peche [2005] proved sine-kernel for **hermitian** Wigner matrices with **Gaussian component** of the form

$$H = H_0 + \sqrt{t}V \in \mathbb{C}^{N\times N}, \qquad H_0 \quad \text{is Wigner, } V \text{ is GUE, } t > 0.$$

Key explicit formula: [Brézin-Hikami 1996, Johansson 2001] The two-point corr. fn. of H in terms of p_0 (e.v. distribition of H_0) is

$$p^{(2)}(x_1, x_2) = \int p_0(\mathbf{y})\mathrm{d}\mathbf{y} \int \det\Big(K_{t,N}(x_i, x_j; \mathbf{y})\Big)^2_{i,j=1}$$

with

$$K_{t,N}\Big(E, E + \frac{\tau}{N\varrho}; \mathbf{y}\Big) = \int_\gamma \mathrm{d}z \int_\Gamma \mathrm{d}w\; g_N(z, w; \tau) e^{N(f_N(w) - f_N(z))}$$

$$f_N(z) = \frac{1}{2t}(z^2 - 2Ez) + \frac{1}{N}\sum_j \log(z - y_j), \qquad g_N = \dots$$

Remark: Available only for the **hermitian** case.

To identify the saddles at energy E, one needs to solve for z

$$0 = f_N'(z) = \frac{1}{t}(z - E) + \frac{1}{N}\sum_j \frac{1}{z - y_j}$$

where y_j's are the eigenvalues of the Wigner matrix.

Approximately

$$0 = \frac{1}{t}(z - E) + \frac{1}{N}\sum_j \frac{1}{z - y_j} \iff 0 = \frac{1}{t}(z - E) + \int \frac{\varrho_{sc}(y)\mathrm{d}y}{z - y} \qquad (*)$$

and here Im $z \approx t$.

Thus (*) can be justified if semicircle law is valid on scale $O(t)$. Choosing $t = N^{-1+\varepsilon}$, the sine-kernel is proved for matrices

$$H = H_0 + \sqrt{t}V, \qquad t = N^{-1+\varepsilon}$$

Theorem [E-Peche-Ramirez-Schlein-Yau, 2009] Sine kernel holds for **hermitian** Wigner ensemble if the distribution of the matrix elements has subexp decay and some smoothness (no Gaussian component).

Method: Extend Johansson's result to a tiny Gaussian component $t = N^{-1+\varepsilon}$ using **local semicircle law** in the saddle point analysis of the contour integral. To remove this tiny Gaussian component: use **reverse heat flow:** Given a smooth distribution $u(x)\mathrm{d}x$, choose

$$g_t = \Big(1 - t\Delta + \frac{(-t\Delta)^2}{2!} + \dots\Big)u \approx e^{-t\Delta}u$$

then

$$e^{t\Delta}g_t$$

will have a Gaussian component and will be close to u.

Method of the reverse heat flow

Let u_0 be the density of the matrix elements of H_0, then the matrix elements of $H = H_0 + \sqrt{t}V$ have density

$$u_t = e^{tL}u_0, \qquad L = \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}$$

The probability density of all matrix elements, $F_0 = u_0^{\otimes N^2}$ at $t = 0$ and $F_t = (e^{tL}u_0)^{\otimes N^2}$ satisfy

$$\|F_t - F_0\|_1 \approx N^2\|e^{tL}u_0 - u_0\|_1 \approx CN^2 t \gg 1, \quad t \sim N^{-1+\varepsilon}$$

Key observation Compare F with $e^{t\mathcal{L}}G$ for **some** G that satisfied the local semicircle law; G may even depend on t.

Simple way to find a better G: For smooth u_0, choose

$$G = g_t^{\otimes N^2}, \quad g_t = (1 - tL)u_0 \quad \text{then} \quad e^{tL}g_t - u_0 = O(t^2)$$

Related results

- Four-Moment Theorem [Tao-Vu 2009]: if the first four moments of the single entry distribution of two Wigner matrices match, then the local e.v. statistics coincide.

  Idea: Expand the function $x_i(x_{jk}, j = 1, \ldots, N; k = 1, \ldots, N)$ in Taylor series of x_{jk} (recall $N^{-1/2}x_{jk}$ is the matrix element distribution). Input: Local semicircle law + eigenvector delocalization

  With Johansson's result, it yields universality for **hermitian** Wigner matrix under a third moment and a support condition.

- Universality for **hermitian** case with no condition apart from subexp decay [ERSTVY, 2009] (Combine the two methods)

All results used **explicit hermitian formulas**; method is not robust.

End of the historical detour: back to new method.

KEY STEPS (recall)

Step 1. Good local semicircle law including a control near the edge.

Method: System of self-consistent equations for the Green function, control the error by large deviation methods.

Step 2. Universality for Gaussian divisible matrices with a small ($\sim N^{-\varepsilon}$) Gaussian component.

Method: Modify DBM to speed up its local relaxation, then show that the modification is irrelevant for statistics involving differences of eigenvalues.

Step 3. Universality for arbitrary Wigner matrices.

Method: Remove the small Gaussian component in Step 2 by resolvent perturbation theory and moment matching.

Step 1: Strong semicircle law for general Wigner matrices

Theorem [E-Yau-Yin, 2010] Let $\mathbb{E}h_{ij} = 0$, $\mathbb{E}|h_{ij}|^2 = \sigma_{ij}^2$, where $B := (\sigma_{ij}^2)$ is double stochastic, $c \le N\sigma_{ij}^2 \le C$. Assume

$$\mathbb{P}\Big(|h_{ij}| \ge x\sigma_{ij}\Big) \le \vartheta^{-1} e^{-x^{\vartheta}}, \qquad x > 0$$

Then for any $z = E + i\eta$ with $|E| \le 5$, $N^{-1} \ll \eta \le 10$ we have

$$\mathbb{P}\Big(\max_i |G_{ii}(z) - m_{sc}(z)| \gg \frac{1}{\sqrt{N\eta}}\Big) \ll N^{-\infty}$$

$$\mathbb{P}\Big(\max_{ij} |G_{ij}(z)| \gg \frac{1}{\sqrt{N\eta}}\Big) \ll N^{-\infty}$$

$$\mathbb{P}\Big(|m(z) - m_{sc}(z)| \gg \frac{1}{N\eta}\Big) \ll N^{-\infty}$$

where $G_{ij} = \frac{1}{H-z}(i,j)$ and $m(z) = \frac{1}{N}\mathrm{Tr} G = \frac{1}{N}\sum_i G_{ii}$.
(Here $\ll$ means $\log N$ corrections)

Corollary 1. [Counting functions]

$$\mathfrak{n}(E) := \frac{1}{N}\#\{\lambda_j \le E\}, \qquad\qquad n_{sc}(E) := \int_{-\infty}^{E} \varrho_{sc}(x)\mathrm{d}x$$

then for any E

$$\mathbb{P}\Big(|\mathfrak{n}(E) - n_{sc}(E)| \gg \frac{1}{N}\Big) \ll N^{-\infty}$$

Corollary 2. [Eigenvalue locations] Let γ_j be the classical location of the j-th eigenvalue, i.e.

$$\int_{-\infty}^{\gamma_j} \varrho_{sc}(x)\mathrm{d}x = \frac{j}{N}$$

Then for any $1 \le j \le N/2$

$$\mathbb{P}\Big(|\lambda_j - \gamma_j| \gg \frac{1}{N^{2/3} j^{1/3}}\Big) \ll N^{-\infty}$$

All these estimates are optimal (up to log corrections)

From Stieltjes transform to counting function

Let $\varrho(\lambda)\mathrm{d}\lambda$ be a signed measure on a compact interval in $\mathbb{R}$, say $[-3,3]$ and suppose its Stieltjes transform satisfies

$$|\mathrm{Im}\, m(x+iy)| \le \frac{U}{Ny}, \qquad y>0,\ |x|+y \le 10$$

Then for any $E_1, E_2 \in [-3,3]$ we have

$$\Big| \int f_{E_1,E_2,\eta}(\lambda)\varrho(\lambda)\mathrm{d}\lambda \Big| \le \frac{CU|\log\eta|}{N}$$

where $f_{E_1,E_2,\eta}$ is a smoothed char. function of $[E_1,E_2]$ on scale η.

