
Two proofs of the sharp Hardy-Littlewood-Sobolev inequality

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Declaration of Authorship

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

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Chapter 1

Introduction

The Hardy-Littlewood-Sobolev inequality in its general form states the following:

Theorem 1.1. *For every $0 < \lambda < d$ and for every $f \in L^p(\mathbb{R}^d)$ and $g \in L^q(\mathbb{R}^d)$ with $p, q > 1$ and $1/p + \lambda/d + 1/q = 2$ there exists a sharp constant $C(d, \lambda, p)$, such that*

$$\left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x) |x - y|^{-\lambda} g(y) dx dy \right| \leq C(d, \lambda, p) \|f\|_p \|g\|_q. \quad (1.1)$$

For arbitrary p and q an estimate of the upper bound for the constant $C(d, \lambda, p)$ was given by [Hardy-Littlewood] and [Sobolev], but till now no sharp value is known. In the special case $p = q$, however, the sharp constant was found by Lieb [Lieb, 1983] along with the optimising functions:

Theorem 1.2. *Under the conditions of Theorem 1.1 in case $p = q = 2d/(2d - \lambda)$*

$$C(d, \lambda, p) = C(d, \lambda) = \pi^{\lambda/2} \frac{\Gamma(d/2 - \lambda/2)}{\Gamma(d - \lambda/2)} \left(\frac{\Gamma(d/2)}{\Gamma(d)} \right)^{-1 + \lambda/d} \quad (1.2)$$

and the equality in (1.1) occurs if and only if $g(x) = c_1 f(x)$ and $f(x) = c_2 h(x/\mu^2 - a)$, where

$$h(x) = \left(\frac{1}{1 + |x|^2} \right)^{(2d-\lambda)/2}, \quad (1.3)$$

$a \in \mathbb{R}^d$, $c_1, c_2, \mu \in \mathbb{R} \setminus \{0\}$.

In this thesis we will go through two different proofs of the Theorem 1.2 - the sharp version of the HLS inequality. The first one was proposed by Lieb and exploits the invariance of the inequality under specific transformations. The second proof, valid only for the $\lambda = d - 2$, $d \geq 3$ case, exploits some interesting properties of a differential equation, describing fast diffusion, and is due to [Carlen-Carrillo-Loss]. We will start with the former, reproducing results from [Lieb-Loss].

Chapter 2

Invariance proof

2.1 Outline of the proof

We assume the general HLS inequality and proceed with the observation, that in the special case $p = q$ we may take $g(x) \equiv f(x)$ and look for the sharp bound for the following functional, which we logically call the HLS functional:

$$H(f) := \frac{\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x) |x - y|^{-\lambda} f(y) dx dy}{\|f\|_p^2}. \quad (2.1)$$

This functional has two interesting properties, on which the ongoing proof is based.

First of all it is invariant under the group of conformal transformations, i.e. applying specific conformal transformation to the function f does not change its value. Conformal transformations are by definition transformations that preserve angles between any two lines (if we have two curves in \mathbb{R}^d intersecting at some point then we look at the angle between its tangent vectors in this point). It is known (see, e.g., [Dubrovin-Fomenko-Novikov, Th.15.2]), that translations, rotations and reflections, scaling, inversion on the unit sphere and its combinations build the whole group of conformal transformations, so none of these will change anything in the HLS inequality. See section 2.3.

On the other hand there exist transformations, which change the value of the HLS functional. One of them is the symmetric-decreasing rearrangement - a transformation, which constructs from any function a symmetric-decreasing, i.e. radially symmetric ($|x| = |y| \Rightarrow f(x) = f(y)$) and nonincreasing one (the precise definition will be given in section 2.4). If we apply the rearrangement to the function f , the integral in the numerator of the functional will never decrease (this is the Riesz rearrangement inequality), but the norm of f will be preserved. It follows, that looking for a sharp constant, one can optimise within the class of symmetric-decreasing functions.

The main idea is to combine both properties. Take an arbitrary function $f \in L^p(\mathbb{R}^d)$ and rearrange it. The HLS functional will stay the same or get bigger. Now apply some conformal transformation, which output is no longer symmetric-decreasing (translation is one example, but another one will be used in the proof) - the functional will not change its value. Repeat

the procedure infinitely many times: at each step apply the rearrangement and then the same conformal transformation. If the functional explodes, then there exists no constant $C(d, \lambda, p)$ and no HLS inequality. In fact the functional never explodes, but converges in course of this procedure to some value that does not depend on f at all. This value is indeed the sharp constant $C(d, \lambda, p)$. It will be shown in section 2.5.

So the next 3 sections can be seen as preliminaries, and the proof starts from section 2.5. In the following we assume, that all functions are real-valued.

