



DIPLOMARBEIT

A Derivation of The Nonlinear Boltzmann
Equation
From a Newtonian Hard Ball System

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Die Arbeit wurde bisher keiner anderen Prüfungsbehörde vorgelegt und auch noch nicht veröffentlicht.

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CONTENTS

1. Historical Introduction	2
2. Preliminaries	4
3. The BBGKY Hierarchy for Correlation Functions of n Hard Spheres	9
4. The Point Particle Configuration	14
5. The Boltzmann-Grad Limit	15
6. Propagation of Chaos	17
Appendix A.	19
Appendix B.	32
References	38

1. HISTORICAL INTRODUCTION

The Boltzmann equation is the backbone of the kinetic theory of dilute gases. But not only this, properly generalized, it is also a useful tool to study electron transport in solids and plasmas, as well as neutron transport in nuclear reactors, photon transport in superfluids, and radiative transfer in planetary and stellar atmospheres. The advantage of this equation is that one can handle a huge number of particles ($N \sim 10^{23}$), because it approximates a many-body dynamics by a one-particle equation. One problem of many-body dynamics, is the problem of recollisions. Recolliding particles are highly correlated and therefore mathematically difficult to grasp. If the gas is dilute, then recollisions are rare. Thus we will consider a limiting regime where the effective density goes to zero. This is the so-called low density limit or Boltzmann Grad limit.

We use a scaling where the particles have a diameter of the order $\mathcal{O}(\varepsilon)$. When we talk about a dilute gas, we will be talking about a particle number of order $\mathcal{O}(\varepsilon^{-2})$. The particles are of diameter εa , contained in a fixed box $\Pi \subset \mathbb{R}^3$ of volume one. In particular, the average distance to the nearest particle is of order $\varepsilon^{\frac{2}{3}}$, which is much bigger than the average size of the particles. Hence, the gas is dilute.

To prevent a lowering of the collision rate as we shrink the diameter εa of the particles, we have to increase the number of particles in Π by a factor ε^{-2} . This ensures that we have, in average, one collision per particle per unit time, i.e. if we follow the trace of a particle for a unit time interval, which is the tube $(\varepsilon a)^2 \pi \times [0, 1]$, we expect that the number of other particles in this tube, is of order one. As one takes the low density limit $\varepsilon \rightarrow 0$, the Boltzmann equation exhibits the Markovian character of the dynamics in a dilute gas.

The basic equation,

$$(1.1) \quad \frac{\partial}{\partial t} f_t(q, p) + p \cdot \nabla_q f_t(q, p) = N a^2 \int_{\mathbb{R}^3} dp_2 \int_{\{(p_2 - p_1) \cdot \hat{\omega} \geq 0\}} d\hat{\omega} \hat{\omega} \cdot (p_2 - p_1) \\ \times [f_t(q, p'_1) f_t(q, p') - f_t(q, p_1) f_t(q, p)],$$

was first established in 1872 by Ludwig Boltzmann for the time evolution of the one particle density function $f_t(q, p)$. The precise notations will be introduced in the following chapter. The Boltzmann equation is not a first principle equation, such as Newton's equations. On one hand, its phenomenological derivation is based on momentum and energy conservation, using the collision equations for hard balls. On the other hand, it has its foundations on the principle of the propagation of chaos. This principle postulates a certain statistical independence Ansatz, in which the incoming momenta of

colliding particles are independent. This Ansatz is certainly wrong in full generality because, due to recollisions, particles build up correlations as they evolve in time. However, if we use a dilute gas limiting regime it turns out to be correct. Because the formulation required mathematical subtleties it took more than 100 years until O. Lanford could give in 1975, a rigorous derivation of the Boltzmann equation using the BBGKY hierarchy.

Since then there has been a strong interest in this subject and therefore a considerable advance in research concerning the Boltzmann equation in the last 50 years, it is natural that there exist lots of publications about this topic, which only explain some aspects of the Boltzmann equation. This thesis will discuss the concept of the derivation of the Boltzmann equation, and will give the rigorous proof, following mainly O. Lanford's proof. However, it will establish the BBGKY hierarchy directly with probabilistic methods, as shown in a paper by H. Spohn, 2006 [4].

2. PRELIMINARIES

In this first section we will introduce notations, and we will explain the structure of the sets we are dealing with.

Let $\Pi \subseteq \mathbb{R}^3$ be the domain where the particles are confined. We assume Π to be of volume one, bounded, closed and connected, and its boundary $\partial\Pi$ to be smooth, to assure that the collisions of the particles with the boundary are well defined. Let $x_i = (q_i, p_i)$, $i = 1, \dots, N$, be the representation of the i -th hard sphere, with diameter a , in the phase space $\Pi \times \mathbb{R}^3$. Here $q_i \in \Pi$ describes the center of the i -th hard sphere, and $p_i \in \mathbb{R}^3$ is its incoming momentum. For now we will restrict ourself to at most N hard spheres, $N = 1, 2, \dots$, even though we will relax this condition later in the proof. Since we will need the n -tuple (x_1, \dots, x_n) frequently in the proof we abbreviate it by x .

The n -particle phase space, $n = 1, \dots, N$, is defined as a subset of $(\Pi \times \mathbb{R}^3)^n$ such that no particles overlap with each other or the boundary. I.e.

$$(2.1) \quad \Gamma(n) = \left\{ (x_1, \dots, x_n) \in (\Pi \times \mathbb{R}^3)^n \mid |q_i - q_j| \geq \frac{a}{2} \right. \\ \left. \text{for } i = 1, \dots, n \text{ and } \forall q \in \partial\Pi, \text{ and } |q_i - q_j| \geq a \text{ for } i \neq j = 1, \dots, n \right\}.$$

Collisions between two particles will be treated according to the following equations, which are derived from the momentum- and energy conservation:

$$(2.2) \quad p'_i = p_i - [(p_i - p_j) \cdot \hat{\omega}] \hat{\omega} \quad p'_j = p_j + [(p_i - p_j) \cdot \hat{\omega}] \hat{\omega}$$

$i \neq j = 1, \dots, n$. Here p_i denotes the incoming, and p'_i the outgoing momentum of the i -th particle, and $\hat{\omega}$ denotes the unit contact vector between two colliding particles i and j , $i < j$, such that $\hat{\omega}$ is pointing toward the j -th particle. Particle collisions with the boundary of Π are described by,

$$(2.3) \quad p' = p - 2n(q)[n(q) \cdot p],$$

where $n(q)$ is the unit outward-normal of $\partial\Pi$ at the the point of collision $q \in \partial\Pi$, p denotes the incoming, and p' denotes the outgoing momentum.

The equations (2.2) and (2.3), only describe collisions between two particles, respectively one particle colliding with the boundary. An equation which describes simultaneous collisions between three or more particles, as well as simultaneous collisions of at least two particles at the boundary, can not be derived purely from energy and momentum conservation since energy- and momentum conservation do not determine the outgoing momenta uniquely. Thus the N -particle dynamics is not well defined for these collision instants and we have to eliminate multiple collisions. Therefore we remove all phase space points $x = (x_1, \dots, x_n)$ which will lead to multiple collisions in course of time, considering both past and future. We also remove all phase points such that there are cluster points of collision instants with

the boundary, otherwise we would have infinitely many collision instants of a particle in a finite time span. Fortunately, these sets are of measure zero, confer to [2], Theorem 4.2.1. From this point, we can work with a set $\Gamma_n^* \subset \Gamma_n$, which is defined to exclude all phase space points mentioned above. A heuristic explanation for $\Gamma_n \setminus \Gamma_n^*$ being a zero measure set is the following. Let $x(t) = (q_1(t), \dots, p_n(t))$ be then the evolution of the phase point x at time t under the n -hard ball dynamics, and

$$M(t) = \bigcup_{i,j,k} \{x \in \Gamma_n : |q_i(t) - q_j(t)| = a, |q_i(t) - q_k(t)| = a\},$$

be the set of all phase space configurations which lead to triple collisions at time t . This set has codimension 2. If one builds the set

$$\bigcup_{t \in \mathbb{R}} M(t) = \Gamma_n \setminus \Gamma_n^*,$$

it still has codimension 1 and therefore is a zero measure set.

Now we define the flow of n interacting particles on Γ_n^* for any time $t \in \mathbb{R}$ by $T_t^{(n)}(x_1, \dots, x_n) = (x_1(t), \dots, x_n(t))$. It consists of the free flow and the collisions between the particles. Γ_n^* is invariant under $T_t^{(n)}$. If t_0 is a collision time, we have two different limits,

$$T_{t_0 \pm}^{(n)}(x_1, \dots, x_n) = \lim_{\varepsilon \rightarrow 0, \varepsilon > 0} T_{t_0 \pm \varepsilon}^{(n)}(x_1, \dots, x_n),$$

since we have to distinguish between incoming and outgoing momenta. Otherwise we have a unique limit

$$T_t^{(n)}(x_1, \dots, x_n) = \lim_{\varepsilon \rightarrow 0, \varepsilon > 0} T_{t \pm \varepsilon}^{(n)}(x_1, \dots, x_n).$$

We are prepared to define a collision history δ^x , of n particles. It describes the flow of n interacting particles backwards in time. We start at time t with n nonoverlapping particles at $(x_1(t), \dots, x_n(t)) = (x_1, \dots, x_n) = x$. These particles evolve backwards in time, colliding not only with each other, but also with additional particles which were not considered until the collision. These "new" particles extend the configuration until the collision history δ^x ends at time $t = 0$ with $n + m$ particles at $(x_1(0), \dots, x_{n+m}(0))$. Thus we adjoin fresh particles along the backward collision history.

If one follows the n particles in the physical forward time direction starting at time zero, this procedure corresponds to removing extra particles after they collide with one of the other particles still present at that time. To explain a collision history δ^x in detail, we need the following specifications:

- (i) a time $t > 0$ with a time t configuration $x = x(t)$.

- (ii) number $m, m = 0, 1, 2, \dots$, of particles adjoined. $n + m$ is the number of particles present in the time 0 configuration.
- (iii) times of adjoining
 $t_1, \dots, t_m, 0 < t_m < \dots < t_1 < t$.
- (iv) label of the particle at which the new hard sphere is adjoined,
 $j_1, j_1 = 1, \dots, n,$
 \vdots
 $j_m, j_m = 1, \dots, n + m - 1.$
- (v) contact vectors and momenta of particles adjoined
 $\hat{\omega}_1, \dots, \hat{\omega}_m \in S^2,$
 $\hat{p}_1, \dots, \hat{p}_m \in \mathbb{R}^3.$

Of course all these variables depend on a particular collision history δ^x . We stress that in case of a collision we use the incoming momentum to describe the location of the particles in the phase space.

A collision history evolves backwards in time, starting with n nonoverlapping particles at time t . $(x_1, \dots, x_n) = x$ evolves backwards, for the time span $(t - t_1)$, according to the n -body flow $T_t^{(n)}$. At the time t_1 a fresh particle is adjoined to the system touching one of the “old” particles, i.e. we introduce a new particle at time t_1 , described by $(q_{n+1}, p_{n+1}) = (q_{j_1} + a\hat{\omega}_1, p_{n+1})$, $i = 1, \dots, n$. It is understood that we allow only nonoverlapping particles. The index j_1 identifies the particle which collides with the first fresh particle. The fresh particle will have the momentum $\hat{p}_1 = p_{n+1}$. If $(p_{j_1} - \hat{p}_1) \cdot \hat{\omega}_1 \leq 0$, a new configuration is constructed according to the collision rule (2.2). If $(p_{j_1} - \hat{p}_1) \cdot \hat{\omega}_1 > 0$, the two particles move smoothly apart. The new configuration, $(x_1(t_1), \dots, x_{n+1}(t_1))$, evolves backward for the time span $(t_1 - t_2)$ according to $T_t^{(n+1)}$. At the time t_2 a fresh particle with momentum \hat{p}_2 is adjoined. If we iterate this construction, it will terminate at time zero with the initial configuration $(x_1(0), \dots, x_{n+m}(0))$.

The set of all collision histories for a given time t configuration $x = (x_1, \dots, x_n)$, and a given timespan $[0, t]$ is denoted by $\Lambda_a(x_1, \dots, x_n, [0, t])$. Here $\Lambda_a(x_1, \dots, x_n, [0, t])$ is to be seen as a subset of

$$\bigcup_{m=1}^{\infty} \bigcup_{j_1=1}^n \dots \bigcup_{j_m=1}^{n+m-1} \Delta_t^m \times (\mathbb{R}^3 \times S^2)^m,$$

where $\Delta_t^m \subset \mathbb{R}_+^m$ is the simplex $\{(t_1, \dots, t_m) \in \mathbb{R}_+^m \mid 0 < t_1 < \dots < t_m < t\}$. We remove overlapping particles, as well as configurations which will lead, in course of time, to multiple collisions and cluster points of collision instants, from this letter set. We use δ^x to denote the elements of $\Lambda_a(x_1, \dots, x_n, [0, t])$.

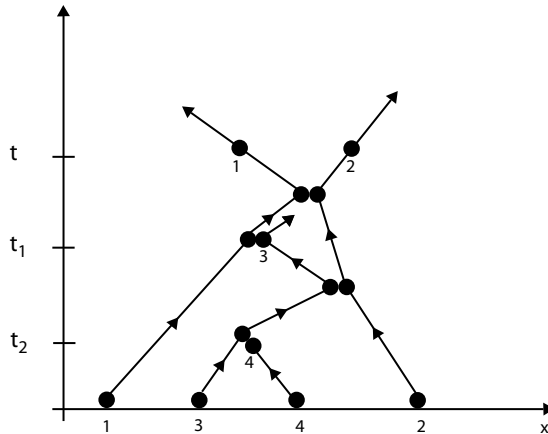


FIGURE 1. Collision history involving 4 hard spheres

Similarly, we define collision histories for point particles as

$$(2.4) \quad \Lambda_0(x_1, \dots, x_n, [0, t]) \subset \bigcup_{m \geq 0} \bigcup_{j_1=1}^n \bigcup_{j_m=1}^{n+m-1} \Delta_t^m \times (\mathbb{R}^3 \times S^2)^m,$$

where we remove only the multiple collision configurations. In particular, $\Lambda_0(x_1, \dots, x_n, [0, t])$ is a subset of full measure of the triple union on the right hand side of (2.4). There will be no problems concerning the overlap of particles. We stress the $(x, [0, t])$ dependence of Λ_a and Λ_0 .