Helffer-Sjöstrand formula: Let $f \in C^1(\mathbb{R})$. Let $\chi(y)$ be a cutoff function in $[-1,1]$. Define the quasianalytic extension of f as

$$\widetilde{f}(x+iy) = (f(x) + iyf'(x))\chi(y),$$

then

$$f(\lambda) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{\partial_{\bar{z}}\widetilde{f}(x+iy)}{\lambda - x - iy}\mathrm{d}x\mathrm{d}y = \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{iyf''(x)\chi(y) + i(f(x)+iyf'(x))\chi'(y)}{\lambda - x - iy}\mathrm{d}$$

Detour: similar results for universal matrices

Theorem [E-Yau-Yin, 2010] Let $\mathbb{E}h_{ij} = 0$, $\mathbb{E}|h_{ij}|^2 = \sigma_{ij}^2$, where $B := (\sigma_{ij}^2)$ is double stochastic. Let $M := (\max_{ij} \sigma_{ij}^2)^{-1}$. Assume

$$\mathbb{P}\Big(|h_{ij}| \geq x\sigma_{ij}\Big) \leq \vartheta^{-1} e^{-x^\vartheta}, \qquad x > 0$$

Then for any $z = E + i\eta$ with $M\eta \gg 1$ and $\Big||E| - 2\Big| \geq \kappa_0 > 0$:

$$\mathbb{P}\Big(\max_i |G_{ii}(z) - m_{sc}(z)| \gg \frac{1}{\sqrt{M\eta}}\Big) \ll N^{-\infty}$$

$$\mathbb{P}\Big(\max_{ij} |G_{ij}(z)| \gg \frac{1}{\sqrt{M\eta}}\Big) \ll N^{-\infty}$$

$$\forall \varepsilon, K \qquad \mathbb{P}\Big(|m(z) - m_{sc}(z)| \gg \frac{N^\varepsilon}{M\eta}\Big) \ll N^{-K}$$

For E away from the edge,

$$\mathbb{P}\Big(|\mathfrak{n}(E) - n_{sc}(E)| \geq \frac{N^\varepsilon}{M}\Big) \ll N^{-K}$$

Step 2: Local ergodicity of Dyson Brownian Motion

Assumption I. [Convexity] The Hamiltonian of the invariant measure $\mu \sim e^{-N\mathcal{H}(\mathbf{x})}$ on $\mathbb{R}^N$ has the representation

$$\mathcal{H}(\mathbf{x}) = \beta \left[\sum_{i=1}^{N} U(x_i) - \frac{1}{N} \sum_{i<j} \log |x_j - x_i| \right]$$

$$\partial f_t = \mathcal{L} f_t, \qquad \mathcal{L} = \frac{1}{2N} \sum_i \Delta_i - \nabla \mathcal{H} \cdot \nabla$$

Assumption II [Macroscopic limit of density] There exists a continuous, compactly supported density function $\varrho(x) \geq 0$, $\int_{\mathbb{R}} \varrho = 1$, such that for any fixed $a, b \in \mathbb{R}$

$$\lim_{N\to\infty} \sup_{t\geq 0} \left| \int \frac{1}{N} \sum_{j=1}^{N} \mathbf{1}(x_j \in [a,b]) f_t(\mathbf{x}) \mathrm{d}\mu(\mathbf{x}) - \int_a^b \varrho(x) \mathrm{d}x \right| = 0.$$

Assumption III [Eigenvalues are close to their classical location] Let γ_j denote the classical location of the j-th eigenvalue (given by ϱ). Suppose there is $\varepsilon > 0$ such that

$$\sup_{t \geq N^{-2\varepsilon}} N^{-1} \sum_{j=1}^{N} \int (x_j - \gamma_j)^2 f_t(\mathrm{dx}) \mu(\mathrm{dx}) \leq C N^{-1-2\varepsilon}$$

I.e., typical spacing between the random points x_j and their classical location is less than $N^{-1/2-\varepsilon}$.

Assumption IV. [Upper bound on local density] For any $I \in \mathbb{R}$, let

$$\mathcal{N}_I := \sum_{i=1}^{N} \mathbf{1}(x_i \in I)$$

denote the number of points in I. For any compact subinterval $I \subset \{E \,:\, \varrho(E) > 0\}$, $|I| \geq N^{-1+\sigma}$

$$\sup_{\tau \geq N^{-2\varepsilon}} \int \mathbf{1}\Big\{\mathcal{N}_I \geq KN|I|\Big\} f_\tau \mathrm{d}\mu \leq C_n K^{-n}, \qquad n = 1, 2, \ldots$$

Theorem [E-Schlein-Yau-Yin, 2009] Suppose Assumption I [convexity] holds. Suppose at time $t_0 = N^{-2\varepsilon}$, the density f_{t_0} satisfy a bounded entropy condition, i.e., for some m

$$S_\mu(f_{t_0}) := \int f_{t_0} \log f_{t_0} \mathrm{d}\mu \leq CN^m$$

and suppose that Assumptions II, III and IV hold for the solution of $\partial f_t = L f_t$. Let $E \in \mathbb{R}$ and $b > 0$ such that

$$\min\{\varrho(x) \ : \ x \in [E-b, E+b]\} > 0.$$

Then $\forall \delta > 0$, $\forall k \geq 1$ and for any nice test fn $O : \mathbb{R}^k \to \mathbb{R}$ we have

$$\lim_{N\to\infty} \sup_{t \geq N^{-2\varepsilon+\delta}} \frac{1}{2b} \int_{E-b}^{E+b} \mathrm{d}E' \int_{\mathbb{R}^k} \mathrm{d}\alpha_1 \ldots \mathrm{d}\alpha_k \, O(\alpha_1, \ldots, \alpha_k)$$
$$\times \frac{1}{\varrho(E)^k} \Big(p_{t,N}^{(k)} - p_{\mu,N}^{(k)} \Big) \Big(E' + \frac{\alpha_1}{N\varrho(E)}, \ldots, E' + \frac{\alpha_k}{N\varrho(E)} \Big) = 0$$

where $\varepsilon > 0$ is the exponent from Assumption III.

The method gives an effective error bound with trade-off between ε and b, in particular $b = N^{-power}$ is allowed.

Local relaxation flow

We modify the dynamics by adding a fictitious term to $\mathcal{H}$:

$$\widetilde{\mathcal{H}} = \mathcal{H} + W, \qquad W = \sum_j \frac{(x_j - \gamma_j)^2}{R^2}$$

where $R \ll 1$ and γ_j are the classical location of ev's, defined by

$$\int_{-\infty}^{\gamma_j} \varrho_{sc}(x)\mathrm{d}x = \frac{j}{N}$$

New equilibrium measure: $\omega \sim e^{-N\widetilde{\mathcal{H}}}$.

We then have

$$\Big\langle \mathbf{v}, \nabla^2 \widetilde{\mathcal{H}}(\mathbf{x})\mathbf{v} \Big\rangle \geq \frac{1}{R^2}\|\mathbf{v}\|^2 + \frac{1}{N}\sum_{i<j}\frac{(v_i - v_j)^2}{(x_i - x_j)^2}, \qquad \mathbf{v} \in \mathbb{R}^N.$$

i.e. the relaxation time is $t \sim R^2 \ll 1$ (Bakry-Emery). We also keep the second term from log $|x_i - x_j|$.

Theorem 1. [Relaxation of the modified dynamics] Suppose

$$\left\langle \mathbf{v}, \nabla^2 \widetilde{\mathcal{H}}(\mathbf{x}) \mathbf{v} \right\rangle \geq \frac{1}{R^2} \|\mathbf{v}\|^2 + \frac{1}{N} \sum_{i<j} \frac{(v_i - v_j)^2}{(x_i - x_j)^2}, \qquad \mathbf{v} \in \mathbb{R}^N.$$

Consider the forward equation

$$\partial_t q_t = \widetilde{\mathcal{L}} q_t, \qquad t \geq 0,$$

with initial condition $q_0 = q$ and with reversible measure ω. Then

$$\partial_t D_\omega(\sqrt{q_t}) \leq -\frac{1}{R^2} D_\omega(\sqrt{q_t}) - \frac{1}{2N^2} \int \sum_{i,j=1}^N \frac{(\partial_i \sqrt{q_t} - \partial_j \sqrt{q_t})^2}{(x_i - x_j)^2} \mathrm{d}\omega,$$

$$\frac{1}{2N^2} \int_0^\infty \mathrm{d}s \int \sum_{i,j=1}^N \frac{(\partial_i \sqrt{q_s} - \partial_j \sqrt{q_s})^2}{(x_i - x_j)^2} \mathrm{d}\omega \leq D_\omega(\sqrt{q})$$

and the logarithmic Sobolev inequality holds

$$S_\omega(q) \leq C R^2 D_\omega(\sqrt{q})$$

Thus the time to equilibrium is of order R^2:

$$S_\omega(q_t) \leq e^{-Ct/R^2} S_\omega(q_0).$$

Theorem 2. [Gap distribution estimated by Dir. form] Let $q \in L^\infty$ be a density, $\int q \mathrm{d}\omega = 1$. Then for any $\tau > 0$ we have

$$\Big| \int \frac{1}{N} \sum_i G(N(x_i - x_{i+1})) q \mathrm{d}\omega - \int \frac{1}{N} \sum_i G(N(x_i - x_{i+1})) \mathrm{d}\omega \Big|$$

$$\leq C \Big(\frac{D_\omega(\sqrt{q})\tau}{N} \Big)^{1/2} + C\sqrt{S_\omega(q)} e^{-c\tau/R^2}.$$

Similar result holds for $\mathcal{G}_{i,\mathbf{m}}(\mathbf{x})$.