2.2 Positive definiteness of the functional

We answer the question, why looking for a sharp bound it is enough to consider only those functionals, where $g(x) \equiv f(x)$.

Proposition 2.1. *Let $0 < \lambda < d$, $p = 2d/(2d - \lambda)$ and $f, g \in L^p(\mathbb{R}^d)$. Define*

$$H'(f, g) := \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x) |x - y|^{-\lambda} g(y) dx dy.$$

Then $H'(f, f) \geq 0$ with equality if and only if $f \equiv 0$ and

$$|H'(f, g)|^2 \leq H'(f, f)H'(g, g) \tag{2.2}$$

with equality for $g \not\equiv 0$ if and only if $f = cg$, $c \in \mathbb{R}$.

Proof. Let $h \in C_c^\infty(\mathbb{R}^d)$ be radially symmetric with $h(x) \geq 0$ for all x . Define the convolution $k(x) := (h * h)(x) = \tilde{k}(|x|)$. Now redefine h multiplying it with a constant, so that $\int_0^\infty t^{\lambda-1} \tilde{k}(t) dt$ is normalised, and define a new integral $I(x) := \int_0^\infty t^{\lambda-1} k(tx) dt$. With $t' := t|x|$ we get

$$I(x) = \int_0^\infty t^{\lambda-1} |x|^{1-\lambda} k(t') \frac{1}{|x|} dt' = |x|^{-\lambda} \int_0^\infty t'^{\lambda-1} \tilde{k}(t') dt' = |x|^{-\lambda} \tag{2.3}$$

or $|x - y|^{-\lambda} = I(x - y)$.

We will also need another expression for $|x - y|^{-\lambda}$:

$$\begin{aligned} I(x - y) &= \int_0^\infty t^{\lambda-1} \int_{\mathbb{R}^d} h(tx - ty - z) h(z) dz dt \\ &= \int_0^\infty t^{d+\lambda-1} \int_{\mathbb{R}^d} h(t(x - y - z)) h(tz) dz dt \\ &= \int_0^\infty t^{d+\lambda-1} \int_{\mathbb{R}^d} h(t(z - y)) h(t(z - x)) dz dt, \end{aligned} \tag{2.4}$$

where in the third equality we redefine $z := x - z$ and use $h(x) = h(-x)$.

The integral can be now rewritten in the following way:

$$\begin{aligned}
H'(f, f) &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x) I(x-y) f(y) dx dy \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x) f(y) \left[\int_0^\infty t^{d+\lambda-1} \int_{\mathbb{R}^d} h(t(z-y)) h(t(z-x)) dz dt \right] dx dy \\
&= \int_0^\infty t^{\lambda-d-1} \left[\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} t^d f(x) t^d f(y) \int_{\mathbb{R}^d} h(t(z-y)) h(t(z-x)) dz dx dy \right] dt \\
&= \int_0^\infty t^{\lambda-d-1} \left[\int_{\mathbb{R}^d} t^d f(y) \int_{\mathbb{R}^d} h(t(z-y)) \int_{\mathbb{R}^d} t^d f(x) h(t(z-x)) dx dz dy \right] dt \\
&= \int_0^\infty t^{\lambda-d-1} \left[\int_{\mathbb{R}^d} g_t(z) \int_{\mathbb{R}^d} t^d f(y) h(t(z-y)) dy dz \right] dt \\
&= \int_0^\infty t^{\lambda-d-1} \int_{\mathbb{R}^d} |g_t(z)|^2 dz dt, \tag{2.5}
\end{aligned}$$

where $g_t(z) := \int_{\mathbb{R}^d} t^d f(x) h(t(z-x)) dx$ and Fubini in the lines 3, 4 and 5 is justified by the upper bound estimate in the general HLS inequality, i.e.

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |f(x)| |x-y|^{-\lambda} |f(y)| dx dy \leq C \|f\|_p^2 < \infty.$$

It now follows that $H'(f, f) \geq 0$.

For the case of equality assume $H'(f, f) = 0$. It implies $g_t(z) \equiv 0$ for a.e. $t \in (0, \infty)$. Note, that g_t can be rewritten as a convolution $h_t * f$, where $h_t(y) := t^d h(ty)$, so we have $h_t * f(z) \equiv 0$ for a.e. $t \in (0, \infty)$. Now we make use of the Approximation by C^∞ -functions theorem (see [Lieb-Loss, Th.2.16]). $h_t \in L^1(\mathbb{R}^d)$ and $I := \int_{\mathbb{R}^d} h_t(y) dy$ does not depend on t . $f \in L^p(\mathbb{R}^d)$ and we get

$$h_t * f \rightarrow If \text{ strongly in } L^p(\mathbb{R}^d) \text{ as } t \rightarrow \infty,$$

taking only those t for which $g_t \equiv 0$. From this we can conclude, that $f(x) = 0$ a.e. and $f \equiv 0$.