We can keep the notations of a hard ball collision history in order to explain a point particle collision history. A collision history of point particles starts at time t with a configuration $x = (x_1, \dots, x_n)$. This phase point evolves backwards in time for a time span $t - t_1$, according to the free flow

$$(2.5) \quad x(s) = (q_1(s) - (s - t_1)p_1(s), p_1(s), \dots, q_n(s) - (s - t_1)p_n(s), p_n(s)),$$

for $s \in [t_1, t]$. At time t_1 and at point q_{j_1} a new particle is adjoined with the momentum p_{n+1} , i.e. we set $q_{n+1}(t_1) := q_{j_1}(t_1)$ and $p_{n+1}(t_1) := p_{n+1}$. If $(p_{j_1}(t_1) - p_{n+1}) \cdot \hat{\omega} \leq 0$, then a new configuration (x_1, \dots, x_{n+1}) at time t_{1-} is constructed according to the collision rule (2.2). It evolves backward in time, for the time span $t_1 - t_2$ according to the free flow of $n + 1$ particles, given similarly to (2.5). At $q_{j_2}(t_2)$ a new particle is adjoined with the momentum p_{n+2} at time t_2 , etc. Iteration will lead to the initial configuration $(x_1(0), \dots, x_{n+m}(0))$. Note, the point particle dynamics has no recollisions the only collisions occur when fresh particles are adjoined.

Clearly $\Lambda_0(x_1, \dots, x_n, [0, t]) \supset \Lambda_a(x_1, \dots, x_n, [0, t])$. We equip the space $\Lambda_0(x_1, \dots, x_n, [0, t])$ with the measure $d\delta$, which is an abbreviation for the product of three measures,

- (i) the counting measure for $m, j_i, i = 1, \dots, m$,
- (ii) the Lebesgue measure for $t_i, \hat{p}_i, i = 1, \dots, m$
- (iii) and the surface measure on S^2 for $\hat{\omega}_i, i = 1, \dots, m$.

When we are dealing with hard spheres with diameter a we will consider the restriction of this measure to $\Lambda_a(x, [0, t])$.

Each collision history will be equipped with a weight, given by the weight function

$$(2.6) \quad W_a(x_1, \dots, x_n, \delta) = \prod_{k=1}^m \{a^2 \hat{\omega}_k \cdot (\hat{p}_k - p_{j_k}(t_k, x_1, \dots, x_n, \delta))\},$$

in case of the hard ball system, and

$$(2.7) \quad W_0(x_1, \dots, x_n, \delta) = \prod_{k=1}^m \{a^2 \hat{\omega}_k \cdot (\hat{p}_k - p_{j_k}(t_k, x_1, \dots, x_n, \delta))\},$$

in case of the point particle system. We stress that $p_{j_k}(t_k, x_1, \dots, x_n, \delta)$ denotes in (2.6) the hard ball dynamics, whereas in (2.7) it denotes the point particle dynamics. To avoid confusion we will indicate if we talk about hard ball dynamics, or about point particle dynamics.

In both cases, we outline the dependence of the j -th particle in the phase space, $x_j(s, x_1, \dots, x_n, \delta) = (q_j(s, x_1, \dots, x_n, \delta), p_j(s, x_1, \dots, x_n, \delta))$, $0 \leq s \leq t$, on the particular collision history δ and the configuration (x_1, \dots, x_n) at the final time t . We evaluate this point at time s . For brevity we will use from now on $x_j(s, x, \delta)$.

3. THE BBGKY HIERARCHY FOR CORRELATION FUNCTIONS OF n HARD SPHERES

This section will give an idea how to derive evolution equations for the correlation functions for n particles, directly by using probabilistic arguments, as done in a paper by Spohn in [4]. This system of evolution equations is called the BBGKY hierarchy. It is named after Born, Bogoliubov, Green, Kirkwood and Yvan, who established this system. The rigorous proofs will follow in Appendix A.

We start with a

Definition 1. A function $f_n : \Gamma_n \rightarrow \mathbb{R}$ is said to be continuous along trajectories of $T_t^{(n)}$ on Γ_n^* , if $\forall (x_1, \dots, x_n) \in \Gamma_n^*$

$$\lim_{t \rightarrow 0} f_n(T_t^{(n)}(x_1, \dots, x_n)) = f_n(x_1, \dots, x_n),$$

$t \in \mathbb{R}$.

I.e. in case of a collision between the i -th and the j -th particle ($i < j$), with the contact vector $\hat{\omega} \in S^2$, we have for all $(x_1, \dots, (q_i, p_i), \dots, (q_i + a\hat{\omega}, p_j), \dots, x_n) \in \Gamma_n^*$,

$$\begin{aligned} f_n(x_1, \dots, (q_i, p_i), \dots, (q_i + a\hat{\omega}, p_j), \dots, x_n) \\ = f_n(x_1, \dots, (q_i, p'_i), \dots, (q_i + a\hat{\omega}, p'_j), \dots, x_n) \end{aligned}$$

and similarly for collisions with the boundary of Π , we have for all $q \in \partial\Pi$ and all $(x_1, \dots, (q_i, p_i), \dots, (q + \frac{a}{2}\hat{\omega}, p_j), \dots, x_n) \in \Gamma_n^*$,

$$f_n(x_1, \dots, (q_i, p_i), \dots, (q + \frac{a}{2}\hat{\omega}, p_j), \dots, x_n) = f_n(x_1, \dots, (q_i, p'_i), (q + \frac{a}{2}\hat{\omega}, p'_j), \dots, x_n).$$

The momenta p_j and p'_j are understood as in (2.2) respectively (2.3).

Additionally, we define the following sets:

$$(3.1) \quad \Gamma_{N-n}(x_1, \dots, x_n) = \begin{cases} \{(x_{n+1}, \dots, x_N) \in \Gamma_{N-n} \mid |q_i - q_j| \geq a, \\ \text{for } i = 1, \dots, n \text{ and } j = n+1, \dots, N\} & \text{for } (x_1, \dots, x_n) \in \Gamma_n^* \\ \emptyset, & \text{otherwise.} \end{cases}$$

$\Gamma_{N-n}(x_1, \dots, x_N)$ represents the set of all particles in the phase space Γ_{N-n} , such that there is no overlap between these latter particles, and the particles in Γ_n^* . I.e. this set describes the configurations of the adjoinable particles.

Let

$$\Omega_j(x_1, \dots, x_n, p_{n+1}) = \begin{cases} \{\hat{\omega} \in S^2 \mid ((x_1, \dots, x_n, (q_j + a\hat{\omega}, p_{n+1})) \in \Gamma_{n+1}^* \subset S^2\}, \\ \text{for } j = 1, \dots, n, (x_1, \dots, x_n) \in \Gamma_n^*, \text{ and } p_{n+1} \in \mathbb{R} \\ \emptyset, & \text{otherwise.} \end{cases}$$

This is the set of contact vectors $\hat{\omega}$ in S^2 , such that a fresh particle can be adjoined to the j -th particle of $(x_1, \dots, x_n) \in \Gamma_n^*$, with the contact vector $\hat{\omega}$. However, it has the restrictions that this new particle does not overlap with any of the other particles, and we do not have "bad" collisions, i.e. $(x_1, \dots, x_n, (q_j + a\hat{\omega}, p_{n+1})) \in \Gamma_{n+1}^*$.

Ω_j can be split into two subsets, $\Omega_j = \Omega_{j+} \cup \Omega_{j-}$, where

$$\Omega_{j\pm}(x_1, \dots, x_n, p_{n+1}) := \{\hat{\omega} \in \Omega_j(x_1, \dots, x_n, p_{n+1}) \mid \hat{\omega} \cdot (p_{n+1} - p_j) \gtrless 0 (< 0)\}.$$

We introduce a probability measure P on Γ_N^* , describing the initial state $t = 0$, with the following properties:

(i) P has a density,

$$P(dx_1, \dots, dx_N) = f_N(x_1, \dots, x_N) dx_1 \dots dx_N$$

(ii) f_N is symmetric in its argument.

(iii) With $h_\beta(p) = \left(\frac{\beta}{2\pi}\right)^{\frac{3}{2}} \exp\left(-\beta\frac{p^2}{2}\right)$, $\beta > 0$, the *normalized Maxwellian*, f_N is bounded on Γ_N^* by,

$$f_N(x_1, \dots, x_N) \leq c \prod_{j=1}^N h_\beta(p_j)$$

for some $c, \beta > 0$.

(iv) f_N is continuous along trajectories of $T_t^{(N)}$ on Γ_N^* (cf. Definition 1), and zero on $\Gamma_N \setminus \Gamma_N^*$.

(v) The time evolved measure P_t has a density $f_N(t)$ given by $f_N(t) = 0$ on $\Gamma_N \setminus \Gamma_N^*$ and $f_N(x_1, \dots, x_N, t) = f_N(T_{-t}^{(N)}(x_1, \dots, x_N))$ for $(x_1, \dots, x_N) \in \Gamma_N^*$.

The canonical equilibrium measure on $(\Pi \times \mathbb{R}^3)^N$ will be denoted by

$$(3.2) \quad P_{eq}(dx_1 \dots dx_N) = \prod_{j=1}^N h_\beta(p_j) dx_j,$$

where $dx_j = dq_j dp_j$.

It has a density,

$$P_{eq}(dx_1, \dots, dx_N) = f_{eq,N}(x_1, \dots, x_N) dx_1 \dots dx_N$$

Since we want P_{eq} to be a probability measure, we scale it such that $|\Pi| = 1$.

From this point on we set, without loss of generality, $t > 0$. The correlation functions are defined by,

$$(3.3) \quad \rho_n(x_1, \dots, x_n, t) = N \dots (N-n+1) \int_{\Gamma_{N-n}(x_1, \dots, x_n)} dx_{n+1} \dots dx_N f_N(x_1, \dots, x_N, t).$$

By the definition of Γ_{N-n} one has $\rho_n(t) = 0$ on $\Gamma_n \setminus \Gamma_n^*$.

Let $\Delta \subset \Gamma_n^*$ be a Borel set. If we integrate $\rho_n(x_1, \dots, x_n, t)$ over Δ we get, by symmetry and hard core exclusion,

$$(3.4) \quad \int_{\Delta} dx_1 \dots dx_n \rho_n(x_1, \dots, x_n, t) = N \dots (N - n + 1) P\{(x_1(t), \dots, x_n(t)) \in \Delta\},$$

From this identity we get the following proposition,

Proposition 1. *If $\Delta \subset \Gamma_n^*$ is a Borel set, and $n = 1, \dots, N$, we have,*

$$\begin{aligned} \int_{\Delta} dx_1 \dots dx_n \rho_n(x_1, \dots, x_n, t) &= \int_{\Delta} dx_1 \dots dx_n \rho_n(T_{-t}^{(n)}(x_1, \dots, x_n)) \\ &+ \sum_{m=1}^{\infty} N \dots (N - n + 1) P\{(x_1(t_m+), \dots, x_n(t_m+)) \in T_{t_m-t+}(\Delta), t_m \leq t\} \\ &- \sum_{m=1}^{\infty} N \dots (N - n + 1) P\{(x_1(t_m-), \dots, x_n(t_m-)) \in T_{t_m-t-}(\Delta), t_m \leq t\}. \end{aligned}$$

Here t_m is the collision time of the m -th fresh particle (we still have a backward time evolution).

Proof. For the proof cf. Appendix A. □

We want to convert the latter equation into an equation written in terms of correlation functions. In order to do this, ρ_n has to be continuous along trajectories, but this clearly follows from our assumptions on P . We define the *collision* operator of the correlation function by,

$$(3.5) \quad (C_{j,n+1}\rho_n(s))(x_1, \dots, x_n) = a^2 \int dp_{n+1} \int_{\Omega_j(x_1, \dots, x_n, p_{n+1})} d\hat{\omega} \hat{\omega} \cdot (p_{n+1} - p_j) \rho_{n+1}(x_1, \dots, x_n, (q_j + a\hat{\omega}, p_{n+1}), s),$$

and obtain the following proposition, which states the BBGKY-hierarchy in the integrated form.

Proposition 2. *For Δ and n as above we have,*

$$(3.6) \quad \int_{\Delta} dx_1 \dots dx_n \rho_n(x_1, \dots, x_n, t) = \int_{\Delta} dx_1 \dots dx_n \rho_n(T_{-t}^{(n)}(x_1, \dots, x_n)) \\ + \sum_{j=1}^n \int_0^t \int_{\Delta} ds dx_1 \dots dx_n [C_{j,n+1}\rho_n(s)](T_{-t+s}^{(n)}(x_1, \dots, x_n)).$$

Proof. Cf. Appendix A □

Since $\rho_n = 0$ for $n > N$ we obtain after iterating this equation,

$$(3.7) \quad \int_{\Delta} dx_1 \dots dx_n \rho_n(x_1, \dots, x_n, t) = \\ \sum_{m=0}^{N-n} \sum_{j_1=0}^n \dots \sum_{j_m=0}^{n+m-1} \int_0^t \dots \int_0^{t_m} \int_{\Delta} dt_1 \dots dt_m dx_1 \dots dx_n \\ \times (C_{j_1, n+1} \dots (C_{j_m, n+m} (\rho_{n+m} \circ T_{t_m}^{(n+m)})) \circ T_{t_m - t_{m-1} +}^{(n+m-1)} \dots) (T_{t_1 - t}^{(n)}(x_1, \dots, x_n)).$$

To prove the existence of the Boltzmann-Grad limit, i.e. the low density limit as the diameter of the particles tend to zero, it is convenient to rewrite this equation with the notation of the collision histories. We have the following proposition.