As a comparison we show what entropy control alone can reach

Theorem: We have, for any $\tau > 0$, and $|E| < 2$

$$\Big| \frac{1}{N} \sum_i \int G\Big(N(x_i - E)\Big) \Big[q - 1\Big] \mathrm{d}\omega \Big| \leq C \Bigg(S_\omega(q) \Big[\tau + e^{-c\tau/R^2} \Big] \Bigg)^{1/2}$$

$$= CR\sqrt{S_\omega(q)}$$

after optimizing for τ in the last step. N problem!

We will use this for $q = g_\tau$, where $g_t \mathrm{d}\omega = f_t \mathrm{d}\mu$, or, $g_t = f_t/\psi$. Comparison of dynamics:

$$\begin{array}{llll} \partial_t f_t = \mathcal{L} f_t & f_t\mu \to \mu & \text{slow} & \text{[true DBM]} \\ \partial_t q_t = \widetilde{\mathcal{L}} q_t & q_t\omega \to \omega & \text{fast} & \text{[approx]} \end{array}$$

Initial state $f_0 = q_0\psi$. Write $f_t = g_t\psi$, compare $f_t\mu = g_t\omega$ with $q_t\omega$.

Theorem 3: [Comparison of dynamics] Set $\tau = R^2 N^{\varepsilon'}$. Let $g_t = f_t/\psi$. Assume that $S_\omega(g_0) \le CN^m$ with some fixed m. Then the entropy and the Dirichlet form satisfy the estimates:

$$S_\omega(g_{\tau/2}) \le CNR^{-2}Q, \qquad D_\omega(\sqrt{g_\tau}) \le CNR^{-4}Q.$$

Recall:

$$Q := \sup_{t\ge 0} \sum_j \int (x_j - \gamma_j)^2 f_t \mathrm{d}\mu.$$

Note: g_t does NOT evolve according to the $\widetilde{\mathcal{L}}$ dynamics (then the estimate would be exp. small).

Still, differentiating entropy (similar to Boltzmann's $\partial_t S = -D$)

$$\partial_t S_\omega(g_t) = -\frac{2}{N}\sum_j \int (\partial_j \sqrt{g_t})^2 \, \mathrm{d}\omega + \sum_j \int b_j \partial_j g_t \, \mathrm{d}\omega, \qquad b_j = \nabla W_j = \frac{x_j - \gamma_j}{R^2}$$

From the Schwarz inequality

$$\partial_t S_\omega(g_t) \le -D_\omega(\sqrt{g_t}) + CN \sum_j \int b_j^2 g_t \, \mathrm{d}\omega \le -D_\omega(\sqrt{g_t}) + CNQR^{-4} \qquad (*)$$

Together with LSI, we have

$$\partial_t S_\omega(g_t) \le -CR^{-2} S_\omega(g_t) + CNQR^{-4}$$

which, after integrating from $t = t_0$ to $\tau/2$, proves the bound on $S_\omega(g_\tau)$, using that $\tau = R^2 N^{\varepsilon'}$ and

$$S_\omega(g_0) = S_\mu(f_{t_0}|\psi) = S_\mu(f_{t_0}) - \log(Z/\widetilde{Z}) + N \int W f_{t_0} \mathrm{d}\mu \le CN^m.$$

In fact, we have

$$S_\omega(g_s) \le CR^2 NQR^{-4} = CNQR^{-2}$$

for any $s \ge \tau/2$. The estimate for D is integration of (*) from $\tau/2$ to τ.

Step 3: Green function comparison theorem

Theorem: Let $H^{(v)}$ and $H^{(w)}$ be two generalized $N \times N$ Wigner matrices, with matrix elements h_{ij} given by the random variables $N^{-1/2}v_{ij}$ and $N^{-1/2}w_{ij}$, respectively, with v_{ij} and w_{ij} satisfying the uniform subexponential decay condition.

Define interpolating matrices, $H_0 = H^{(v)}, H_1, \dots H_{N^2} = H^{(w)}$, i.e. order the elements in some way and let H_γ such that its matrix elements h_{ij} follow the v-distribution up to γ and then they follow the w-distribution;

Assume that for any $y \geq N^{-1+\varepsilon'}$ and $|E| < 2$,

$$\left| \left(\frac{1}{H_\gamma - E - iy} \right)_{kk} \right| \lesssim 1$$

Assume moment matching

$$\mathbb{E} v_{ij}^k = \mathbb{E} w_{ij}^k, \qquad k = 1, 2, 3, \qquad \left| \mathbb{E} v_{ij}^4 - \mathbb{E} w_{ij}^4 \right| \leq N^{-\delta}$$

Let F be a nice function and $G^{(v)}(z) = (H^{(v)} - z)^{-1}$ be the resolvent. Then

$$\left| \mathbb{E}F\left(\frac{1}{N}\mathrm{Tr}\left[\prod_{j=1}^{k_1} G^{(v)}(z_j^1)\right], \ldots, \frac{1}{N}\mathrm{Tr}\left[\prod_{j=1}^{k_n} G^{(v)}(z_j^n)\right]\right) - \mathbb{E}F\left(G^{(v)} \to G^{(w)}\right) \right| \ll 1$$

for any collection of spectral parameters, $z_j^m = E_j^m \pm i\eta$, with $\eta \geq N^{-1-\varepsilon}$, $|E_j^m| \leq 2 - 2\kappa$.

Similar statement holds for matrix elements.

Moral of the statement: if four moments (almost) match, local statistics are the same.

The four moment condition first appeared in Tao-Vu's Four mo-memt theorem, stating that observables of the form

$$\sum_{i_1<i_2<...<i_k} F\Big(N\lambda_{i_1}, N\lambda_{i_2}, \ldots, N\lambda_{i_k}\Big)$$

have the same expectation under the two ensembles, if the first four moments of the single entry distribution match. Their proof also used local semicircle law and it was quite involved due to resonances [they used level repulsion].

Our proof is a simple resolvent perturbation, exploiting the stability of the resolvent; $|G(z)| \le \eta^{-1}$ (never blows up).

Matching lemma

We have proved so far universality for Wigner matrices with a single entry distribution having an $N^{-\varepsilon}$ Gaussian convolution.

We also proved that local stat are stable if four moments (almost) match.

To prove the universality for any single entry distribution, we need to approximate it by another one with a small Gaussian components so that the first four moments almost match. This is and elementary lemma.

Let

$$m_k(\xi) = \mathbb{E}\xi^k$$

be the k-th moment of a random variable ξ.

Matching lemma: Let m_3 and m_4 be two numbers such that

$$m_4 - m_3^2 - 1 \geq 0, \;\; m_4 \leq C_2$$

Let ξ^G be a Gaussian random variable with mean 0 and variance 1. Then for any sufficient small $\gamma > 0$ there exists a real random variable ξ_γ with subexponential decay and independent of ξ^G, such that the first four moments of

$$\xi' = (1-\gamma)^{1/2}\xi_\gamma + \gamma^{1/2}\xi^G$$

are $m_1(\xi') = 0$, $m_2(\xi') = 1$, $m_3(\xi') = m_3$ and $m_4(\xi')$, and

$$|m_4(\xi') - m_4| \leq C\gamma$$

Note that $m_4 - m_3^2 - 1 \geq 0$ holds for the moments of any random variable with $m_1 = 0$, $m_2 = 1$.

SUMMARY: KEY STEPS

Step 1. Good local semicircle law including a control near the edge.

Method: System of self-consistent equations for the Green function, control the error by large deviation methods.

Step 2. Universality for Gaussian divisible matrices with a small ($\sim N^{-\varepsilon}$) Gaussian component.

Method: Modify DBM to speed up its local relaxation, then show that the modification is irrelevant for statistics involving differences of eigenvalues.

Step 3. Universality for arbitrary Wigner matrices.

Method: Remove the small Gaussian component in Step 2. by perturbation theory and moment matching.

OPEN PROBLEMS

1. Extend universality to band matrices for $W \gg \sqrt{N}$

2. Prove delocalization for band matrices, $W \gg \sqrt{N}$

3. Extend the method to β-ensembles.

Random band matrices: H is symmetric with independent but not identically distributed entries with mean zero and variance

$$\mathbb{E}\, N|h_{k\ell}|^2 = e^{-|k-\ell|/W}$$

Conjecture

Narrow band, $W \ll \sqrt{N} \Longrightarrow$ localization, Poisson statistics

Broad band, $W \gg \sqrt{N} \Longrightarrow$ delocalization, GOE statistics

Even the Gaussian case is completely open.