Now for $g \neq 0$ consider a function $\bar{f} := f - \frac{H'(f,g)}{H'(g,g)}g$, which is in $L^p(\mathbb{R}^d)$ as $\frac{H'(f,g)}{H'(g,g)}$ is a number. We compute

$$0 \leq H'(\bar{f}, \bar{f}) = H'(f, f) - 2 \frac{|H'(f, g)|^2}{H'(g, g)} + \frac{|H'(f, g)|^2}{H'(g, g)}$$

from which (2.2) follows. We have already proved that for equality $\bar{f} \equiv 0$ should hold, which implies $f = cg$ for some $c \in \mathbb{R}$. \square

Multiplying (2.2) with $(\|f\|_p \|g\|_p)^{-2}$ we get

$$\frac{|H'(f, g)|^2}{(\|f\|_p \|g\|_p)^2} \leq \frac{H'(f, f)}{\|f\|_p^2} \frac{H'(g, g)}{\|g\|_p^2} \tag{2.6}$$

and we may deduce, that looking for a sharp constant it is enough to study the HLS functional defined by (2.1), i.e. take $g(x) \equiv f(x)$.

2.3 Conformal invariance of the HLS functional

In this section we will discuss one by one all the conformal transformations and prove that the HLS functional is invariant under them, i.e. for every conformal γ we will define its action γ^* on an L^p -function f and prove that $H(\gamma^*f) = H(f)$. However we will start with simpler transformations and will generalise only in section 2.3.5. We will also study the stereographic projection, as it will be useful to perform some transformations on the projections of functions and then reproject them back to \mathbb{R}^d (actually $\mathbb{R}^d \cup \{\infty\}$).

Note, that all results in this section do not exploit $g(x) \equiv f(x)$ and are true for the general HLS inequality.

We define conformal transformations as transformations that preserve angles.

2.3.1 Euclidean group

The simplest conformal transformations are members of the Euclidean group, i.e. translations $(\tau_a f)(x) := f(x - a)$ and rotations and reflections $(Rf)(x) := f(R^{-1}x)$, where $a \in \mathbb{R}^d$ and $R \in O(d)$.

The invariance of Lebesgue integrals under translations is a theorem (see e.g. [Königsberger, p.252]), so both the integral and the norm do not change their value. To prove the invariance under R we use the substitution rule. $R \in O(d)$ has an inverse and for the integrable $f(x) |x - y|^{-\lambda} f(y)$

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x) |x - y|^{-\lambda} f(y) dx dy = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(R^{-1}x) |x - y|^{-\lambda} f(R^{-1}y) |\det R^{-1}|^2 dx dy,$$

because R does not change the distance between two points. With $\det R^{-1} = \pm 1$ we get the invariance of the integral. Analogously $\|f(x)\|_p = \|f(R^{-1}x)\|_p$.

2.3.2 Scaling

Scaling $S(x) := sx$ is also a conformal transformation. It is a simultaneous stretching in all directions, so it changes distances, but not angles. To prove this consider two differentiable curves $x(t)$ and $y(t)$, which intersect in some point, for simplicity at time 0: $x(0) = y(0) = z$. φ_{xy} , the angle between the curves in z is defined as an angle between its tangent lines at 0: $\dot{x}(0)$ and $\dot{y}(0)$. After scaling we get another curves $u(t) := sx(t)$ and $v(t) := sy(t)$, which also intersect at time 0. We want to prove, that the angle φ_{uv} between $\dot{u}(0)$ and $\dot{v}(0)$ stays the same.

First we compute

$$|\dot{u} - \dot{v}|(0) = \lim_{t \rightarrow 0} \frac{1}{t} |sx(t) - sz + sz - sy(t)| = s |\dot{x} - \dot{y}|(0)$$

It follows, in particular, $|\dot{u}| = s |\dot{x}|$; analogously $\dot{u} = s\dot{x}$, and

$$\cos \varphi_{uv} = \frac{\dot{u} \cdot \dot{v}}{|\dot{u}| |\dot{v}|} = \frac{\dot{x} \cdot \dot{y}}{|\dot{x}| |\dot{y}|} = \cos \varphi_{xy}.$$