Proposition 3. *For every Borel set $\Delta \subset \Gamma_n^*$, and every $n \in \mathbb{N}$,*

$$(3.8) \quad \int_{\Delta} dx_1 \dots dx_n \rho_n(x_1, \dots, x_n, t) = \\ \int_{\Delta} dx_1 \dots dx_n \int_{\Lambda_a(x_1, \dots, x_n; [0, t])} d\delta \\ \times W_a(x_1, \dots, x_n, \delta) \rho_{n+m(\delta)}(x_1(0, x, \delta), \dots, x_{n+m(\delta)}(0, x, \delta)).$$

Proof. Cf. Appendix A. □

This means we have,

$$(3.9) \quad \rho_n(x_1, \dots, x_n, t) = \int_{\Lambda_a(x_1, \dots, x_n; [0, t])} d\delta \\ \times W_a(x_1, \dots, x_n, \delta) \rho_{n+m(\delta)}(x_1(0, x, \delta), \dots, x_{n+m(\delta)}(0, x, \delta)),$$

$dx_1 \dots dx_n$ a.s..

Since we want to look at particles with a diameter tending to zero, we substitute a by εa . We indicate, with the superscript ε , that the correlation functions ρ_n^ε , as well as the phase space configurations $x_j^\varepsilon(t, x, \delta)$ of the hard core dynamics with radius εa , are dependent on ε . We get,

$$(3.10) \quad \rho_n^\varepsilon(x_1, \dots, x_n, t) = \int_{\Lambda_{\varepsilon a}(x_1, \dots, x_n; [0, t])} d\delta \\ \times W_{\varepsilon a}(x_1, \dots, x_n, \delta) \rho_{n+m}^\varepsilon(x_1^\varepsilon(0, x, \delta), \dots, x_{n+m}^\varepsilon(0, x, \delta)).$$

In order to keep the number of collisions per particle of order 1, we need to increase the number of particles N as ε^{-2} , while we reduce the size of the particles. Thus we expect the correlation function $\rho_n^\varepsilon(t)$ to be of order ε^{-2n} . This leads us to the definition of the rescaled correlation functions,

$$(3.11) \quad r_n^\varepsilon(t) := \varepsilon^{2n} \rho_n^\varepsilon(t)$$

which are expected to be of order 1. I.e. we have

$$(3.12) \quad r_n^\varepsilon(x_1, \dots, x_n, t) = \int_{\Lambda_{\varepsilon a}(x_1, \dots, x_n; [0, t])} d\delta \\ \times W_a^\varepsilon(x_1, \dots, x_n, \delta) r_{n+m(\delta)}^\varepsilon(x_1^\varepsilon(x_1, \dots, x_n, \delta), \dots, x_{n+m(\delta)}^\varepsilon(x_1, \dots, x_n, \delta)),$$

$n = 0, 1, \dots$ where the weight function W_a^ε is given by

$$(3.13) \quad W_a^\varepsilon(x_1, \dots, x_n, \delta) = \prod_{k=1}^m \{a^2 \hat{\omega}_k \cdot (\hat{p}_k - p_{j_k}^\varepsilon(t_k, x, \delta))\}.$$

Note that $W_a^\varepsilon(x_1, \dots, x_n, \delta)$ is ε -dependent only through the $p_{j_k}^\varepsilon(t_k, x, \delta)$.

To take the limit $\varepsilon \rightarrow 0$, in (3.12), we use the dominated convergence theorem, on $\Lambda_0(x, [0, t])$. Note that $\Lambda_{\varepsilon a}(x, [0, t]) \subset \Lambda_0(x, [0, t])$ and $\Lambda_{\varepsilon a}(x, [0, t]) \nearrow \Lambda_0(x, [0, t])$ as $\varepsilon \rightarrow 0$. Therefore we need an integrable bound, independent of ε , of the integrand. We recall the normalized Maxwellian, $h_\beta(p) := (\frac{\beta}{2\pi})^{\frac{3}{2}} \exp[-\frac{\beta p^2}{2}]$, as a preparation of the following proposition.

Proposition 4. *Let $r_n^\varepsilon(x, t)$ be defined as in (3.11), and let $r_n^\varepsilon(x, 0)$ satisfy the Maxwellian bound,*

$$(3.14) \quad |r_n^\varepsilon(x, 0)| \leq M \prod_{j=1}^n z h_\beta(p_j),$$

for some $M, z, \beta > 0$ independently of ε .

Then there exist a $t_0(z, \beta)$ such that for $0 \leq t < t_0(z, \beta)$,

$$(3.15) \quad |r_n^\varepsilon(x, t)| \leq M \prod_{j=1}^n z(t) h_{\beta(t)}(p_j),$$

for some functions $z(t)$ and $\beta(t)$, independently of ε . We will define $z(t)$ and $\beta(t)$ within the proof.

Moreover we have the bound $0.2 \frac{\sqrt{m\beta}}{\sqrt{3\pi z}} < t_0(z, \beta) < 0.4 \frac{\sqrt{m\beta}}{\sqrt{3\pi z}}$ for the threshold time $t_0(z, \beta)$.

Proof. Cf. Appendix B. □

4. THE POINT PARTICLE CONFIGURATION

As a goal of this paper we want to describe an increasing number of particles, with a decreasing diameter. The limiting dynamics describes therefore point particles. In this chapter we consider the set of point particle collision histories, $\Lambda_0(x, [0, t])$, cf. (2.4), with the corresponding weight function,

$$(4.1) \quad W_0(x_1, \dots, x_n, \delta) = \prod_{k=1}^m \{a^2 \hat{\omega}_k \cdot (\hat{p}_k - p_{j_k}(t_k, x_1, \dots, x_n, \delta))\}.$$

Note that $W_0(x, \delta)$ is defined as in (2.7), and the p_{j_k} are obtained by the point particle dynamics.

We use now the unrestricted product measure $d\delta$ on $\Lambda_0(x, [0, t])$. We introduce a family of functions $\{r_n | n = 1, 2, \dots\}$ such that $r_n : (\Pi \times \mathbb{R}^3)^n \rightarrow \mathbb{R}$ is continuous, and $|r_n(x)| \leq M \prod_{j=1}^n z h_\beta(p_j)$. From this point on we will set $r_n(x, 0) = r_n(x)$.

For $t \in [0, t_0(z, \beta))$ we define

$$(4.2) \quad r_n(x, t) = \int_{\Lambda_0(x, [0, t])} d\delta W_0(x_1, \dots, x_n, \delta) \\ \times r_{n+m(\delta)}(x_1(0, x, \delta), \dots, x_{n+m(\delta)}(0, x, \delta)),$$

where we recall, that $x_i(0, x, \delta)$ notes the position of the i -th particle at time 0, starting with the configuration $x = (x_1, \dots, x_n)$ at the final time t and evolving backwards in time by the collision history δ^x , following the point particle dynamics.

Proposition 5. *Let $r_n^\varepsilon(x, t)$ be defined as in (4.2), and $r_n^\varepsilon(x, 0)$ satisfies the Maxwellian bound,*

$$(4.3) \quad |r_n^\varepsilon(x, 0)| \leq M \prod_{j=1}^n z h_\beta(p_j),$$

for some $M, z, \beta > 0$ independently of ε .

Then there exist a $t_0(z, \beta)$ such that for $0 \leq t < t_0(z, \beta)$,

$$(4.4) \quad |r_n^\varepsilon(x, t)| \leq M \prod_{j=1}^n z(t) h_{\beta(t)}(p_j),$$

for some functions $z(t)$ and $\beta(t)$, independently of ε . We will define $z(t)$ and $\beta(t)$ within the proof.

Moreover we have the bound $0.2 \frac{\sqrt{m\beta}}{\sqrt{3\pi z}} < t_0(z, \beta) < 0.4 \frac{\sqrt{m\beta}}{\sqrt{3\pi z}}$ for the threshold time $t_0(z, \beta)$.

Proof. We will prove this proposition together with Proposition 4 in Appendix B. \square

5. THE BOLTZMANN-GRAD LIMIT

In order to take the Boltzmann-Grad limit, it is necessary to further restrict $\Lambda_{\varepsilon a}(x; [0, t])$. We define $\tilde{\Lambda}_{\varepsilon a}(x; [0, t]) \subset \Lambda_{\varepsilon a}(x; [0, t])$ such that for all $\delta \in \tilde{\Lambda}_{\varepsilon a}(x; [0, t])$ there are no other collisions in the hard sphere flow, except those when a fresh hard sphere is adjoined.

This is not a strong restriction, since $\Lambda_{\varepsilon a}(x; [0, t]) \setminus \tilde{\Lambda}_{\varepsilon a}(x; [0, t])$ is a set of vanishing measure as $\varepsilon \rightarrow 0$ with respect to $r_n^\varepsilon(0, x, \delta)$. To see this we first recall that we require the Maxwellian bound $|r_n^\varepsilon(x, 0)| \leq M \prod_{j=1}^n z h_\beta(p_j)$. Additionally we have a weight function W_a^ε which does not blow up as $\varepsilon \rightarrow 0$. Hence, the claim follows from $\tilde{\Lambda}_{\varepsilon a}(x; [0, t]) \nearrow \Lambda_0(x, [0, t])$ as $\varepsilon \rightarrow 0$. The reason for this latter convergence is that we have a fixed number of particles $n = 1, 2, \dots$ with a decreasing diameter, i.e one has to focus the particles better and better to obtain a collision within $[0, t_0(\beta, z)]$, as $\varepsilon \rightarrow 0$. Thus the replacement of $\Lambda_{\varepsilon a}(x; [0, t])$ by $\tilde{\Lambda}_{\varepsilon a}(x; [0, t])$ in (3.12) gives only a marginal error. Note that for $\delta \in \tilde{\Lambda}_{\varepsilon a}(x; [0, t])$, the weight functions W_a and W_0 , cf. (2.6) and (2.7), are exactly equal.

For the same reason we define

$$\Gamma_n = (\Pi \times \mathbb{R}^3)^n, \quad \Gamma_n(0) = \{(x_1, \dots, x_n) \in \Gamma_n \mid q_i \neq q_j \text{ for } i \neq j = 1, \dots, n\}$$

and

$$\Gamma_n(t) = \{x \in \Gamma_n \mid q_i - p_{is} \neq q_j - p_{js} \text{ for all } 0 \leq s \leq t, i \neq j = 1, \dots, n\}.$$

These sets make sure that each particle, of the point particle phase space configuration $(x_1, \dots, x_n) \in \Gamma_n$, starts at time 0, at a distinguished location in Π , and that there is no collision in the time interval $[0, t]$, except those where a fresh particle is adjoined.

For $\delta \in \tilde{\Lambda}_{\varepsilon a}(x; [0, t])$, the weight $W_a^\varepsilon(x, \delta)$, defined in (3.13) is independent of ε , and the time zero configuration, $(x_1^\varepsilon(0), \dots, x_{n+m}^\varepsilon(0))$, has a simple ε dependence. The momenta $p_j(0)$, $j = 1, \dots, n + m$, are independent of ε and the positions $q_1^\varepsilon(0), \dots, q_{n+m(\delta)}^\varepsilon(0)$, obtained by the point particle dynamics, differ only of the order ε from their limit value. Hence, to show that $r_{n+m(\delta)}^\varepsilon(x_1^\varepsilon(0, x, \delta), \dots, x_{n+m(\delta)}^\varepsilon(0, x, \delta))$ has a limit for $\varepsilon \rightarrow 0$, it suffices to require that $r_{n+m(\delta)}^\varepsilon$ has a uniform limit on a compact set. In the appendix we will show the required regularity properties. All these preparations explain the conditions of the following theorem, which is in fact the main technical result.

Theorem 1. *Let $0 \leq t < t_0(z, \beta)$, and let the family of rescaled correlation functions $\{r_n^\varepsilon \mid n = 1, 2, \dots\}$, of the hard ball dynamics at time 0, satisfy the Maxwellian bound (3.15). Suppose there exists a family of continuous functions $\{r_n \mid n = 1, 2, \dots\}$, where $r_n : \Gamma_n \rightarrow \mathbb{R}$, such that for every $n = 1, 2, \dots$*

and every compact set $K \subset \Gamma_n(0)$,

$$(5.1) \quad \lim_{\varepsilon \rightarrow 0} \operatorname{ess-sup}_{(x_1, \dots, x_n) \in K} |r_n^\varepsilon(x_1, \dots, x_n) - r_n(x_1, \dots, x_n)| = 0.$$

Then,

$$\lim_{\varepsilon \rightarrow 0} \operatorname{ess-sup}_{(x_1, \dots, x_n) \in K} |r_n^\varepsilon(x_1, \dots, x_n, t) - r_n(x_1, \dots, x_n, t)| = 0.$$

for every $n = 1, 2, \dots$ and every compact $K \subset \Gamma_n(t)$.