Some proofs for Step 1: Local semicircle law

$H^{(i)}$ denotes the $(N-1)\times(N-1)$ minor of H after removing the i-th column/row and similarly for $H^{(ij)}$ etc. Let $\mathbf{a}^i$ be the i-th column. The Green functions are:

$$G = \frac{1}{H-z}, \qquad G^{(i)} = \frac{1}{H^{(i)}-z}$$

Then from Schur decomposition we have

$$G_{ii} = \left(\frac{1}{H-z}\right)_{ii} = \frac{1}{h_{ii} - z - \mathbf{a}^i \cdot G^{(i)}\mathbf{a}^i}$$

With the notation $K_{ij} = h_{ij} - z\delta_{ij} - \mathbf{a}^i \cdot G^{(ij)}\mathbf{a}^j$ we also have

$$G_{ij} = -G_{jj}G_{ii}^{(j)}K_{ij}$$

and a different set of formulas (i, j, k different)

$$G_{ii} = G_{ii}^{(j)} + \frac{G_{ij}G_{ji}}{G_{jj}}, \qquad G_{ij} = G_{ij}^{(k)} + \frac{G_{ik}G_{kj}}{G_{kk}}$$

Rule of thumb: $G_{ii} \sim O(1)$, G_{ij} is small.

Crudest local semicircle law (for $\sigma_{ij}^2 = \frac{1}{N}$)

$$G_{ii} = \frac{1}{h_{ii} - z - \mathbb{E}_i\, \mathbf{a}^i \cdot G^{(i)} \mathbf{a}^i - Z_i}$$

with

$$Z_i := \mathbf{a}^i \cdot G^{(i)} \mathbf{a}^i - \mathbb{E}_i\, \mathbf{a}^i \cdot G^{(i)} \mathbf{a}^i$$

By a simple computation

$$\mathbb{E}_i\, \mathbf{a}^i \cdot G^{(i)} \mathbf{a}^i = \frac{N-1}{N} m^{(i)},$$

and by the interlacing property of the eigenvalues of H and $H^{(i)}$,

$$|m(z) - m^{(i)}(z)| \le \frac{1}{N\eta}, \qquad \eta = \operatorname{Im} z$$

Putting all together

$$G_{ii} = \frac{1}{-z - m(z) + \Omega_i}, \qquad \Omega_i = h_{ii} - Z_i + O\Big(\frac{1}{N\eta}\Big)$$

$$G_{ii} = \frac{1}{-z - m(z) + \Omega_i}, \qquad \Omega_i = h_{ii} - Z_i + O\Big(\frac{1}{N\eta}\Big)$$

Expand and sum up, with $\Omega = \max_i \Omega_i \ll 1$

$$m(z) = -\frac{1}{z + m(z)} + O(\Omega)$$

By the stability analysis of this equation, and $m_{sc} = -\frac{1}{z+m_{sc}}$,

$$|m - m_{sc}| \le \frac{C\Omega}{\sqrt{\Omega + \kappa}}, \qquad \kappa := \Big||E| - 2\Big|$$

By the large deviation bound on Z_i, we have $\Omega \lesssim (N\eta)^{-1/2}$, thus

$$|m - m_{sc}| \lesssim \min\Big(\frac{1}{\sqrt{N\eta\kappa}}, \frac{1}{(N\eta)^{1/4}}\Big)$$

It can be improved in various directions.

Some large deviation results

Let a_i $(1 \le i \le N)$ be independent complex random variables with mean zero, variance σ^2 and having a uniform subexponential decay

$$\mathbb{P}(|a_i| \ge x\sigma) \le \vartheta^{-1} \exp\big(-x^{\vartheta}\big), \qquad \forall\, x \ge 1,$$

with some $\vartheta > 0$. Let A_i, $B_{ij} \in \mathbb{C}$ $(1 \le i, j \le N)$. Then for any $\xi > 1$ there exists a constant L_0 depending on ϑ and ξ such that we have

$$\mathbb{P}\left\{\left|\sum_{i=1}^N a_i A_i\right| \ge (\log N)^{L_0}\sigma\sqrt{\sum_i |A_i|^2}\right\} \le C\exp\Big[-(\log N)^{\xi}\Big],$$

$$\mathbb{P}\left\{\left|\sum_{i=1}^N \overline{a}_i B_{ii} a_i - \sum_{i=1}^N \sigma^2 B_{ii}\right| \ge (\log N)^{L_0}\sigma^2\sqrt{\sum_i |B_{ii}|^2}\right\} \le C\exp\Big[-(\log N)^{\xi}\Big],$$

$$\mathbb{P}\left\{\left|\sum_{i\ne j} \overline{a}_i B_{ij} a_j\right| \ge (\log N)^{L_0}\sigma^2\sqrt{\sum_{i\ne j} |B_{ij}|^2}\right\} \le C\exp\Big[-(\log N)^{\xi}\Big].$$

For example

$$|Z_i| \lesssim \sqrt{\sum_{j,k\neq i} \left|\sigma_{ij} G^{(i)}_{jk} \sigma_{ki}\right|^2} \lesssim \frac{1}{\sqrt{N\eta}}$$

since

$$\sum_{j,k\neq i} \left|\sigma_{ij} G^{(i)}_{jk} \sigma_{ki}\right|^2 \leq \frac{1}{N^2} \sum_j \left(|G^{(i)}|^2\right)_{jj} = \frac{1}{N\eta}\frac{1}{N} \sum_j \operatorname{Im} G^{(i)}_{jj} \leq \frac{C}{N\eta}$$

using

$$\sum_k |G_{jk}|^2 = \sum G_{jk} \overline{G^*_{kj}} = (GG^*)_{jj} = \frac{1}{\eta} \operatorname{Im} G_{jj}$$

and similarly

$$Z_{ij} = \mathbf{a}^i \cdot G^{(ij)} \mathbf{a}^j \lesssim \frac{1}{\sqrt{N\eta}}$$

Improved local semicircle law. I: Bound on matrix elements

Self-consistent equation for the diagonal elements:

$$G_{ii} = \frac{1}{-z - \sum_j \sigma_{ij}^2 G_{jj} + Y_i}$$

$$Y_i := h_{ii} + A_i - Z_i$$

$$A_i := \sigma_{ii}^2 G_{ii} + \sum_j \sigma_{ij}^2 \frac{G_{ji}G_{ij}}{G_{ii}}$$

$$Z_i := \mathbf{a}^i \cdot G^{(i)}\mathbf{a}^i - \mathbb{E}_i \mathbf{a}^i \cdot G^{(i)}\mathbf{a}^i$$

Further key quantities

$$\Lambda_d := \max_k |G_{kk} - m_{sc}|, \qquad \Lambda_o := \max_{j\neq k} |G_{jk}|, \qquad \Lambda := |m - m_{sc}|$$

(they all depend on $z = E + i\eta$, but suppressed in the notation)

Dichotomy lemma

Away from the spectral edge (for simplicity), we have that

IF $\quad \Lambda_o + \Lambda_d \leq \dfrac{1}{(\log N)^2}, \quad$ THEN $\quad \Lambda_o + \Lambda_d \lesssim \dfrac{1}{\sqrt{N\eta}} \quad$ with high prob

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Proof. For the offdiagonal part, use

$$|G_{ij}| = |G_{ii}||G_{jj}^{(i)}||K_{ij}| \leq C(|h_{ij}| + |Z_{ij}|) \lesssim \frac{1}{\sqrt{N\eta}}$$

For the diagonal part, we will use the self-consistent equation and

$$|Y_i| \leq |h_{ii}| + |A_i| + |Z_i| \lesssim \frac{1}{\sqrt{N}} + \left(\frac{1}{N} + \frac{C}{N\eta}\right) + \frac{1}{\sqrt{N\eta}} \leq \frac{1}{\sqrt{N\eta}}$$

Expansion of the self-consistent equation

With the notation $v_i := G_{ii} - m_{sc}$, we have

$$v_i = \frac{1}{-z - m_{sc} - \Big(\sum_j \sigma_{ij}^2 v_j - Y_i\Big)} - m_{sc}$$

We know that $|z + m_{sc}| \geq c$ and by the assumption $(\ldots) \ll 1$, so we can expand

$$v_i = m_{sc}^2 \Big(\sum_j \sigma_{ij}^2 v_j - Y_i\Big) + O\Big(\sum_j \sigma_{ij}^2 v_j - Y_i\Big)^2$$

using $m_{sc} + \frac{1}{z+m_{sc}} = 0$. (More precise expansion is possible – this is the key to further improvements). Introduce

$$\bar{v} = \frac{1}{N}\sum_i v_i, \qquad \bar{Y} := \frac{1}{N}\sum_i Y_i, \qquad Y := \max |Y_i|, \qquad \zeta := m_{sc}^2$$

$$\bar{v} = \zeta \bar{v} - \zeta \bar{Y} + O(\Lambda_d^2 + Y^2)$$

$$\bar{v} = -\frac{\zeta}{1-\zeta}\bar{Y} + O\Big(\frac{\zeta}{1-\zeta}(\Lambda_d^2 + Y^2)\Big) = O\Big(\frac{\zeta}{1-\zeta}(\Lambda_d^2 + Y)\Big)$$

where the second bound is crude (another key to improvement).

Let $B_{ij} = \sigma_{ij}^2$ be the covariance matrix, and let $Q = I - |\mathbf{e}\rangle\langle\mathbf{e}|$ be the projection onto the orth. complement of the trivial eigenvector.

$$v_i - \bar{v} = \zeta \sum_j B_{ij}(v_j - \bar{v}) - \zeta(Y_i - \bar{Y}) + O(\Lambda_d^2 + Y^2)$$

The error sums up to 0, so we can invert $I - \zeta B$ on the range of Q.

$$\begin{aligned} v_i - \bar{v} &= \sum_j \Big(\frac{\zeta Q}{1-\zeta B}\Big)_{ij}(Y_j - \bar{Y}) + \Big\|\frac{\zeta Q}{1-\zeta B}\Big\|_{\infty\to\infty} O(\Lambda_d^2 + Y^2) \\ &= \Big\|\frac{\zeta Q}{1-\zeta B}\Big\|_{\infty\to\infty} O(\Lambda_d^2 + Y) \qquad \text{(crude)} \end{aligned}$$

$$|v_i| \le \left(\Big\|\frac{\zeta Q}{1-\zeta B}\Big\|_{\infty\to\infty} + \Big|\frac{\zeta}{1-\zeta}\Big|\right) O(\Lambda_d^2 + Y)$$

$$|v_i| \leq \left(\left\| \frac{\zeta Q}{1-\zeta B} \right\|_{\infty\to\infty} + \left| \frac{\zeta}{1-\zeta} \right| \right) O(\Lambda_d^2 + Y)$$

Here use that Spec$(B) \subset [-1+\delta_-, 1-\delta_+] \cup \{1\}$, $\delta_\pm > 0$, thus

$$\begin{aligned}
\left\| \frac{\zeta Q}{1-\zeta B} \right\|_{\infty\to\infty} &\leq \sum_{k\leq n} \|\zeta^{k+1} Q B^k\|_{\infty\to\infty} + \sqrt{N} \sum_{k>n} \|\zeta^{k+1} Q B^k\|_{2\to 2} \\
&\leq n + \sqrt{N} \sum_{k>n} (1-\delta_\pm)^k \\
&\leq \frac{C \log N}{\delta_\pm}, \qquad \text{by choosing} \quad n \sim \frac{\log N}{\delta_\pm}
\end{aligned}$$

Neglect the edge-deterioration (i.e. that $1-\zeta = 1 - m_{sc}^2 \sim \sqrt{\kappa+\eta}$),

$$\Lambda_d \leq C(\Lambda_d^2 + Y)(\log N)$$

Quadratic dichotomy:

$$\text{If } \Lambda_d \ll (\log N)^{-1}, \text{ then } \Lambda_d \leq C(\log N) Y \lesssim \frac{1}{\sqrt{N\eta}}$$

Continuity argument

How to guarantee $\Lambda_d \ll (\log N)^{-1}$?

$$R(z) := (\log N)^{-2}, \qquad S(z) := \frac{(\log N)^C}{\sqrt{N\eta}}, \qquad S(z) < R(z)$$

We have proved that (with high prob)

$$\Lambda_o(z) + \Lambda_d(z) \le R(z) \quad \Longrightarrow \quad \Lambda_o(z) + \Lambda_d(z) \le S(z) \quad (*)$$

For $\eta = \text{Im} z = 10$ (say), one has $\Lambda_o(z) + \Lambda_d(z) \le 3/10$, so expansion is possible ($|z+m| \ge 10 - \frac{1}{10} > 3/10$) and one gets that the LHS of (*) holds for $\eta = 10$.

Now fix E and reduce η, note that $\eta \mapsto \Lambda_o(z) + \Lambda_d(z)$ is cont. so $\Lambda_o(z) + \Lambda_d(z) \le S(z)$ as long as $S(z) < R(z)$.

Improved local semicircle law II.

The key to improve the previous argument is to exploit a CLT-like cancellation in

$$\bar{Y} = \frac{1}{N}\sum_i Y_i, \qquad \text{i.e., mainly in} \quad \frac{1}{N}\sum_i Z_i$$

Recall that

$$Z_i = \mathbf{a}^i \cdot G^{(i)}\mathbf{a}^i - \mathbb{E}_i\, \mathbf{a}^i \cdot G^{(i)}\mathbf{a}^i$$

Here Z_i and Z_j are not independent, but "almost".

Lemma

$$\mathbf{E}\Big|\frac{1}{N}\sum_i Z_i\Big|^p \le p^{Cp}\mathbf{E}\Big[\Lambda_o^{2p} + N^{-p}\Big]$$

or, after plugging in the size of $\Lambda_o \le (N\eta)^{-1/2}$

$$\mathbf{E}\Big|\frac{1}{N}\sum_i Z_i\Big|^p \le p^{Cp}(N\eta)^{-p} \quad \Longrightarrow \quad \bar{Z} \lesssim (N\eta)^{-1}$$

Sketch of the sketch of the proof: $p = 2$

$$\mathbb{E}\Big|\frac{1}{N}\sum_i Z_i\Big|^2 = \frac{1}{N^2}\sum_{i\neq j}\mathbb{E}Z_iZ_j + \frac{1}{N^2}\sum_i \mathbb{E}|Z_i|^2$$

We look only at the first term, this is $\mathbb{E}Z_1Z_2$ (by symmetry)

$$\mathbb{E}Z_1Z_2 = \mathbb{E}\Big[\mathbf{a}^1\cdot G^{(1)}\mathbf{a}^1 - \mathbb{E}_1\mathbf{a}^1\cdot G^{(1)}\mathbf{a}^1\Big]\Big[\mathbf{a}^2\cdot G^{(2)}\mathbf{a}^2 - \mathbb{E}_1\mathbf{a}^2\cdot G^{(2)}\mathbf{a}^2\Big]$$

Make the main terms independent by

$$G^{(1)}_{k\ell} = G^{(12)}_{k\ell} + \frac{G^{(1)}_{k2}G^{(1)}_{2\ell}}{G^{(1)}_{22}} =: P^{(12)}_{k\ell} + P^{(1)}_{k\ell}$$

(superscripts indicate independence of the givens rows/columns)

$$\mathbb{E}Z_1Z_2 = \mathbb{E}\Big[\mathbb{E}\!\!\!|_1\mathbf{a}^1P^{(12)}\mathbf{a}^1 + \mathbb{E}\!\!\!|_1\mathbf{a}^1P^{(1)}\mathbf{a}^1\Big]\Big[\mathbb{E}\!\!\!|_2\mathbf{a}^2P^{(12)}\mathbf{a}^2 + \mathbb{E}\!\!\!|_2\mathbf{a}^2P^{(2)}\mathbf{a}^2\Big]$$

where $\mathbb{E}\!\!\!|_i := I - \mathbb{E}_i$, i.e. acting on a r.v. X, we have $\mathbb{E}\!\!\!|_i X = X - \mathbb{E}_i X$.

$$\mathbb{E}Z_1Z_2 = \mathbb{E}\Big[\, \mathbb{E}\!\!\!|_1\, \mathbf{a}^1 P^{(12)} \mathbf{a}^1 + \mathbb{E}\!\!\!|_1\, \mathbf{a}^1 P^{(1)} \mathbf{a}^1 \Big]\Big[\, \mathbb{E}\!\!\!|_2\, \mathbf{a}^2 P^{(12)} \mathbf{a}^2 + \mathbb{E}\!\!\!|_2\, \mathbf{a}^2 P^{(2)} \mathbf{a}^2 \Big]$$

with $\mathbb{E}\!\!\!|_i := I - \mathbb{E}_i$.

Note that if $A^{(i)}$ is indep of i and Y is arbitrary, then

$$\mathbb{E}\!\!\!|_i \Big[Y A^{(i)} \Big] = A^{(i)}\, \mathbb{E}\!\!\!|_i Y$$

in particular, $\mathbb{E}\!\!\!|_i A^{(i)} = 0$, and

$$\mathbb{E}\Big(\, \mathbb{E}\!\!\!|_1 X^{(2)} \Big)\Big(\, \mathbb{E}\!\!\!|_2 X^{(1)} \Big) = \mathbb{E}\, \mathbb{E}\!\!\!|_1 \Big[X^{(2)}\, \mathbb{E}\!\!\!|_2 X^{(1)} \Big] = 0$$

(with $Y := X^{(2)}$, $A^{(1)} := \mathbb{E}\!\!\!|_2 X^{(1)}$), since $\mathbb{E}\, \mathbb{E}\!\!\!|_i = 0$. Thus

$$\mathbb{E}Z_1Z_2 = \mathbb{E}\Big[\, \mathbb{E}\!\!\!|_1\, \mathbf{a}^1 P^{(1)} \mathbf{a}^1 \Big]\Big[\, \mathbb{E}\!\!\!|_2\, \mathbf{a}^2 P^{(2)} \mathbf{a}^2 \Big]$$

and $\mathbb{E}[\mathbf{a}^1 P^{(1)} \mathbf{a}^1]^2 \sim \sum_{k\ell \neq 1} |P^{(1)}_{k\ell}|^2 \sigma^2_{1k} \sigma^2_{1\ell} \sim \Lambda_o^4$.