Proof. We consider the families of functions $\{r_n^\varepsilon | n = 1, 2, \dots\}$ and $\{r_n | n = 1, 2, \dots\}$ on $\Lambda_0(x, [0, t])$, and recall that $\tilde{\Lambda}_{\varepsilon a}(x; [0, t]) \nearrow \Lambda_0(x, [0, t])$, as $\varepsilon \rightarrow 0$. Thus $\Lambda_0(x, [0, t]) \setminus \tilde{\Lambda}_{\varepsilon a}(x, [0, t])$ is a set of order ε , and vanishes as $\varepsilon \rightarrow 0$. Therefore

$$(5.2) \quad |r_n^\varepsilon(x, t) - r_n(x, t)| \\ = \left| \int_{\Lambda_0(x, [0, t])} d\delta \chi_{\tilde{\Lambda}_{\varepsilon a}(x, [0, t])} W_{\varepsilon a}(x, \delta) r_{n+m(\delta)}^\varepsilon(x_1^\varepsilon(0, x, \delta), \dots, x_{n+m(\delta)}^\varepsilon(0, x, \delta)) + \mathcal{O}(\varepsilon) \right. \\ \left. - \int_{\Lambda_0(x, [0, t])} d\delta W_0(x, \delta) r_{n+m(\delta)}(x_1(0, x, \delta), \dots, x_{n+m(\delta)}(0, x, \delta)) \right| \\ \leq \int_{\Lambda_0(x, [0, t])} d\delta \left| \chi_{\tilde{\Lambda}_{\varepsilon a}(x, [0, t])} W_{\varepsilon a}(x, \delta) r_{n+m(\delta)}^\varepsilon(x_1^\varepsilon(0, x, \delta), \dots, x_{n+m(\delta)}^\varepsilon(0, x, \delta)) \right. \\ \left. + \mathcal{O}(\varepsilon) - W_0(x, \delta) r_{n+m(\delta)}(x_1(0, x, \delta), \dots, x_{n+m(\delta)}(0, x, \delta)) \right|$$

We conclude the prove by using dominated convergence and (5.1). \square

This Theorem states that the correlation functions, properly scaled, do have a limit in a dilute gas as $\varepsilon \rightarrow 0$. The convergence is uniform on compact sets away from the exceptional sets $\Gamma_n \setminus \Gamma_n(t)$. Unfortunately these exceptional sets grow in course of time (note that $\Gamma_n(t_2) \subset \Gamma_n(t_1)$ for $t_1 < t_2 \in [0, t_0(z, \beta)]$).

Since two particles are highly correlated after a collision, we find the trace of this correlation still in the $r_n^\varepsilon(t)$. However, in the limit as ε tends to zero, all these correlations are concentrated in the hypersurfaces $\Gamma_n \setminus \Gamma_n(t)$.

These hypersurfaces, however, have measure zero. To see this, let

$$H(s) = \bigcup_{0 \leq i, j \leq n} \{x \in \Gamma_n : q_i - p_i s = q_j - p_j s, s \in [0, t]\},$$

be the set of all point particle phase space configurations which lead to a collision at time s . This set has codimension 3. If one builds the set

$$\bigcup_{t \in \mathbb{R}} H(t) = \Gamma_n \setminus \Gamma_n(t),$$

it has codimension 2. Hence, $\Gamma_n \setminus \Gamma_n(t)$ is a measure zero set.

6. PROPAGATION OF CHAOS

Let $r_n(t)$ be the limit correlation function, defined as in (4.2). These functions are solutions to a system of coupled linear equations called the *Boltzmann hierarchy*, which we will define below.

We define, for a continuous $f : (\Pi \times \mathbb{R}^3)^{n+1} \rightarrow \mathbb{R}$ with the Maxwellian bound as in (3.15), the collision operator,

$$(6.1) \quad (\tilde{C}_{j,n+1}f)(x_1, \dots, x_n) \\ = \sum_{j=1}^n \int dp_{n+1} \int_{\{\hat{\omega} \cdot (p_{n+1} - p_j) \geq 0\}} d\hat{\omega} a^2 \hat{\omega} \cdot (p_{n+1} - p_j) \\ \times [f(x_1, \dots, x_{j-1}, q_j, p'_j, \dots, q_j, p'_{n+1}) - f(x_1, \dots, x_{j-1}, q_j, p_j, \dots, q_j, p_{n+1})]$$

with p'_j and p'_{n+1} as in (2.2), and the unitary operator,

$$(6.2) \quad (S_n(t)f)(q_1, \dots, p_n) = f(q_1 - p_1 t, p_1, \dots, q_n - p_n t, p_n),$$

induced by the free flow.

Note that $(\tilde{C}_{j,n+1}f)(x_1, \dots, x_n)$ maps functions on $(\Pi \times \mathbb{R}^3)^{n+1}$ to functions on $(\Pi \times \mathbb{R}^3)^n$. Furthermore note that $(\tilde{C}_{j,n+1}f)(x_1, \dots, x_n)$ is actually the same as $(C_{j,n+1}\rho_{n+1})(x_1, \dots, x_n)$, cf. (3.5), in case of the point particle dynamics. In this case we have $\Omega_j(x, p_{n+1}) = S^2$. We use the sets $\{\hat{\omega} \cdot (p_{n+1} - p_j) \geq 0\}$ and $\{\hat{\omega} \cdot (p_{n+1} - p_j) < 0\}$ as a partition of S^2 . The second set is equal to $\{\hat{\omega} \cdot (p'_{n+1} - p'_j) \geq 0\}$. The change of p'_{n+1} and p'_j to p_{n+1} and p_j respectively, is due to energy conservation an isometry. Moreover we have a gain and a loss term of (3.5), which are

$$-a^2 \int dp_{n+1} \int_{\{\hat{\omega} \cdot (p_{n+1} - p_j) \geq 0\}} d\hat{\omega} \hat{\omega} \cdot (p_{n+1} - p_j) f(x_1, \dots, x_{j-1}, q_j, p_j, \dots, q_j, p_{n+1})$$

and

$$a^2 \int dp_{n+1} \int_{\{\hat{\omega} \cdot (p_{n+1} - p_j) \geq 0\}} d\hat{\omega} \hat{\omega} \cdot (p_{n+1} - p_j) f(x_1, \dots, x_{j-1}, q_j, p'_j, \dots, q_j, p'_{n+1})$$

respectively. Piecing all this together gives us (6.1).

Since we imposed continuity and the bound on f , the collision operator is well defined. Using similar arguments as in the derivation of the BBGKY hierarchy for hard spheres (cf. Section 2), we can show that $\{r_n(t) | n = 1, 2, \dots\}$ satisfy the integral equations

$$(6.3) \quad r_n(t) = S_n(t)r_n + \int_0^t ds S_n(t-s) \tilde{C}_{n,n+1} r_{n+1}(s),$$

for $n = 1, 2, \dots$. The equations (6.3) are called the *Boltzmann hierarchy*.

The Boltzmann equation has a property known as the *propagation of chaos*. This property postulates that if the initial correlation functions satisfy

$$(6.4) \quad r_n(x_1, \dots, x_n) = \prod_{j=1}^n f(q_j, p_j),$$

then this factorization property will be preserved in the course of time. I.e. the solution of (6.3) is of the form

$$(6.5) \quad r_n(x_1, \dots, x_n, t) = \prod_{j=1}^n f_t(q_j, p_j),$$

for some f_t . It turns out that if (6.5) is known, then f_t is a solution of the non-linear Boltzmann equation,

$$(6.6) \quad f_t = S_1(t)f + \int_0^t ds S_1(t-s) \tilde{C}_{1,2} f_s \times f_s,$$

with initial datum f . Note that (6.6) is the integral form of the Boltzmann equation (1.1)

The factorization of (6.5) is verified by inserting the initial factorization into (6.3), using the Maxwellian bound and the restriction to short times.

Together with Theorem 1 we have proven,

Corollary 1. *In addition to the assumptions of Theorem 1 let $r_n(x_1, \dots, x_n)$ be factorized, cf. (6.4). Then, for any $K \subset \Gamma_n \setminus \Gamma_n(t)$,*

$$\lim_{\varepsilon \rightarrow 0} \operatorname{ess-sup}_{(x_1, \dots, x_n) \in K} r_n^\varepsilon(x_1, \dots, x_n, t) = \prod_{j=1}^n f_t(q_j, p_j)$$

and $t \rightarrow f_t$ is the unique solution of the Boltzmann equation with initial condition f , for $t \in [0, t_0(z, \beta)]$.

This following section provides the rigorous proofs, which are needed to derive the Boltzmann equation. The proofs are going to be quite technical, however it is useful to see them in all detail, because it helps to get a deeper understanding for the subject.

APPENDIX A

In Section A we will prove the derivation of the BBGKY hierarchy. We closely follow [4].

We consider the hard sphere dynamics of N particles with diameter a . Let $\{t_m \geq 0 \mid m = 1, 2, \dots\}$, be the collision times between the N spheres (collisions with the wall are not counted).

Proposition 1

If $\Delta \subset \Gamma_n^*$ is a Borel set, and $n = 1, \dots, N$ we have,

$$\begin{aligned}
 (A.1) \quad & \int_{\Delta} dx_1 \dots dx_n \rho_n(x_1, \dots, x_n, t) = \int_{\Delta} dx_1 \dots dx_n \rho_n(T_{-t}^{(n)}(x_1, \dots, x_n)) \\
 & + \sum_{m=1}^{\infty} N \dots (N - n + 1) P\{(x_1(t_m+), \dots, x_n(t_m+)) \in T_{t_m-t+}^{(n)}(\Delta), t_m \leq t\} \\
 & - \sum_{m=1}^{\infty} N \dots (N - n + 1) P\{(x_1(t_m-), \dots, x_n(t_m-)) \in T_{t_m-t-}^{(n)}(\Delta), t_m \leq t\}
 \end{aligned}$$

Proof.

$$\begin{aligned}
 (A.2) \quad & P\{(x_1(t), \dots, x_n(t)) \in \Delta\} \\
 & = P\{(x_1(t-), \dots, x_n(t-)) \in \Delta, t_1 \leq t\} \\
 & + P\{(x_1(t+), \dots, x_n(t+)) \in \Delta, t_1 > t\} \\
 & = \sum_{m=0}^{\infty} P\{(x_1(t-), \dots, x_n(t-)) \in \Delta, t_{m+1} > t, t_m \leq t\} \\
 & + P\{(x_1, \dots, x_n) \in T_{-t+}^{(n)}\Delta, t_1 > t\} \\
 & = \sum_{m=0}^{\infty} P\{(x_1(t_m+), \dots, x_n(t_m+)) \in T_{t_m-t+}^{(n)}\Delta, t_{m+1} > t, t_m \leq t\} \\
 & + P\{(x_1, \dots, x_n) \in T_{-t}^{(n)}\Delta\} - P\{(x_1, \dots, x_n) \in T_{-t-}^{(n)}\Delta, t_1 \leq t\}
 \end{aligned}$$

$$\begin{aligned}
(A.3) \quad &= \sum_{m=0}^{\infty} P\{(x_1(t_m+), \dots, x_n(t_m+)) \in T_{t_m-t+}^{(n)} \Delta, t_m \leq t\} \\
&- \sum_{m=0}^{\infty} P\{(x_1(t_m+), \dots, x_n(t_m+)) \in T_{t_m-t+}^{(n)} \Delta, t_{m+1} \leq t\} \quad (*) \\
&+ P\{(x_1, \dots, x_n) \in T_{-t}^{(n)} \Delta\} \\
&- \sum_{m=0}^{\infty} P\{(x_1(t_m-), \dots, x_n(t_m-)) \in T_{t_m-t-}^{(n)} \Delta, t_m \leq t\} \\
&+ \sum_{m=0}^{\infty} P\{(x_1(t_m-), \dots, x_n(t_m-)) \in T_{t_{m+1}-t-}^{(n)} \Delta, t_{m+1} \leq t\} \quad (**)
\end{aligned}$$

To justify (A.2) we need to find a summable bound. It is easy to see that the m -th term is bounded by $c^n P_{eq}\{t_m \leq t\}$ (cf. (3.2)) which is, as we will see in the following Lemma, summable. Note that the lines (*) and (**) cancel each other, because as sets

$$\begin{aligned}
(A.4) \quad &\{(x_1, \dots, x_N) \in \mid (x_1(t_m+), \dots, x_n(t_m+)) \in T_{t_m-t+}^{(n)} \Delta, t_{m+1} \leq t\} \\
&= \{(x_1, \dots, x_N) \in \mid (x_1(t_{m+1}-), \dots, x_n(t_{m+1}-)) \in T_{t_{m+1}-t-}^{(n)} \Delta, t_{m+1} \leq t\}.
\end{aligned}$$

□

Lemma 1. *Let $0 \leq t_m < t$, $m = 1, 2, \dots$. Then*

$$\begin{aligned}
(A.5) \quad \sum_{m=1}^{\infty} P_{eq}\{t_m \leq t\} &= t \int dx_1 \int dp_2 \int_{\Omega_{1-(x_1, p_2)}} d\hat{\omega} a^2 \hat{\omega} \cdot (p_1 - p_2) \\
&\quad \times \rho_{eq,2}(x_1, q_1 + a\hat{\omega}, p_2),
\end{aligned}$$

where $\rho_{eq,2}$ is the correlation function induced by the measure P_{eq} .

Note that $\sum_{m=1}^{\infty} P_{eq}\{t_m \leq t\}$ is the expected number of collisions

$$\mathbb{E}_{eq} N = \sum_{m=1}^{\infty} m P_{eq}\{N = m\} = \sum_{m=1}^{\infty} P_{eq}\{N \geq m\},$$

where the random variable N is the number of collisions within $[0, t]$.

Proof. We use a construction which interprets the hard sphere dynamics as a flow under a ceiling function, cf. [5], to prove our Lemma.

We use a general fact in dynamical systems. Let B be a base space and $T : B \rightarrow B$ be an invertible map which preserves a finite measure μ . Further let a so-called ceiling function $h : B \rightarrow \mathbb{R}_+$ be integrable. Our phase space is then

$$\Gamma = \{x \in B, y \in \mathbb{R}_+ \mid 0 \leq y \leq h(x)\}.$$

This set describes the region below the graph of the ceiling function. For $t \in \mathbb{R}$ we define the flow $T_t(x, y)$ piecewise, such that $T_t : B \times \mathbb{R}_+ \rightarrow B \times \mathbb{R}_+$, where $T_t(x, y) = (x, y + t)$ until the first time t_1 such that $(x, y + t_1) = (x, h(x))$. Then we set at (t_1+) , $T_{t_1+}(x, y) = (Tx, 0)$. This construction is then continued into future and past. The dynamics T_t is discontinuous exactly at times t_1, t_2, \dots where the flow hits the ceiling function h . This dynamics actually act only on Γ . On this set T_t is invariant and it preserves measure $\mu \times dy$. Under the above conditions we can state the following lemma.