Some proofs for Step 2: Bakry-Emery argument

Simple calculation shows

$$\partial_t S(f_t) = \int (Lf_t) \log f_t \mathrm{d}\mu + \int f_t \frac{Lf_t}{f_t} \mathrm{d}\mu = -\frac{1}{2} \int \frac{(\nabla f_t)^2}{f_t} \mathrm{d}\mu = -4D(\sqrt{f_t}), \tag{1}$$

using $\int Lf_t \mathrm{d}\mu = 0$.

Now we compute $\partial_t D(\sqrt{f_t})$. Let $h := \sqrt{f}$ for simplicity, then

$$\partial_t h_t = \frac{1}{2h_t} \partial h_t^2 = \frac{1}{2h_t} Lh_t^2 = Lh_t + \frac{1}{2h_t} (\nabla h_t)^2.$$

In the last step we used that $Lh^2 = (\nabla h)^2 + 2hLh$ that can be seen directly from $L = \frac{1}{2}\Delta - \frac{1}{2}(\nabla \mathcal{H})\nabla$.

We compute (dropping the t subscript for brevity and $\int = \int \mathrm{d}\mu$)

$$\partial_t D(\sqrt{f_t}) = \frac{1}{2}\partial_t \int (\nabla h)^2 \mathrm{d}\mu \qquad\qquad h = \sqrt{f}$$

$$= \int (\nabla h)(\nabla L h) + \frac{1}{2}\int (\nabla h)\cdot \nabla \frac{(\nabla h)^2}{h}$$

$$= \int (\nabla h)[\nabla, L]h + \int (\nabla h) L(\nabla h) + \frac{1}{2}\int \sum_{ij} \partial_i h \Big[\frac{2(\partial_j h)\partial_i\partial_j h}{h} - \frac{(\partial_j h)^2 \partial_i h}{h^2}\Big]$$

$$= -\frac{1}{2}\int (\nabla h)(\nabla^2 \mathcal{H})\nabla h - \frac{1}{2}\int \sum_{ij} (\partial_i \partial_j h)^2$$

$$+ \frac{1}{2}\int \sum_{ij} \Big[\frac{2(\partial_j h)(\partial_i h)\partial_{ij} h}{h} - \frac{(\partial_j h)^2(\partial_i h)^2}{h^2}\Big]$$

$$= -\frac{1}{2}\int (\nabla h)(\nabla^2 \mathcal{H})\nabla h - \frac{1}{2}\int \sum_{ij} \Big(\partial_{ij} h - \frac{(\partial_i h)(\partial_j h)}{h}\Big)^2, \qquad (2)$$

where we used the commutator

$$[\nabla, L] = -\frac{1}{2}(\nabla^2 \mathcal{H})\nabla.$$

Local relaxation flow

We modify the dynamics by adding a fictitious term to $\mathcal{H}$:

$$\widetilde{\mathcal{H}} = \mathcal{H} + W, \qquad W = \sum_j \frac{(x_j - \gamma_j)^2}{R^2}$$

where $R \ll 1$ and γ_j are the classical location of ev's, defined by

$$\int_{-\infty}^{\gamma_j} \varrho_{sc}(x)\mathrm{d}x = \frac{j}{N}$$

New equilibrium measure: $\omega \sim e^{-N\widetilde{\mathcal{H}}}$.

We then have

$$\Big\langle \mathbf{v}, \nabla^2 \widetilde{\mathcal{H}}(\mathbf{x})\mathbf{v} \Big\rangle \geq \frac{1}{R^2} \|\mathbf{v}\|^2 + \frac{1}{N} \sum_{i<j} \frac{(v_i - v_j)^2}{(x_i - x_j)^2}, \qquad \mathbf{v} \in \mathbb{R}^N.$$

i.e. the relaxation time is $t \sim R^2 \ll 1$ (Bakry-Emery). We also keep the second term from log $|x_i - x_j|$.

Theorem 1. [Relaxation of the modified dynamics] Suppose

$$\left\langle \mathbf{v}, \nabla^2 \widetilde{\mathcal{H}}(\mathbf{x})\mathbf{v} \right\rangle \geq \frac{1}{R^2}\|\mathbf{v}\|^2 + \frac{1}{N}\sum_{i<j}\frac{(v_i - v_j)^2}{(x_i - x_j)^2}, \qquad \mathbf{v} \in \mathbb{R}^N.$$

Consider the forward equation

$$\partial_t q_t = \widetilde{\mathcal{L}} q_t, \qquad t \geq 0,$$

with initial condition $q_0 = q$ and with reversible measure ω. Then

$$\partial_t D_\omega(\sqrt{q_t}) \leq -\frac{1}{R^2} D_\omega(\sqrt{q_t}) - \frac{1}{2N^2}\int \sum_{i,j=1}^{N} \frac{(\partial_i \sqrt{q_t} - \partial_j \sqrt{q_t})^2}{(x_i - x_j)^2} \mathrm{d}\omega,$$

$$\frac{1}{2N^2}\int_0^\infty \mathrm{d}s \int \sum_{i,j=1}^{N} \frac{(\partial_i \sqrt{q_s} - \partial_j \sqrt{q_s})^2}{(x_i - x_j)^2} \mathrm{d}\omega \leq D_\omega(\sqrt{q})$$

and the logarithmic Sobolev inequality holds

$$S_\omega(q) \leq C R^2 D_\omega(\sqrt{q})$$

Thus the time to equilibrium is of order R^2:

$$S_\omega(q_t) \leq e^{-Ct/R^2} S_\omega(q_0).$$

Theorem [Reformulation]

$$Q := \sup_{t \geq t_0} \sum_j \int (x_j - \gamma_j)^2 f_t \mathrm{d}\mu, \qquad t_0 = N^{-2\varepsilon}$$

and assume that $Q \leq CN^m$ with some exponent m. Fix $n \geq 1$ and an array of positive integers, $\mathbf{m} = (m_1, m_2, \ldots, m_n) \in \mathbb{N}_+^n$. Let $G : \mathbb{R}^n \to \mathbb{R}$ be a nice fn and define

$$\mathcal{G}_{i,\mathbf{m}}(\mathbf{x}) := G\Big(N(x_i - x_{i+m_1}), N(x_{i+m_1} - x_{i+m_2}), \ldots, N(x_{i+m_{n-1}} - x_{i+m_n})\Big).$$

Then for any sufficiently small $\varepsilon' > 0$, and for $\tau \geq 3t_0$ we have

$$\Big| \int \frac{1}{N} \sum_i \mathcal{G}_{i,\mathbf{m}}(\mathbf{x}) f_\tau \mathrm{d}\mu - \int \frac{1}{N} \sum_i \mathcal{G}_{i,\mathbf{m}}(\mathbf{x}) \mathrm{d}\mu \Big| \leq CN^{\varepsilon'} \sqrt{Q\tau^{-1}} + Ce^{-cN^{\varepsilon'}}.$$

Since $Q \leq N^{-2\varepsilon}$ in applications, the rhs is small if $\tau \gg N^{-2\varepsilon}$.

From this theorem and the upper bound one can easily get the limit of correlation functions. Note that the observable $\mathcal{G}$ depends on differences.

Theorem 2. [Gap distribution estimated by Dir. form] Let $q \in L^\infty$ be a density, $\int q \mathrm{d}\omega = 1$. Then for any $\tau > 0$ we have

$$\Big| \int \frac{1}{N} \sum_i G(N(x_i - x_{i+1})) q \mathrm{d}\omega - \int \frac{1}{N} \sum_i G(N(x_i - x_{i+1})) \mathrm{d}\omega \Big|$$

$$\leq C \Big(\frac{D_\omega(\sqrt{q})\tau}{N} \Big)^{1/2} + C\sqrt{S_\omega(q)} e^{-c\tau/R^2}.$$

Similar result holds for $\mathcal{G}_{i,\mathbf{m}}(\mathbf{x})$.

As a comparison we show what entropy control alone can reach

Theorem: We have, for any $\tau > 0$, and $|E| < 2$

$$\Big| \frac{1}{N} \sum_i \int G\Big(N(x_i - E)\Big) \Big[q - 1\Big] \mathrm{d}\omega \Big| \leq C \Bigg(S_\omega(q) \Big[\tau + e^{-c\tau/R^2} \Big] \Bigg)^{1/2}$$

$$= CR\sqrt{S_\omega(q)}$$

after optimizing for τ in the last step. N problem!

Proof. Run the $\widetilde{\mathcal{L}}$ flow with q as initial condition. Split

$$\int \frac{1}{N}\sum_i G(N(x_i - x_{i+1}))\Big[q-1\Big]\mathrm{d}\omega = \int \frac{1}{N}\sum_i G(\ldots)\Big[(q-q_\tau)+(q_\tau-1)\Big]\mathrm{d}\omega$$

For the second term use entropy,

$$\int \frac{1}{N}\sum_i |G(\ldots)||q_\tau - 1|\mathrm{d}\omega \le C\sqrt{S_\omega(q_\tau)} \le C\sqrt{S_\omega(q)}e^{-c\tau/R^2}$$

For the first term, integrate its derivative

$$\int \frac{1}{N}\sum_i G(\ldots)(q-q_\tau)\mathrm{d}\omega = \int_0^\tau \mathrm{d}s \int \frac{1}{N}\sum_i G(\ldots)\partial_s q_s \mathrm{d}\omega$$

$$= \int_0^\tau \mathrm{d}s \int \frac{1}{N}\sum_j (\partial_j q_s)\; \partial_j \Big[\frac{1}{N}\sum_i G(N(x_i - x_{i+1}))\Big]$$

$$= \int_0^\tau \mathrm{d}s \int \frac{1}{N}\sum_i (\partial_i q_s - \partial_{i+1} q_s)\; G'(N(x_i - x_{i+1}))$$

$$= \int_0^\tau \mathrm{d}s \int \frac{1}{N}\sum_i (\partial_i \sqrt{q_s} - \partial_{i+1}\sqrt{q_s})\; \sqrt{q_s}\; G'(N(x_i - x_{i+1}))$$

After a Schwarz

$$\Big|\int \frac{1}{N}\sum_i G(N(x_i - x_{i+1}))\Big[q - q_\tau\Big]\mathrm{d}\omega\Big|$$

$$= \Big|\int_0^\tau \mathrm{d}s \int \frac{1}{N}\sum_i (\partial_i\sqrt{q_s} - \partial_{i+1}\sqrt{q_s})\ \sqrt{q_s}\ G'(N(x_i - x_{i+1}))\Big|$$

$$\leq \left[\int_0^\tau \mathrm{d}s \int \sum_i \frac{[\partial_i\sqrt{q_s} - \partial_{i+1}\sqrt{q_s}]^2}{N^2(x_i - x_{i+1})^2}\mathrm{d}\omega\right]^{1/2}$$

$$\times \left[\int_0^\tau \mathrm{d}s \int \sum_i G'(N(x_i - x_{i+1}))^2 (x_i - x_{i+1})^2\, q_s \mathrm{d}\omega\right]^{1/2}$$

$$\leq \left[\int_0^\tau \mathrm{d}s \int \sum_{i,j} \frac{[\partial_i\sqrt{q_s} - \partial_j\sqrt{q_s}]^2}{N^2(x_i - x_j)^2}\mathrm{d}\omega\right]^{1/2} \sqrt{\frac{\tau}{N}}$$

The first term is estimated by $D_\omega(\sqrt{q})$ from the additional term in Bakry-Emery. $\square$

We will use this for $q = g_\tau$, where $g_t \mathrm{d}\omega = f_t \mathrm{d}\mu$, or, $g_t = f_t/\psi$. Comparison of dynamics:

$$\begin{array}{llll} \partial_t f_t = \mathcal{L} f_t & f_t \mu \to \mu & \text{slow} & \text{[true DBM]} \\ \partial_t q_t = \widetilde{\mathcal{L}} q_t & q_t \omega \to \omega & \text{fast} & \text{[approx]} \end{array}$$

Compare $f_t \mu = g_t \omega$ with ω for $t \geq 3t_0$.

Theorem 3: [Comparison of dynamics] Set $\tau = R^2 N^{\varepsilon'} \geq 3t_0$. Let $g_t = f_t/\psi$. Assume that $S_\mu(f_{t_0}) \leq CN^m$ with some fixed m. Then the entropy and the Dirichlet form satisfy the estimates:

$$S_\omega(g_{\tau/2}) \leq CNR^{-2}Q, \qquad D_\omega(\sqrt{g_\tau}) \leq CNR^{-4}Q.$$

Recall:

$$Q := \sup_{t \geq t_0} \sum_j \int (x_j - \gamma_j)^2 f_t \mathrm{d}\mu.$$

Note: g_t does NOT evolve according to the $\widetilde{\mathcal{L}}$ dynamics (then the estimate would be exp. small).

Still, differentiating entropy (similar to Boltzmann's $\partial_t S = -D$)

$$\partial_t S_\omega(g_t) = -\frac{2}{N}\sum_j \int (\partial_j \sqrt{g_t})^2 \, \mathrm{d}\omega + \sum_j \int b_j \partial_j g_t \, \mathrm{d}\omega, \qquad b_j = \nabla W_j = \frac{x_j - \gamma_j}{R^2}$$

From the Schwarz inequality

$$\partial_t S_\omega(g_t) \leq -D_\omega(\sqrt{g_t}) + CN \sum_j \int b_j^2 g_t \, \mathrm{d}\omega \leq -D_\omega(\sqrt{g_t}) + CNQR^{-4} \qquad (*)$$

Together with LSI, we have

$$\partial_t S_\omega(g_t) \leq -CR^{-2} S_\omega(g_t) + CNQR^{-4}$$

which, after integrating from $t = t_0$ to $\tau/2$, proves the bound on $S_\omega(g_\tau)$, using that $\tau = R^2 N^{\varepsilon'}$ and

$$S_\omega(g_0) = S_\mu(f_{t_0}|\psi) = S_\mu(f_{t_0}) - \log(Z/\widetilde{Z}) + N \int W f_{t_0} \mathrm{d}\mu \leq CN^m.$$

In fact, we have

$$S_\omega(g_s) \leq CR^2 NQR^{-4} = CNQR^{-2}$$

for any $s \geq \tau/2$. The estimate for D is integration of (*) from $\tau/2$ to τ.

Completion of the proof of Step 2.

Theorem [Recall]

$$Q := \sup_{t \geq t_0} \sum_j \int (x_j - \gamma_j)^2 f_t \mathrm{d}\mu, \qquad t_0 = N^{-2\varepsilon}$$

$$\mathcal{G}_{i,\mathbf{m}}(\mathbf{x}) := G\Big(N(x_i - x_{i+m_1}), N(x_{i+m_1} - x_{i+m_2}), \ldots, N(x_{i+m_{n-1}} - x_{i+m_n})\Big).$$

Then for any sufficiently small $\varepsilon' > 0$, and for $\tau \geq 3t_0$ we have

$$\Big| \int \frac{1}{N} \sum_i \mathcal{G}_{i,\mathbf{m}}(\mathbf{x}) f_\tau \mathrm{d}\mu - \int \frac{1}{N} \sum_i \mathcal{G}_{i,\mathbf{m}}(\mathbf{x}) \mathrm{d}\mu \Big| \leq C N^{\varepsilon'} \sqrt{Q\tau^{-1}} + Ce^{-cN^{\varepsilon'}}.$$

Proof: Recall that $t_0 = N^{-2\varepsilon}$, $\tau = R^2 N^{\varepsilon'}$. Choose $q = g_\tau = f_\tau/\psi$ in Theorem 2. Then Thm 2 and 3 give:

$$\Big| \int \frac{1}{N} \sum_i \mathcal{G}_{i,\mathbf{m}}(\mathbf{x}) f_\tau \mathrm{d}\mu - \int \frac{1}{N} \sum_i \mathcal{G}_{i,\mathbf{m}}(\mathbf{x}) \mathrm{d}\omega \Big| \leq C N^{\varepsilon'} \sqrt{Q\tau^{-1}} + Ce^{-cN^{\varepsilon'}}.$$

Use the same for $f_0 = 1$ (so $f_t = 1$) to compare $f_\tau \mu$ with μ. $\square$

Step 3: Green function comparison theorem

Theorem Let $H^{(v)}$ and $H^{(w)}$ be two generalized $N \times N$ Wigner matrices, with matrix elements h_{ij} given by the random variables $N^{-1/2}v_{ij}$ and $N^{-1/2}w_{ij}$, respectively, with v_{ij} and w_{ij} satisfying the uniform subexponential decay condition.