Lemma 2. *Let $h, B, \Gamma, T, T_t, \mu$ and $\mu \times dy$ be as above, then*

$$\sum_{m=1}^{\infty} (\mu \times dy)\{t_m \leq t\} = t\mu(B)$$

Proof. The Poincaré recurrence theorem states that

$$\sum_{j=0}^{\infty} h(T^{-j}x) = \infty \mu(dx) \text{ a.s.}$$

Therefore

$$(A.6) \quad B_k(t) = B_k := \left\{ x \in B \mid \sum_{j=0}^{k-2} h(T^{-j}x) \leq t, \sum_{j=0}^{k-1} h(T^{-j}x) \geq t \right\},$$

forms a partition of B , for $k = 1, 2, \dots$. Here B_k is the set of those points whose trajectories under the T_t dynamics reaches exactly $k - 1$ times the ceiling function h , within the time interval $[0, t]$. Thus we get

$$\begin{aligned} (A.7) \quad & \sum_{m=1}^{\infty} (\mu \times dy)\{t_m \leq t\} = (\mu \times dy)\{t_1 \leq t\} + \sum_{m=2}^{\infty} (\mu \times dy)\{t_m \leq t\} \\ & = \left[(\mu \times dy)\{\Gamma\} - (\mu \times dy)\{t_1 \geq t\} \right] + \sum_{m=2}^{\infty} \left[(\mu \times dy)\{\Gamma\} - (\mu \times dy)\{t_m \geq t\} \right] \\ & = \left[\sum_{k=1}^{\infty} \int_{B_k} h(x) d\mu - \int_{B_1} (h(x) - t) d\mu \right] \\ & + \left[\sum_{m=2}^{\infty} \left\{ \sum_{k=m+1}^{\infty} \int_{B_k} h(T^{-k+2}x) d\mu - \int_{B_m} \sum_{j=0}^{m-2} (h(T^{-j}x) - t) d\mu \right\} \right] \\ & = \left[t\mu(B_1) + \sum_{k=2}^{\infty} \int_{B_k} \mu h(x) \right] + \left[\sum_{m=2}^{\infty} \left\{ \int_{B_m} \mu \left(t - \sum_{j=0}^{m-2} h(T^{-j}x) \right) \right. \right. \\ & \quad \left. \left. + \sum_{k=m+1}^{\infty} \int_{B_k} \mu h(T^{-k+2}x) \right\} \right] = t \sum_{k=1}^{\infty} \mu(B_k) = t\mu(B). \end{aligned}$$

□

We use this result in the following way. In our context of hard spheres we have the base,

$$B = \{x \in \Gamma_n^* \mid \exists (i, j), i \neq j, \exists \hat{\omega} \in S^2, \text{ such that } q_j = q_i + a\hat{\omega}, (p_j - p_i) \cdot \hat{\omega} > 0\}.$$

The measure P_{eq} induces on B the invariant surface measure μ , given by

$$(A.8) \quad \left\{ \sum_{i \neq j=1}^N a^2 dq_j d\hat{\omega} \hat{\omega} \cdot (p_j - p_i) \prod_{k=1, k \neq i, j}^N dq_k \right\} \prod_{k=1}^N h_\beta(p_k) dp_k,$$

on the set where $q_j = q_i + a\hat{\omega}$. The product measure $\mu(dx) \times dy := P_{eq}$, is invariant under T_t . We refer to the times t_m , $m = 1, 2, \dots$, in which we reset the flow, to the times of collision. The ceiling function $h(x)$ is defined to be the time until the next collision (not counting collisions with the wall). Note that $h(x) > 0$ for all $x \in \Gamma_n^*$, because we do not have multiple collisions. Furthermore note that $\int h d\mu > 0$. The resetting transformation will be seen as a collision. Finally T is the discrete flow which acts only on the instants of a collision, i.e. it jumps from collision to collision. Using this specifications we get,

$$(A.9) \quad \sum_{m=1}^{\infty} P_{eq} \{t_m \leq t\} = t\mu(B).$$

From this result and from the definition of P_{eq} we derive

$$(A.10) \quad \sum_{m=1}^{\infty} P_{eq} \{t_m \leq t\} = t \int dx_1 \int dp_2 \int_{\Omega_1 - (x_1, p_2)} d\hat{\omega} a^2 \hat{\omega} \cdot (p_1 - p_2) \\ \times \rho_{eq,2}(x_1, (q_1 + a\hat{\omega}, p_2)),$$

□

Our aim is to express (A.1) in terms of correlation functions. For this purpose we first have to prove some regularity of these functions.

Lemma 3. *Under our assumptions on P , cf. Section 3, for every $s \in \mathbb{R}$, $\rho_n(s) = 0$ on $\Gamma_n \setminus \Gamma_n^*$ and $\rho_n(s)$ is continuous along trajectories of $T_t^{(n)}$ on Γ_n^* .*

Proof. By the assumptions (iv) and (v) on P , $f_N(s)$ has the same continuity properties as $f_N(0)$, we may set $s = 0$. For brevity we use $x = (x_1, \dots, x_n)$, $y = (x_{n+1}, \dots, x_N)$. For $x \in \Gamma_n^*$ let $\Lambda(x, t) \subset \Lambda$ be a spatial region traced out by the particle's motion $x(s)$, $0 \leq s \leq t$, with the initial condition x . Correspondingly we define $\Lambda(y, t) \subset \Lambda$ for $y \in \Gamma_{N-n}^*$, where $\Gamma_{N-n}^* \subset \Gamma_{N-n}$ such that we remove the zero measure set of phase points with undefined dynamics up to time t . For $x \in \Gamma_n^*$ let

$$(A.11) \quad \Gamma_{N-n}^*(x, t) = \{y \in \Gamma_{N-n}^* \mid \Lambda(x, t) \cap \Lambda(y, t) = \emptyset\},$$

be the set in which the trace of $x \in \Gamma_n^*$ and the trace of $y \in \Gamma_{N-n}^*$, up to time t , have no points in common. For $x \in \Gamma_n^*$ we define the $N - n$ particle flow $T_t^{(x)}$ on $\Gamma_{N-n}(x)$, where $\Gamma_{N-n}(x)$ is defined in (3.1). Following these considerations and using Liouville's equation for the map $T_t^{(x(t))}$ for a fixed t , we conclude

$$(A.12) \quad \begin{aligned} \rho_n(T_{t\pm}^{(n)}x) &= \int_{\Gamma_{N-n}(x(t))} dy N \dots (N - n + 1) f_N(T_{t\pm}^{(n)}x, y) \\ &= \int_{\Gamma_{N-n}(x(t))} dy N \dots (N - n + 1) f_N(T_{t\pm}^{(n)}x, T_t^{(x(t))}y). \end{aligned}$$

Let τ be such that $0 \leq t \leq \tau$. Then we have two possibilities for $x \in \Gamma_n^*$ and $y \in \Gamma_{N-n}^*$:

- (1) The trace of x and the trace of y have no points in common, i.e. $y \in \Gamma_{N-n}(x, t)$. On this set we have $(T_{t\pm}^{(n)}x, T_{t\pm}^{(x(t))}y) = T_{t\pm}^{(N)}(x, y)$.
- (2) The traces of x and y have at least on point in common, i.e. $y \in \Gamma_{N-n} \setminus \Gamma_{N-n}(x, t)$.

Note that we restrict (x, y) to be in Γ_N^* , and recall that $\Gamma_N \setminus \Gamma_N^*$ is a zero measure set. Again we denote with $\Gamma_{N-n}^*(x, t)$ the set in which we removed the multiple collisions. Therefore, for all $x \in \Gamma_n^*$ and $\tau \geq t$, we have

$$(A.12) = \int_{\Gamma_{N-n}^*(x, \tau)} dy f_N(T_{t\pm}^{(n)}x, T_t^{(x(t))}y) + \int_{\Gamma_{N-n}(x\tau) \setminus \Gamma_{N-n}^*(x, \tau)} dy f_N(T_{t\pm}^{(n)}x, T_t^{(x(t))}y) = \int_{\Gamma_{N-n}^*(x, \tau)} dy f_N(T_t^{(N)}(x, y)) + \int_{\Gamma_{N-n}(x\tau) \setminus \Gamma_{N-n}^*(x, \tau)} dy f_N(T_{t\pm}^{(n)}x, T_t^{(x(t))}y).$$

Thus we get for,

$$(A.13) \quad \begin{aligned} &|\rho(T_{t\pm}^{(n)}x) - \rho_n(x, 0)| \\ &= N \dots (N - n + 1) \left| \int_{\Gamma_{N-n}^*(x, \tau)} dy f_N(T_t^{(N)}(x, y)) \right. \\ &+ \left. \int_{\Gamma_{N-n}(x(\tau)) \setminus \Gamma_{N-n}^*(x, \tau)} dy f_N(T_{t\pm}^{(n)}x, T_t^{(x(t))}y) - \int_{\Gamma_{N-n}(x)} dy f_N(x, y, 0) \right| \\ &\leq N \dots (N - n + 1) \left\{ \int_{\Gamma_{N-n}^*(x, \tau)} dy |f_N(T_t^{(n)}(x, y)) - f_N(x, y, 0)| \right. \\ &+ c \int_{\Gamma_{N-n}(x(\tau)) \setminus \Gamma_{N-n}^*(x, \tau)} dy f_{eq, N}(x, y) + c \int_{\Gamma_{N-n}(x) \setminus \Gamma_{N-n}^*(x, \tau)} dy f_{eq, N}(x, y, 0) \left. \right\}. \end{aligned}$$

Here we used the two possibilities for a time evolution in $\rho(T_{t\pm}^{(n)}x)$, i.e. the particles described by x have "bad" collisions with fresh particles within $[0, \tau]$, or they do not. The last two terms are bounded by a constant times τ , due to Lemma 1. For fixed τ , using dominated convergence and assumption (iv), the first term vanishes in the limit $t \rightarrow 0$. Finally we let τ tend to zero. \square

We have proven that $\rho_n(x)$ is continuous through collisions, now we prove the pointwise continuity of the map $t \mapsto r_n(t)$.

Lemma 4. *Under our assumptions on P , for all $x \in \Gamma_n^*$ the map $t \mapsto \rho_n(x, t)$ is continuous, i.e.*

$$(A.14) \quad \lim_{t \rightarrow 0} \rho_n(x, s + t) = \rho_n(x, s).$$

Proof. By our assumptions (iv) and (v) on P , $f_N(s)$ has the same continuity properties at any s as at $s = 0$. Hence we may set $s = 0$. For every $x \in \Gamma_n^*$ we have

$$(A.15) \quad \begin{aligned} \rho_n(x, t) - \rho_n(x, 0) &= \int_{\{y|(x,y) \in \Gamma_N^*\}} dy f_N(x, y, t) - \int_{\{y|(x,y) \in \Gamma_N^*\}} dy f_N(x, y, 0) \\ &= \int_{\{y|(x,y) \in \Gamma_N^*\}} dy (f_N(T_{-t}^{(N)}(x, y)) - f_N(x, y, 0)) \end{aligned}$$

We conclude the proof using dominated convergence together with assumption (iv) on P . \square

Now we are ready to prove

Proposition 2 *For $\Delta \subset \Gamma_n^*$ and $n = 1, \dots, N$,*

$$(A.16) \quad \begin{aligned} \int_{\Delta} dx_1 \dots dx_n \rho_n(x_1, \dots, x_n, t) &= \int_{\Delta} dx_1 \dots dx_n \rho_n(T_{-t}^{(n)}(x_1, \dots, x_n)) \\ &\quad + \sum_{j=1}^n \int_0^t \int_{\Delta} ds dx_1 \dots dx_n [C_{j,n+1} \rho_n(s)](T_{-t}^{(n)}(x_1, \dots, x_n)). \end{aligned}$$

Proof. We consider the third term on the right hand side of (A.1). For $0 \leq s < t$ want to compute the limit $\varepsilon \rightarrow 0$ of

$$(A.17) \quad \sum_{m=1}^{\infty} \frac{1}{\varepsilon} P \{ (x_1(t_m^-), \dots, x_n(t_m^-)) \in T_{t_m-t}^{(n)} \Delta, t_m \in [s, s + \varepsilon] \}$$

$$= \sum_{k=1}^{\infty} \frac{1}{\varepsilon} P_s \{ \text{particles with labels } 1, \dots, n \text{ collide exactly } k \text{ times with}$$

particles with the label $n + 1, \dots, N$ during the time interval $[0, \varepsilon]$,

$$\text{at the times } \tau \text{ of collision } x(\tau^-) \in T_{s+\tau-t}^{(n)} \Delta \}.$$

Lemma 5. *For our equilibrium measure P_{eq} , we have*

$$(A.18) \quad \sum_{k=2}^{\infty} \frac{1}{\varepsilon} P_{eq} \{ t_k \leq \varepsilon \} = \sum_{k=2}^{\infty} \frac{1}{\varepsilon} k P_{eq} \{ t_k = \varepsilon \}.$$

tends to zero as $\varepsilon \rightarrow 0$.

Proof. Here we use the time evolved measure P_s , defined by assumption (v) on P . By assumption (iii), on P , the sum for $k \geq 2$ is bounded by

$$(A.19) \quad \sum_{k=2}^{\infty} k \frac{c}{\varepsilon} P_{eq} \{ \text{particles have exactly } k \text{ collisions in } [0, \varepsilon] \}.$$

By the proof of Lemma 1 we have,

$$(A.19) = \sum_{m=1}^{\infty} P_{eq} \{ t_m \leq \varepsilon \} = \varepsilon \mu(B) \geq \varepsilon \mu(B_1).$$

Hence,

$$(A.20) \quad \mu(B) - \mu(B_1) = \frac{1}{\varepsilon} \left(\sum_{m=1}^{\infty} P_{eq} \{ t_m \leq \varepsilon \} - P_{eq} \{ t_1 \leq \varepsilon, t_2 > \varepsilon \} \right)$$

$$\geq \frac{1}{\varepsilon} \left(\sum_{m=1}^{\infty} P_{eq} \{ t_m \leq \varepsilon \} - P_{eq} \{ t_1 \leq \varepsilon \} \right) = \frac{1}{\varepsilon} \sum_{m=2}^{\infty} P_{eq} \{ t_m \leq \varepsilon \} \geq 0$$

Recall that $B_1 = B_1(\varepsilon)$ is time dependent. The limit of $B_1(\varepsilon)$, as $\varepsilon \rightarrow 0$, is B . Thus

$$\lim_{\varepsilon \rightarrow 0} \mu(B) - \mu(B_1(\varepsilon)) = 0$$

□

Therefore we have,

$$(A.21) \quad \begin{aligned} & \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} P_{eq} \{ \text{particles have exactly one collisions within the time interval } [0, \varepsilon] \} \\ &= \int dx_1 \int dp_2 \int_{\Omega_1 - (x_1, p_2)} d\hat{\omega} a^2 \hat{\omega} \cdot (p_1 - p_2) \rho_{eq,2}(x_1, (q_1 + a\hat{\omega}, p_2)) \end{aligned}$$

and again according to Lemma 1

$$(A.22) \quad \begin{aligned} & \sum_{k=1}^{\infty} k P_{eq} \{ \text{particles have exactly } k \text{ collisions within the time interval } [0, \varepsilon] \} \\ &= \varepsilon \int dx_1 \int dp_2 \int_{\Omega_1 - (x_1, p_2)} d\hat{\omega} a^2 \hat{\omega} \cdot (p_1 - p_2) \rho_{eq,2}(x_1, (q_1 + a\hat{\omega}, p_2)). \end{aligned}$$

Therefore the expression (A.19) vanishes as we take the limit $\varepsilon \rightarrow 0$.

Hence, we are left with the term $k = 1$ from equation (A.17). We label the particle of collision with $n + 1$. I.e. we have,

$$(A.23) \quad \begin{aligned} & (N-n) \frac{1}{\varepsilon} P_s \{ \text{particles with labels } 1, \dots, n \text{ collide exactly once with the particle} \\ & n+1 \text{ and do not collide with any other particle during the interval } [0, \varepsilon], \\ & \text{at the time } \tau \text{ of collision } x(\tau-) \in T_{s+\tau-t-}^{(n)} \Delta \} = \sum_{j=1}^n (N-n) \frac{1}{\varepsilon} P_s \{ B_j(\varepsilon) \}, \end{aligned}$$

where

$B_j(\varepsilon) = \{ \text{the only collision during the time interval } [0, \varepsilon] \text{ is between the particle } j \text{ and the particle } n+1, \text{ at time } \tau \text{ of collision } x(\tau-) \in T_{s+\tau-t-} \Delta \}.$

Let

$$(A.24) \quad \begin{aligned} A_j(\varepsilon) &:= \{ (x, \dots, x_{n+1}) \in \Gamma_{n+1}^* \mid T_t^{(n+1)}(x_1, \dots, x_{n+1}) \text{ for } 0 \leq t \leq \varepsilon, \\ & \text{has as only collision the one between the } j\text{-th and the } n+1\text{-th particle,} \\ & \text{at the time } \tau \text{ of collision } (x_1(\tau-), \dots, x_n(\tau-)) \in T_{s+\tau-t-}^{(n)} \Delta \}. \end{aligned}$$

Then, continuing (A.23), we have

$$\begin{aligned}
\text{(A.25)} \quad & \sum_{j=1}^n (N-n) \frac{1}{\varepsilon} P_s \{B_j(\varepsilon)\} = \sum_{j=1}^n (N-n) \frac{1}{\varepsilon} \int_{B_j(\varepsilon)} dx_1 \dots dx_N f_N(x_1, \dots, x_N, s) \\
& = \sum_{j=1}^n (N-n) \frac{1}{\varepsilon} \int_{A_j(\varepsilon)} dx_1 \dots dx_{n+1} \int_{\Gamma_{N-(n+1)}(x_1, \dots, x_{n+1})} dx_{n+2} \dots dx_N f_N(x_1, \dots, x_N, s) \\
& \quad + \sum_{j=1}^n (N-n) \frac{1}{\varepsilon} \left[\int_{B_j(\varepsilon)} dx_1 \dots dx_N f_N(x_1, \dots, x_N, s) \right. \\
& \quad \left. - \int_{A_j(\varepsilon)} dx_1 \dots dx_{n+1} \int_{\Gamma_{N-(n+1)}(x_1, \dots, x_{n+1})} dx_{n+2} \dots dx_N f_N(x_1, \dots, x_N, s) \right].
\end{aligned}$$

The term in the brackets is again bounded by

$$\text{(A.26)} \quad \frac{c}{\varepsilon} P_{eq} \{\text{there is more than one collision during the time interval } [0, \varepsilon]\},$$

and vanishes as $\varepsilon \rightarrow 0$. After multiplying with the factor $N(N+1)\dots(N-n-1)$ as in (A.1) we are left with

$$\text{(A.27)} \quad \sum_{j=1}^n (N-n) \frac{1}{\varepsilon} P_s \{B_j(\varepsilon)\} = \sum_{j=1}^n \frac{1}{\varepsilon} \int_{A_j(\varepsilon)} dx_1 \dots dx_{n+1} \rho_{n+1}(x_1, \dots, x_{n+1}, s) + \mathcal{O}(\varepsilon).$$

Let τ , $0 \leq \tau \leq \varepsilon$, be the time of collision. Then on $A_j(\varepsilon)$

$$\text{(A.28)} \quad q_{n+1} = q_j + a\hat{\omega} + \tau(p_j - p_{n+1}).$$

We perform this substitution in the integral (A.27). The change in the volume element is

$$\text{(A.29)} \quad dx_j dx_{n+1} = a^2 \hat{\omega} \cdot (p_j - p_{n+1}) dx_j d\tau d\hat{\omega} dp_{n+1}.$$

We flow on $A_j(\varepsilon)$ all $n+1$ coordinates from time 0 to time τ . Then

$$\begin{aligned}
\text{(A.30)} \quad & \sum_{j=1}^n \frac{1}{\varepsilon} \int_{A_j(\varepsilon)} dx_1 \dots dx_{n+1} \rho_{n+1}(x_1, \dots, x_{n+1}, s) \\
& = \sum_{j=1}^n \frac{1}{\varepsilon} \int_0^\varepsilon d\tau a^2 \int_{T_{s+\tau-t-\Delta}} dx_1 \dots dx_n \int dp_{n+1} \int_{\Omega_{j-(x, p_{n+1})}} d\hat{\omega} \hat{\omega} \cdot (p_j - p_{n+1}) \\
& \quad \times \chi_{\Xi}(x, \hat{\omega}, p_{n+1}) \rho_{n+1}(T_{-\tau}^{(n+1)}(x, (q_j + a\hat{\omega}, p_{n+1})), s) + \mathcal{O}(\varepsilon) \\
& = \sum_{j=1}^n \frac{1}{\varepsilon} \int_0^\varepsilon d\tau a^2 \int_{T_{s+\tau-t-\Delta}} dx_1 \dots dx_n \int dp_{n+1} \int_{\Omega_{j-(x, p_{n+1})}} d\hat{\omega} \hat{\omega} \cdot (p_j - p_{n+1}) \\
& \quad \times \rho_{n+1}(T_{-\tau}^{(n+1)}(x, (q_j + a\hat{\omega}, p_{n+1})), s) + \mathcal{O}(\varepsilon).
\end{aligned}$$

In the second integral χ_{Ξ} is the indicator function of the set

$$\Xi := \{(x, \hat{\omega}, p_{n+1}) \mid T_{\tau'}^{(n+1)}(x, (q_j + a\hat{\omega}, p_{n+1})) \text{ for } -\tau \leq \tau' \leq \varepsilon - \tau \text{ has only one collision}\}$$

. The error term is again bounded by

$$(A.31) \quad \frac{c}{\varepsilon} P_{eq} \{\text{there is more than one collision during the time interval } [0, \varepsilon]\}.$$

Only P_{eq} refers now to the equilibrium measure of $n + 1$ particles.

To obtain the limit as $\varepsilon \rightarrow 0$ of (A.30) we have to show that the map

$$\begin{aligned} \tau \mapsto & \int_{T_{s+\tau-t-\Delta}} dx_1 \dots dx_n \int dp_{n+1} \int_{\Omega_{j-(x, p_{n+1})}} d\hat{\omega} \hat{\omega} \cdot (p_j - p_{n+1}) \\ & \times \rho_{n+1}(T_{-\tau}^{(n+1)}(x, (q_j + a\hat{\omega}, p_{n+1})), s) \end{aligned}$$

is continuous at $\tau = 0$.

As a first step we will show that

$$(A.32) \quad \mu(T_{\tau} \Delta \setminus \Delta) \rightarrow 0 \text{ as } \tau \rightarrow 0.$$

To see this we define

$S_{\tau} := \{t_1(x) > \tau\}$. Then we have for $x \in S_{\tau}$ that $T_t^{(n)}x$ is continuous for $|t| \leq \tau$. Thus

$$\begin{aligned} \mu(T_t^n \Delta \setminus \Delta) &= \mu(S_{\tau} \cap (T_t^n \Delta \setminus \Delta)) + \mu(S_{\tau}^c \cap (T_t^n \Delta \setminus \Delta)) \\ &\leq \mu(S_{\tau} \cap (T_t^n \Delta \setminus \Delta)) + \mu(S_{\tau}^c) \leq \mu(S_{\tau} \cap (T_t^n \Delta \setminus \Delta)) + c\tau. \end{aligned}$$

The last inequality follows by Lemma 1.

Now we bound as,

$$\begin{aligned}
(A.33) \quad & \left| \int_{T_{s+\tau-t}^{(n)}\Delta} dx_1 \dots dx_n \int dp_{n+1} \int_{\Omega_{j-}(x, p_{n+1})} d\hat{\omega} \hat{\omega} \cdot (p_j - p_{n+1}) \right. \\
& \quad \times \rho_{n+1}(T_{-\tau}^{(n+1)}(x, (q_j + a\hat{\omega}, p_{n+1})), s) \\
& \quad - \int_{T_{s-t}^{(n)}\Delta} dx_1 \dots dx_n \int dp_{n+1} \int_{\Omega_{j-}(x, p_{n+1})} d\hat{\omega} \hat{\omega} \cdot (p_j - p_{n+1}) \\
& \quad \left. \times \rho_{n+1}(x, (q_j + a\hat{\omega}, p_{n+1}), s) \right| \\
\leq & \int_{(T_{s+\tau-t}^{(n)}\Delta \cup T_{s-t}^{(n)}\Delta) \setminus (T_{s+\tau-t}^{(n)}\Delta \cap T_{s-t}^{(n)}\Delta)} dx_1 \dots dx_n \int dp_{n+1} \int_{\Omega_{j-}(x, p_{n+1})} d\hat{\omega} \hat{\omega} \cdot (p_j - p_{n+1}) \\
& \quad \times \rho_{n+1}(T_{-\tau}^{(n+1)}(x, (q_j + a\hat{\omega}, p_{n+1})), s) \\
& \quad + \int_{T_{s-t}^{(n)}\Delta} dx_1 \dots dx_n \int dp_{n+1} \int_{\Omega_{j-}(x, p_{n+1})} d\hat{\omega} \hat{\omega} \cdot (p_j - p_{n+1}) \\
& \quad \times \left| \rho_{n+1}(T_{-\tau}^{(n+1)}(x, (q_j + a\hat{\omega}, p_{n+1})), s) - \rho_{n+1}(x, (q_j + a\hat{\omega}, p_{n+1}), s) \right|.
\end{aligned}$$

We bound ρ_{n+1} , in the first term, by $f_{eq, n+1}$. By the dominated convergence and (A.32) this term vanishes as $\tau \rightarrow 0$. In the second term we integrate only over points $(x, (q_j + a\hat{\omega}, p_{n+1})) \in \Gamma_{n+1}^*$. Hence, the integrand is continuous by Lemma 3, and vanishes as $\tau \rightarrow 0$.