Define interpolating matrices, $H_0 = H^{(v)}, H_1, \ldots H_{N^2} = H^{(w)}$, i.e. order the elements in some way and let H_γ such that its matrix elements h_{ij} follow the v-distribution up to γ and then they follow the w-distribution;

Assume that for any $y \geq N^{-1+\varepsilon'}$ and $|E| < 2$,

$$\left| \left(\frac{1}{H_\gamma - E - iy} \right)_{kk} \right| \lesssim 1$$

Assume moment matching

$$\mathbb{E} v_{ij}^k = \mathbb{E} w_{ij}^k, \qquad k = 1, 2, 3, \qquad \left| \mathbb{E} v_{ij}^4 - \mathbb{E} w_{ij}^4 \right| \leq N^{-\delta}$$

Let F be a nice function and $G^{(v)}(z) = (H^{(v)} - z)^{-1}$ be the resolvent. Then

$$\left| \mathbb{E} F\left(\frac{1}{N} \mathrm{Tr}\left[\prod_{j=1}^{k_1} G^{(v)}(z_j^1) \right], \ldots, \frac{1}{N} \mathrm{Tr}\left[\prod_{j=1}^{k_n} G^{(v)}(z_j^n) \right] \right) - \mathbb{E} F\left(G^{(v)} \to G^{(w)} \right) \right| \ll 1$$

for any collection of spectral parameters, $z_j^m = E_j^m \pm i\eta$, with $\eta \geq N^{-1-\varepsilon}$, $|E_j^m| \leq 2 - 2\kappa$.

Similar statement holds for matrix elements.

Moral of the statement: if four moments (almost) match, local statistics are the same.

Sketch of the proof of the comparison theorem

$$\mathbb{E}\,F\left(\frac{1}{N}\mathrm{Tr}\frac{1}{H^{(v)}-z}\right)-\mathbb{E}\,F\left(\frac{1}{N}\mathrm{Tr}\frac{1}{H^{(w)}-z}\right)$$
$$=\sum_{\gamma=1}^{\gamma(N)}\left[\mathbb{E}\,F\left(\frac{1}{N}\mathrm{Tr}\frac{1}{H_\gamma-z}\right)-\mathbb{E}\,F\left(\frac{1}{N}\mathrm{Tr}\frac{1}{H_{\gamma-1}-z}\right)\right].$$

$H_{\gamma-1}$ with H_γ differ only in the (i,j) and (j,i) matrix elements:

$$H_{\gamma-1}=Q+\frac{1}{\sqrt{N}}V, \qquad V:=v_{ij}|\mathbf{e}_i\rangle\langle\mathbf{e}_j|+v_{ji}|\mathbf{e}_j\rangle\langle\mathbf{e}_i|$$

$$H_\gamma=Q+\frac{1}{\sqrt{N}}W, \qquad W:=w_{ij}|\mathbf{e}_i\rangle\langle\mathbf{e}_j|+w_{ji}|\mathbf{e}_j\rangle\langle\mathbf{e}_i|,$$

$$Q_{ij}=Q_{ji}=0$$

Define the Green functions

$$R=\frac{1}{Q-z}, \qquad S=\frac{1}{H_\gamma-z}.$$

Step 1: We have

$$S_{kk} = \left(\frac{1}{H_\gamma - z}\right)_{kk} \lesssim 1 \quad \Longrightarrow \quad R_{kk} = \left(\frac{1}{Q - z}\right)_{kk} \lesssim 1$$

by resolvent perturbation:

$$R = S + \frac{1}{N^{1/2}} SVS + \ldots + \frac{1}{N^{9/5}} (SV)^9 S + \frac{1}{N^5} (SV)^{10} R.$$

Note that offdiagonal elements can be estimated by diagonal, morally

$$|G_{kl}| \lesssim \max |G_{kk}|$$

V has only at most two nonzero element, so $(SVS)_{k\ell} = S_{ki} v_{ij} S_{j\ell} + S_{kj} v_{ji} S_{i\ell}$. In general, $(SVSV...VS)_{kk}$ is a finite sum of products of S_{kl} and v_{ij}. For the last term, use the trivial bound $|R_{ij}| \leq \eta^{-1}$.

Step 2. Compare the resolvents of $H_{\gamma-1}$ and H_γ with the resolvent $R = (Q-z)^{-1}$:

$$S = R - \frac{1}{\sqrt{N}} RVR + \frac{1}{N}(RV)^2 R - \frac{1}{N^{3/2}}(RV)^3 R + \frac{1}{N^2}(RV)^4 R - \frac{1}{N^{5/2}}(RV)^5 S,$$

so we can write

$$\frac{1}{N}\mathrm{Tr} S = \widehat{R} + \xi, \qquad \xi = \sum_{m=1}^{4} \frac{1}{N^{m/2}} \widehat{R}^{(m)} + \frac{1}{N^{5/2}}\Omega$$

with

$$\widehat{R} = \frac{1}{N}\mathrm{Tr} R, \qquad \widehat{R}^{(m)} = (-1)^m \frac{1}{N}\mathrm{Tr}(RV)^m R, \qquad \Omega = -\frac{1}{N}\mathrm{Tr}(RV)^5 S.$$

Each of them is a sum of a few terms, e.g.

$$\widehat{R}^{(2)} = \frac{1}{N}\sum_k \Big[R_{ki}v_{ij}R_{jj}v_{ji}R_{ik} + R_{ki}v_{ij}R_{ji}v_{ij}R_{ik} + R_{kj}v_{ji}R_{ii}v_{ij}R_{jk} + R_{kj}v_{ji}R_{ij}v_{ji}R_{ik}$$

((i,j) fixed!) and similar formulas hold for the other terms.

Taylor expand:

$$\begin{aligned}\mathbb{E}F\left(\frac{1}{N}\mathrm{Tr}\frac{1}{H_\gamma - z}\right) =& \mathbb{E}F\left(\widehat{R}+\xi\right)\\ =& \mathbb{E}\left[F(\widehat{R}) + F'(\widehat{R})\xi + F''(\widehat{R})\xi^2 + \ldots + F^{(5)}(\widehat{R}+\xi')\xi^5\right]\\ =& \sum_{m=0}^{5}\frac{1}{N^{m/2}}\mathbb{E}A^{(m)},\end{aligned}$$

where $\xi' \in [0, \xi]$; the $A^{(m)}$'s are defined as

$$A^{(0)} = F(\widehat{R}), \quad A^{(1)} = F'(\widehat{R})\widehat{R}^{(1)}, \quad A^{(2)} = F''(\widehat{R})(\widehat{R}^{(1)})^2 + F'(\widehat{R})\widehat{R}^{(2)}, \quad etc.$$

The expectation values of the terms $A^{(m)}$, $m \le 4$, with respect to v_{ij} are determined by the first four moments of v_{ij}, for example

$$\begin{aligned}\mathbb{E}\,A^{(2)} =& F''(\widehat{R})\Big[\frac{1}{N}\sum_k R_{ki}R_{jj}R_{ik} + \ldots\Big]\mathbb{E}\,|v_{ij}|^2\\ &+F'(\widehat{R})\Big[\frac{1}{N^2}\sum_{k,\ell} R_{ki}R_{jk}R_{\ell j}R_{i\ell} + \ldots\Big]\mathbb{E}\,|v_{ij}|^2.\end{aligned}$$

The same expansion holds for the resolvent of $H^{(\gamma-1)}$ but w_{ij} replaces v_{ij}. If the four moments match, then the first four (fully expanded) terms in

$$\mathbb{E}F\left(\frac{1}{N}\mathrm{Tr}\frac{1}{H_\gamma - z}\right) = \sum_{m=0}^{5}\frac{1}{N^{m/2}}\mathbb{E}A^{(m)}$$

coincide. The difference is given by the fifth (error) term, which is of order $N^{-5/2}$:

$$A^{(5)} = F'(\widehat{R})\Omega + F^{(5)}(\widehat{R}+\xi')(\widehat{R}^{(1)})^5 + \ldots, \qquad \Omega = -\frac{1}{N}\mathrm{Tr}(RV)^5 S.$$

It involves S, that is correlated with V, but Schwarz it out.

If the fourth moments match only up to $N^{-\delta}$, then

$$\mathbb{E}\,F\left(\frac{1}{N}\mathrm{Tr}\frac{1}{H_\gamma - z}\right) - \mathbb{E}\,F\left(\frac{1}{N}\mathrm{Tr}\frac{1}{H_{\gamma-1} - z}\right) \le CN^{-5/2} + CN^{-2-\delta}.$$

Sum up the telescopic sum with N^2 terms, we get

$$\mathbb{E}\,F\left(\frac{1}{N}\mathrm{Tr}\frac{1}{H^{(v)} - z}\right) - \mathbb{E}\,F\left(\frac{1}{N}\mathrm{Tr}\frac{1}{H^{(w)} - z}\right) \le CN^{-1/2} + CN^{-\delta}.$$

SUMMARY

- All results for general Wigner matrices, no explicit formula used
- Local semicircle law for the DOS on the optimal scale $\sim 1/N$
- Eigenvectors are fully delocalized away from the spectral edges.
- Level repulsion for k eigenvalues with optimal exponent.
- Universality of local eigenvalue statistics for general Wigner matrices with subexponentially decaying matrix elements.

OPEN: Universality for band matrices and random Schrödinger.