Altogether we have shown that the measure

$$\begin{aligned}
(A.34) \quad & \sum_{m=1}^{\infty} N \dots (N - n + 1) P\{(x_1(t_m-), \dots, x_n(t_m-)) \in T_{t_m-t-}(\Delta), t_m \in ds\}
\end{aligned}$$

is absolutely continuous with respect to the Lebesgue measure and has a density given by

$$\begin{aligned}
(A.35) \quad & \sum_{j=1}^n a^2 \int_{T_{s-t}^{(n)}\Delta} dx \int dp_{n+1} \int_{\Omega_{j-}(x, p_{n+1})} d\hat{\omega} \hat{\omega} \cdot (p_j - p_{n+1}) \\
& \quad \times \rho_{n+1}(x, (q_j + a\hat{\omega}, p_{n+1}), s).
\end{aligned}$$

Note that this density is continuous in s , by Lemma 4. The same argumentation applied to the second term in (1) shows that the measure

$$\begin{aligned}
(A.36) \quad & \sum_{m=1}^{\infty} N \dots (N - n + 1) P\{(x_1(t_m+), \dots, x_n(t_m+)) \in T_{t_m-t+}(\Delta), t_m \leq t\}
\end{aligned}$$

has a density given by

$$\begin{aligned}
\text{(A.37)} \quad & \sum_{j=1}^n a^2 \int_{(x_1, \dots, q_j, p'_j, \dots, x_n) \in T_{s-t}^{(n)} \Delta} dx \int dp_{n+1} \int_{\Omega_{j-}(x, p_{n+1})} d\hat{\omega} \hat{\omega} \cdot (p_j - p_{n+1}) \\
& \quad \times \rho_{n+1}(x, (q_j + a\hat{\omega}, p_n + 1), s) \\
& = \sum_{j=1}^n a^2 \int_{(x_1, \dots, q_j, p'_j, \dots, x_n) \in T_{s-t}^{(n)} \Delta} dx_1 \dots dq_j dp'_j \dots dx_n \int dp'_{n+1} \\
& \times \int_{\Omega_{j-}(x_1, \dots, q_j, p'_j, \dots, p'_n + 1)} d\hat{\omega} \hat{\omega} \cdot (p'_j - p'_{n+1}) \rho_{n+1}(x_1, \dots, q_j, p'_j, \dots, x_n, (q_j + a\hat{\omega}, p'_n + 1), s),
\end{aligned}$$

where we used the continuity of $\rho_{n+1}(s)$ through a collision on the domain of our integration. After relabeling (p'_j, p'_{n+1}) as (p_j, p_{n+1}) in (A.37), we subtract (A.35) from (A.37). Since $\Delta \subset \Gamma_n^*$,

$$\begin{aligned}
\text{(A.38)} \quad & \sum_{j=1}^n \int_0^t ds \int_{T_{s-t}^{(n)} \Delta} dx [C_{j, n+1} \rho_{n+1}(s)](x) \\
& = \sum_{j=1}^n \int_0^t ds \int_{\Delta} dx [C_{j, n+1} \rho_{n+1}(s)](T_{s-t}^{(n)} x).
\end{aligned}$$

□

To obtain the integrated form of the BBGKY hierarchy, and therefore obtain (3.7), we have to iterate (A.38). However, we have to prove first that for a fixed t , the map

$$t \mapsto \sum_{j=1}^n \int_0^t ds [C_{j, n+1} \rho_{n+1}(s)](T_{-t}^{(n)} T_{\tau+}^{(n)} x)$$

is continuous along trajectories. We have

$$\begin{aligned}
\text{(A.39)} \quad & \left| \sum_{j=1}^n \int_0^t ds [C_{j, n+1} \rho_{n+1}(s)](T_{-t}^{(n)} T_{\tau+}^{(n)} x) \right. \\
& \quad \left. - \sum_{j=1}^n \int_0^t ds [C_{j, n+1} \rho_{n+1}(s)](T_{-t+\tau}^{(n)} x) \right| \\
& \leq \sum_{j=1}^n \left| \left\{ \int_0^{\tau} ds + \int_t^{t+\tau} ds \right\} \left| [C_{j, n+1} \rho_{n+1}(s - \tau)](T_{-t+\tau}^{(n)} x) \right| \right| \\
& \quad + \sum_{j=1}^n \int_0^t ds \left| [C_{j, n+1} \rho_{n+1}(s - \tau) - C_{j, n+1} \rho_{n+1}(s)](T_{-t+\tau}^{(n)} x) \right|.
\end{aligned}$$

The first term is bounded by $cf_{eq,n}^{(\beta)}$, and thus vanishes as $\tau \rightarrow 0$. For the second term we use dominated convergence. By the assumed continuity of $s \mapsto \rho_{n+1}(x_1, \dots, x_{n+1}, s)$ for $x \in \Gamma_n^*$, and by the definition of $C_{j,n+1}$ (3.5) the integrand vanishes pointwise in the limit $\tau \rightarrow 0$.

Hence, we obtain

$$(A.40) \quad \int_{\Delta} dx_1 \dots dx_n \rho_n(x_1, \dots, x_n, t) = \\ \sum_{m=0}^{N-n} \sum_{j_1=0}^n \dots \sum_{j_m=0}^{n+m-1} \int_0^t \dots \int_0^{t_m} \int_{\Delta} dt_1 \dots dt_m dx_1 \dots dx_n \\ \times (C_{j_1, n+1} \dots (C_{j_m, n+m} (\rho_{n+m} \circ T_{t_m+}^{(n+m)})) \circ T_{t_m - t_{m-1}+}^{(n+m-1)} \dots) (T_{t_1 - t+}^{(n)}(x_1, \dots, x_n)).$$

Since the Boltzmann-Grad limit not quite apparent in this form (A.40), we convert it into an integral over collision histories.

Proposition 3

For every Borel set $\Delta \subset \Gamma_n^*$, and every $n \in \mathbb{N}$,

$$(A.41) \quad \int_{\Delta} dx_1 \dots dx_n \rho_n(x_1, \dots, x_n, t) = \\ \int_{\Delta} dx_1 \dots dx_n \int_{\Lambda_a(x_1, \dots, x_n, [0, t])} d\delta \\ \times W_a(x_1, \dots, x_n, \delta) \rho_{n+m(\delta)}(x_1(0, x, \delta), \dots, x_{n+m(\delta)}(0, x, \delta)).$$

Proof. If we write out (A.40) explicitly, using the definitions of $C_{j,n+1}$, W_a and the definition of the collision history we obtain the later identity.

In order to interchange the integrals we need a integrable bound. We know $\rho_{n+m} \leq c' f_{eq, n+m}^{(\beta)}$ and therefore we get

$$(A.42) \quad (3.7) \leq \int_{\Delta} dx \int_{\Lambda_a(x, [0, t])} d\delta |W_a(x, \delta)| \\ \times f_{eq, n+m}^{(\beta)}(x_1(0, x, \delta), \dots, x_{n+m}(0, x, \delta)) \leq c'' \int_{\Delta} dx f_{eq, n+m}^{(\beta)}(x),$$

for all t . This last estimate can be found in full detail in F. King's thesis [6]. \square

APPENDIX B

Proposition 4 Let $r_n^\varepsilon(x, t)$ be defined as in (3.11), and $r_n^\varepsilon(x, 0)$ satisfies the Maxwellian bound,

$$(B.1) \quad |r_n^\varepsilon(x, 0)| \leq M \prod_{j=1}^n z h_\beta(p_j),$$

for some $M, z, \beta > 0$ independently of ε .

Then there exist a $t_0(z, \beta)$ such that for $0 \leq t < t_0(z, \beta)$,

$$(B.2) \quad |r_n^\varepsilon(x, t)| \leq M \prod_{j=1}^n z(t) h_{\beta(t)}(p_j),$$

for some functions $z(t)$ and $\beta(t)$, independently of ε . We will define $z(t)$ and $\beta(t)$ within the proof.

Moreover we have the bound $0.2 \frac{\sqrt{m\beta}}{\sqrt{3\pi z}} < t_0(z, \beta) < 0.4 \frac{\sqrt{m\beta}}{\sqrt{3\pi z}}$ for the threshold time $t_0(z, \beta)$.

We split the proof into two lemmata.

Lemma 6. Given $(x_1, \dots, x_n) \in \Gamma_n^*$ and for fixed $m, t_1, \dots, t_m, \hat{\omega}_1, \dots, \hat{\omega}_m, \hat{p}_1, \dots, \hat{p}_m$ we have the bound

$$\begin{aligned} & \left| \sum_{j_1=1}^n \dots \sum_{j_m=1}^{n+m-1} \left\{ \prod_{k=1}^m \hat{\omega}_k \cdot (\hat{p}_k - p_{j_k}^\varepsilon(t_k)) \right\} r_{n+m}^\varepsilon(q_1^\varepsilon(0), \dots, p_{n+m}^\varepsilon(0)) \right| \\ & \leq M \left(\prod_{k=1}^m \{(n+k-1)|p_{n+k}| + (n+k-1)^{\frac{1}{2}}\} \right) \left(\prod_{j=1}^{n+m} z h_\beta(p_j) \right) \end{aligned}$$

where $\hat{p}_k = p_{n+k}(s)$ for $k = 1, \dots, m$ and $s \in [0, t_k]$. This limit is uniformly in ε .

Proof. Let $\chi_n^\varepsilon : \mathbb{R}^{3n} \rightarrow \mathbb{R}$ be the indicator function of the set $\{q_1, \dots, q_n \mid |q_i - q_j| \geq \varepsilon a, i, j \in \mathbb{N}\}$. The invariance of the equilibrium measure implies that

- (i) $(\chi_n^\varepsilon \prod_{j=1}^n h_\beta) \circ T_{-t}^{(n)} = \chi_n^\varepsilon \prod_{j=1}^n h_\beta$.
If g is a function of energy and $g(q_1, p_1, \dots, q_n, p_n) = g_0(\frac{1}{2} \sum_{j=1}^n p_j^2)$, then by the conservation of energy
- (ii) $g \circ T_{-t}^{(n)} = g$

L.h.s. of (3.12)

$$\begin{aligned}
&\leq \sum_{j_1=1}^n \cdots \sum_{j_{m-1}=1}^{n+m-1} \left\{ \prod_{k=1}^m (|\hat{p}_k| + |p_{j_k}^\varepsilon(t_k)|) \right\} M \chi_{n+m}^\varepsilon(q_1^\varepsilon(0), \dots, q_{n+m}^\varepsilon(0)) \\
&\quad \times \prod_{j=1}^{n+m} z h_\beta(p_j^\varepsilon(0)) \\
&= \sum_{j_1=1}^n \cdots \sum_{j_{m-1}=1}^{n+m-1} \left\{ \prod_{k=1}^{m-1} (|\hat{p}_k| + |p_{j_k}^\varepsilon(t_k)|) \right\} (|\hat{p}_m| + |p_{j_m}^\varepsilon(t_m)|) \\
&\quad \times M \chi_{n+m}^\varepsilon(q_1^\varepsilon(0), \dots, q_{n+m}^\varepsilon(0)) \prod_{j=1}^{n+m} z h_\beta(p_j^\varepsilon(0))
\end{aligned}$$

Using (i),(ii) and the Schwarz inequality we get that the l.h.s. of (3.12)

$$\begin{aligned}
&\leq \sum_{j_1=1}^n \cdots \sum_{j_{m-1}=1}^{n+m-2} \left\{ \prod_{k=1}^{m-1} (|\hat{p}_k| + |p_{j_k}^\varepsilon(t_k)|) \right\} \left((n+m-1)|\hat{p}_m| + \left((n+m-1) \sum_{j=1}^{n+m-1} |p_j^\varepsilon(t_m)|^2 \right)^{\frac{1}{2}} \right) \\
&\quad \times z h_\beta(\hat{p}_m) M \chi_{n+m-1}^\varepsilon(q_1^\varepsilon(t_m), \dots, q_{n+m-1}^\varepsilon(t_m)) \prod_{j=1}^{n+m-1} z h_\beta(p_j^\varepsilon(t_m)) \\
&= \sum_{j_1=1}^n \cdots \sum_{j_{m-1}=1}^{n+m-2} \left\{ \prod_{k=1}^{m-1} (|\hat{p}_k| + |p_{j_k}^\varepsilon(t_k)|) \right\} \\
&\quad \times \left\{ (n+m-1)|\hat{p}_m| + (n+m-1)^{\frac{1}{2}} \left(\sum_{j=1}^{n+m-1} p_j^\varepsilon(t_{m-1})^2 \right)^{\frac{1}{2}} \right\} z h_\beta(\hat{p}_m) \\
&\quad \times M \chi_{n+m-1}^\varepsilon(q_1^\varepsilon(t_{m-1}), \dots, q_{n+m-1}^\varepsilon(t_{m-1})) \times \prod_{j=1}^{n+m-1} z h_\beta(p_j^\varepsilon(t_{m-1}))
\end{aligned}$$

Now we iterate for j_{m-1}, \dots, j_1 , and conclude the l.h.s of (3.12)

$$\leq M \prod_{k=1}^m \left\{ (n+k-1)|p_{n+k}| + (n+k-1)^{\frac{1}{2}} \left(\sum_{j=1}^{n+k-1} p_j^2(0) \right)^{\frac{1}{2}} \right\} \left(\prod_{j=1}^{n+m} z h_\beta(p_j(0)) \right),$$

since we have by the conservation of energy

$$\sum_{j=1}^{n+k-1} (p_j^\varepsilon)^2(0) = \sum_{j=1}^n (p_j^\varepsilon)^2(0) + \sum_{j=1}^{k-1} (\hat{p}_k)^2 = \sum_{j=1}^n p_j^2(0) + \sum_{j=1}^{k-1} (\hat{p}_k)^2 = \sum_{j=1}^{n+k-1} p_j^2(0)$$

□

Lemma 7. *There exist continuous functions $t \rightarrow z(t), t \rightarrow \beta(t)$ such that $\beta(0) = \beta, z(0) = z$, $z(t)$ is monotonously increasing, and $z(t) \rightarrow \infty$ as $t \rightarrow t_0(z, \beta)_-$, $\beta(t)$ is monotonously decreasing, $\beta(t) \rightarrow 0$ as $t \rightarrow t_0(z, \beta)_-$, and such that for $0 \leq t < t_0(z, \beta)$*

$$(B.3) \quad \sum_{m=0}^{\infty} \frac{t^m}{m!} (4\pi a^2)^m \int dp_{n+1} \dots dp_{n+m} \\ \times \left(\prod_{k=1}^m \left\{ (n+k-1)|p_{n+k}| + (n+k-1)^{\frac{1}{2}} \left(\sum_{j=1}^{n+k-1} p_j^2 \right)^{\frac{1}{2}} \right\} \right) \\ \times \prod_{j=1}^{n+m} z h_{\beta}(p_j) \leq \prod_{j=1}^n z(t) h_{\beta(t)}(p_j)$$

We have $0.2 \frac{\sqrt{m\beta}}{\sqrt{3\pi a^2 z}} < t_0(z, \beta) < 0.4 \frac{\sqrt{m\beta}}{\sqrt{3\pi a^2 z}}$.

Proof. We choose smooth functions $g, h : [0, 1] \rightarrow \mathbb{R}_+$ such that g is monotonously increasing and h monotonously decreasing, $g(0) = 1 = h(0)$. The optimal choice of g and h will be given later in the proof.

We define $z_j = g(\frac{j}{m})z$, $\beta_j = h(\frac{j}{m})\beta$, $j = 1, \dots, m$
 $z = z_0 < z_1 < \dots < z_m = g(1)z = z'$
 $\beta = \beta_0 > \dots > \beta_m = h(1)\beta = \beta'$.

Then the integral on the l.h.s. of (B.3) is

$$\left(\frac{\beta_0}{2\pi} \right)^{\frac{3}{2}(n+m)} \int dp_{n+1} \dots dp_{n+m} \\ \times \prod_{k=1}^m \left\{ (n+k-1)|p_{n+k}| + \left((n+k-1) \sum_{j=1}^{n+k-1} p_j^2 \right)^{\frac{1}{2}} \right\} z_0^{n+m} \exp \left[-\frac{1}{2} \beta_0 \sum_{j=1}^{n+m} p_j^2 \right] \\ = \left(\frac{\beta_0}{2\pi} \right)^{\frac{3}{2}(n+m)} \int dp_{n+1} \dots dp_{n+m} \\ \times \left\{ \left[(n+m-1)|p_{n+m}| + \left((n+m-1) \sum_{j=1}^{n+m-1} p_j^2 \right)^{\frac{1}{2}} \right] \right. \\ \left. \times \prod_{k=1}^{m-1} \left\{ (n+k-1)|p_{n+k}| + \left((n+k-1) \sum_{j=1}^{n+k-1} p_j^2 \right)^{\frac{1}{2}} \right\} \right\} \times z_0^{n+m} \exp \left[-\frac{1}{2} \beta_0 \sum_{j=1}^{n+m} p_j^2 \right]$$

We the integration over dp_{n+m} and get

$$\begin{aligned}
&= \left\{ \left(\frac{\beta_0}{2\pi} \right)^{\frac{3}{2}(n+m)} \int dp_{n+1} \dots dp_{n+m-1} \right. \\
&\quad \prod_{k=1}^{m-1} \left\{ (n+k-1) |p_{n+k}| + \left((n+k-1) \sum_{j=1}^{n+k-1} p_j^2 \right)^{\frac{1}{2}} \right\} \\
&\quad \times z_1^{n+m-1} \exp \left[-\frac{1}{2} \beta_1 \sum_{j=1}^{n+m-1} p_j^2 \right] \left. \cdot \left[\left\{ (n+m-1) \left(8\pi \frac{z_0}{\beta_0^2} \right) \right. \right. \right. \\
&+ \left. \left. \left. (n+m-1) \sum_{j=1}^{n+m-1} p_j^2 \right)^{\frac{1}{2}} z_0 \left(\frac{2\pi}{\beta_0} \right)^{\frac{3}{2}} \right\} \left(\frac{z_0}{z_1} \right)^{n+m-1} \exp \left[-\frac{1}{2} (\beta_0 - \beta_1) \sum_{j=1}^{n+m-1} p_j^2 \right] \right]
\end{aligned}$$

Now we have with:

- (i) $\left(\sum_{j=1}^{n+m-1} p_j^2 \right)^{\frac{1}{2}} \exp \left[-\frac{1}{2} (\beta_0 - \beta_1) \sum_{j=1}^{n+m-1} p_j^2 \right] \leq (e(\beta_0 - \beta_1))^{-\frac{1}{2}}$
- (ii) $(n+m-1) \left(\frac{z_0}{z_1} \right)^{n+m-1} \leq (-e \log \left(\frac{z_0}{z_1} \right))^{-1}$
- (iii) $(n+m-1)^{\frac{1}{2}} \left(\frac{z_0}{z_1} \right)^{n+m-1} \leq (-2 \log \frac{z_0}{z_1})^{-\frac{1}{2}}$

These inequalities follow from elementary calculus. We used for $0 \leq x$, and $0 < \alpha < 1$ the inequalities,

$$x^{\frac{1}{2}} e^{-bx} \leq (eb)^{-\frac{1}{2}},$$

$$m\alpha^m \leq (-e \log \alpha)^{-1}, \text{ and}$$

$$m^{\frac{1}{2}} \alpha^m \leq (-2 \log \alpha)^{-\frac{1}{2}},$$

$$\begin{aligned}
&\leq \left\{ \dots \right\} \left[\left(\frac{8\pi z_0}{\beta_0^2} \right) (-e \log \left(\frac{z_0}{z_1} \right))^{-1} + \left(\frac{2\pi}{\beta_0} \right)^{\frac{3}{2}} z_0 (-2 \log \left(\frac{z_0}{z_1} \right))^{-\frac{1}{2}} \cdot (e(\beta_0 - \beta_1))^{-\frac{1}{2}} \right] \\
&\leq \left\{ \dots \right\} \left[\left(\frac{8\pi z_0}{e\beta_0^2} \right) (\log \left(\frac{z_1}{z_0} \right))^{-1} + \frac{2\pi^{\frac{3}{2}} z_0}{e\beta_0^2} (\log \left(\frac{z_1}{z_0} \right))^{-\frac{1}{2}} \left(1 - \frac{\beta_1}{\beta_0} \right)^{-\frac{1}{2}} \right]
\end{aligned}$$

The term in the curly brackets has the same structure as in the previous step. We iterate the bound and integrate successively over $p_{n+m-1}, \dots, p_{n+1}$

and conclude

$$\begin{aligned}
&\leq \left(\frac{\beta_0}{2\pi}\right)^{\frac{3}{2}(n+m)} z^m \prod_{j=1}^m \exp\left[-\frac{1}{2}\beta' p_j^2\right] \prod_{j=1}^n \left[\left(\frac{8\pi z_{j-1}}{e\beta_{j-1}^2}\right) \left(\log\left(\frac{z_j}{z_{j-1}}\right)\right)^{-1} \right. \\
&\quad \left. + \frac{2\pi^{\frac{3}{2}}}{e} \cdot \frac{z_{j-1}}{\beta_{j-1}^2} \left(1 - \frac{\beta_j}{\beta_{j-1}}\right)^{-\frac{1}{2}} \left(\log\frac{z_j}{z_{j-1}}\right)^{-\frac{1}{2}} \right] \\
&= \left(\frac{\beta_0}{2\pi}\right)^{\frac{3}{2}(n+m)} z^m \left[\prod_{j=1}^n \left(\frac{\beta'}{2\pi}\right)^{-\frac{3}{2}} h_{\beta'}(p_j) \right] \prod_{j=1}^m \frac{16\pi}{e} \left\{ \left(\frac{g\left(\frac{(j-1)}{m}\right)z}{2\left(h\left(\frac{(j-1)}{m}\right)\right)^2 \beta^2 \log\left(\frac{g\left(\frac{j}{m}\right)}{g\left(\frac{(j-1)}{m}\right)}\right)} \right) \right. \\
&\quad \left. + \frac{\sqrt{\pi}}{8} \cdot \frac{g\left(\frac{(j-1)}{m}\right)z}{\left(h\left(\frac{(j-1)}{m}\right)\right)^2 \beta^2} \cdot \left(1 - \frac{h\left(\frac{j}{m}\right)}{h\left(\frac{(j-1)}{m}\right)}\right)^{-\frac{1}{2}} \left(\log\frac{g\left(\frac{j}{m}\right)}{g\left(\frac{(j-1)}{m}\right)}\right)^{-\frac{1}{2}} \right\} \\
&= \left[\prod_{j=1}^n \left(\frac{\beta_0}{2\pi}\right)^{\frac{3}{2}m} \left(\frac{\beta'}{2\pi}\right)^{-\frac{3}{2}} z' h_{\beta'}(p_j) \right] \prod_{j=1}^m \frac{16\pi z}{2e\beta^2} \left\{ \left(\frac{g\left(\frac{(j-1)}{m}\right)}{\left(h\left(\frac{(j-1)}{m}\right)\right)^2 \log\left(\frac{g\left(\frac{j}{m}\right)}{g\left(\frac{(j-1)}{m}\right)}\right)} \right) \right. \\
&\quad \left. \times \left(1 + \frac{\sqrt{\pi}}{4} \cdot \frac{g\left(\frac{(j-1)}{m}\right) \left(h\left(\frac{(j-1)}{m}\right)\right)^2 \log\left(\frac{g\left(\frac{j}{m}\right)}{g\left(\frac{(j-1)}{m}\right)}\right)}{\left(h\left(\frac{(j-1)}{m}\right)\right)^2 \left(1 - \frac{h\left(\frac{j}{m}\right)}{h\left(\frac{(j-1)}{m}\right)}\right)^{\frac{1}{2}} \left(\log\frac{g\left(\frac{j}{m}\right)}{g\left(\frac{(j-1)}{m}\right)}\right)^{\frac{1}{2}} g\left(\frac{(j-1)}{m}\right)} \right) \right\} \\
&= \left[\prod_{j=1}^n \left(\frac{\beta_0}{2\pi}\right)^{\frac{3}{2}m} \left(\frac{\beta'}{2\pi}\right)^{-\frac{3}{2}} z' h_{\beta'}(p_j) \right] \prod_{j=1}^m \frac{16\pi z}{2e\beta^2} \left\{ \left(\frac{g\left(\frac{(j-1)}{m}\right)}{\left(h\left(\frac{(j-1)}{m}\right)\right)^2 \log\left(\frac{g\left(\frac{j}{m}\right)}{g\left(\frac{(j-1)}{m}\right)}\right)} \right) \right. \\
&\quad \left. \times \left(1 + \frac{\sqrt{\pi}}{4} \cdot \frac{\left(h\left(\frac{(j-1)}{m}\right)\right)^{\frac{1}{2}} \left(\log\left(\frac{g\left(\frac{j}{m}\right)}{g\left(\frac{(j-1)}{m}\right)}\right)\right)^{\frac{1}{2}}}{\left(h\left(\frac{(j-1)}{m}\right) - h\left(\frac{j}{m}\right)\right)^{\frac{1}{2}}} \right) \right\}
\end{aligned}$$

\implies l.h.s. of (B.3)

$$\begin{aligned}
\text{(B.4)} \quad &\leq \left[\prod_{j=1}^n \left(\frac{\beta}{\beta'}\right)^{\frac{3}{2}} z' h_{\beta'}(p_j) \right] \sum_{m=1}^{\infty} \frac{t^m}{m!} \left(16a^2 z \left(\frac{\pi}{2\beta}\right)^{\frac{1}{2}}\right)^m \left(\frac{m}{e}\right)^m \\
&\quad \times \prod_{j=1}^m \left\{ \left(\frac{g\left(\frac{(j-1)}{m}\right)}{\left(h\left(\frac{(j-1)}{m}\right)\right)^2 m \log\left(\frac{g\left(\frac{j}{m}\right)}{g\left(\frac{(j-1)}{m}\right)}\right)} \right) \right. \\
&\quad \left. \times \left(1 + \frac{\sqrt{\pi}}{4} \cdot \left(\frac{\left(h\left(\frac{(j-1)}{m}\right)\right) m \left(\log\left(\frac{g\left(\frac{j}{m}\right)}{g\left(\frac{(j-1)}{m}\right)}\right)\right)}{m \left(h\left(\frac{(j-1)}{m}\right) - h\left(\frac{j}{m}\right)\right)} \right)^{\frac{1}{2}} \right) \right\}
\end{aligned}$$

The product grows for large m as D^m with

$$\log D = \int_0^1 dx \frac{g(x)^2}{h(x)^2 g'(x)} \left[1 + \frac{\sqrt{\pi}}{4} \left(\frac{h(x)g'(x)}{-h'(x)g(x)} \right)^{\frac{1}{2}} \right], \text{ because}$$

$$\left(\prod_{j=1}^m \{ \dots \} \right)^{\frac{1}{m}} \leq \frac{1}{m} \sum_{j=1}^m \{ \dots \} \rightarrow \int_0^1 \{ \dots \} \leq e^{\int_0^1 \{ \dots \}} = D$$

and because

$$m \log \frac{g\left(\frac{j}{m}\right)}{g\left(\frac{j-1}{m}\right)} \rightarrow g'(j), \text{ for } m \rightarrow \infty.$$

Therefore the sum converges provided

$$t < \left(16a^2 z D \left(\frac{\pi}{2\beta} \right)^{\frac{1}{2}} \right)$$

One can minimize D by choosing appropriate g and h . We will only optimize g and h in the form of the Ansatz

$$g(x) = (1-x)^{-\alpha}, h(x) = (1-x)^{\alpha'}, \quad \alpha, \alpha' > 0$$

This minimization gives us the bound on the radius of convergence. By scaling the radius of convergence can only depend on the ratio $\frac{\sqrt{\beta}}{a^2 z}$. \square

The particular ratio $\frac{\sqrt{\frac{m\beta}{3}}}{\pi a^2 z}$ has a simple physical meaning. $\sqrt{\frac{m\beta}{3}}$ is the average (root mean square) velocity of an ideal gas with Maxwellian velocity distribution h_β . πa^2 is the cross section of a hard sphere of diameter a and, hence, $(\pi a^2 z)^{-1}$ is the mean free path of a hard sphere with density z .

Therefore $\frac{\sqrt{\frac{m\beta}{3}}}{\pi a^2 z}$ is the mean free time, i.e. the average time between two collisions for a typical molecule. The net conclusion is then that the sum over collision histories converges, uniformly in ε , provided $t \leq \frac{1}{5}$ mean free time.

Why did we achieve only a finite radius of convergence? The reason is to be traced back to the estimation of the weight W^ε by its absolute value $|W^\varepsilon|$. On the level of Boltzmann equation this would correspond to replace the negative sign of the loss term by a plus sign. Since the collision term is quadratic, in essence one has the equation

$$\frac{d}{dt} x(t) = x(t)^2$$

whose solution diverges as $t \rightarrow 1_-$.

The finite radius of convergence and therefore the restriction to short times is the central unsolved problem in the derivation of the Boltzmann equation.

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