To get invariance of the HLS functional it is not enough to scale the argument of f , we also need to multiply it with an appropriate constant. If we replace $f(x)$ by $s^{d/p}f(sx)$ the functional will not change its value:

$$\left\|s^{d/p}f(sx)\right\|_p^p = \int_{\mathbb{R}^d} \left|s^{d/p}f(sx)\right|^p = \int_{\mathbb{R}^d} |s|^d |f(x)|^p \frac{1}{|s^d|} dx = \|f(x)\|_p^p$$

and

$$\begin{aligned} & \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} s^{d/p}f(sx) |x-y|^{-\lambda} s^{d/p}f(sy) dx dy = \\ & \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} s^{d/p}f(x) \left|\frac{x}{s} - \frac{y}{s}\right|^{-\lambda} s^{d/p}f(y) \frac{1}{s^d} dx \frac{1}{s^d} dy = \\ & \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x) |x-y|^{-\lambda} f(y) dx dy, \end{aligned} \quad (2.7)$$

because $-2d + 2d/p + \lambda = 0$ from the assumption on p .

2.3.3 Inversion on the unit sphere

Inversion is defined as $I(x) := \frac{x}{|x|^2}$. In order to have the image of 0 we extend \mathbb{R}^d to $\mathring{\mathbb{R}}^d := \mathbb{R}^d \cup \{\infty\}$. ∞ is now an element of the space, it is defined as the element, which is contained in all unbounded open sets (so no $-\infty$ exist). We may now set $I(0) := \infty$ and $I(\infty) := 0$.

$I(x) : \mathring{\mathbb{R}}^d \rightarrow \mathring{\mathbb{R}}^d$ is a conformal transformation. To prove it we exploit the same technique as by scaling. In this case an additional condition is needed: the intersection point of the curves, z , is not the origin. The distance between two inversed points is

$$|I(x) - I(y)| = \left| \frac{x}{|x|^2} - \frac{y}{|y|^2} \right|^2 = \frac{1}{|x|^2} - \frac{2x \cdot y}{|x|^2 |y|^2} + \frac{1}{|y|^2} = \frac{1}{|x|^2} \frac{1}{|y|^2} |x - y|^2$$

The distance between two tangent vectors is

$$\begin{aligned} |\dot{u} - \dot{v}|(0) &= \lim_{t \rightarrow 0} \frac{1}{t} |I(x(t)) - I(z) + I(z) - I(y(t))| = \\ &= \lim_{t \rightarrow 0} \frac{1}{t} \frac{1}{|x(t)|} \frac{1}{|y(t)|} |x(t) - z + z - y(t)| = \frac{1}{|z|^2} |\dot{x} - \dot{y}|(0) \end{aligned}$$

so with $|\dot{u}| = |\dot{x}|/|z|^2$ and $|\dot{v}| = |\dot{y}|/|z|^2$ the angle does not change.

To prove invariance of the HLS functional under inversion it is useful to introduce the stereographic projection. If we lift a function $f \in \mathbb{R}^d$ to the unit sphere \mathbb{S}^d in \mathbb{R}^{d+1} with centre in the origin, the inversion will be just reflection along the \mathbb{R}^d -hyperplane.

2.3.4 Stereographic projection

For completeness we will stay in $\mathring{\mathbb{R}}^d$. Stereographic projection is a map $\mathcal{S} : \mathring{\mathbb{R}}^d \rightarrow \mathbb{S}^d$ with

$$s_i := \frac{2x_i}{1 + |x|^2} \quad \text{for } i = 1, \dots, d; \quad s_{d+1} := \frac{1 - |x|^2}{1 + |x|^2}. \quad (2.8)$$

Take some point $x \in \mathbb{R}^d$ and draw a line in \mathbb{R}^{d+1} through x and $(0, \dots, 0, -1)$ - the image of x will be its intersection point with \mathbb{S}^d . Clearly, if $|x| < 1$ the image is in the upper half of the sphere, if $|x| > 1$ - in the lower. The image of ∞ is $(0, \dots, 0, -1)$ itself.

We still have to prove, that \mathcal{S} is a map to \mathbb{S}^d :

$$\sum_{i=1}^{d+1} s_i^2 = \frac{4 \sum_{i=1}^d x_i^2}{(1 + |x|^2)^2} + \frac{1 - 2 \sum_{i=1}^d x_i^2 + \left(\sum_{i=1}^d x_i^2 \right)}{(1 + |x|^2)^2} = 1.$$

Now the inverse map $\mathcal{S}^{-1} : \mathbb{S}^d \rightarrow \mathbb{R}^d$ is given by

$$x_i := \frac{s_i}{1 + s_{d+1}} \quad \text{for } i = 1, \dots, d. \quad (2.9)$$

It is an interesting fact, that stereographic projection is also conformal. We use the standard Euclidean metric, so only small distances on \mathbb{S}^d make sense, but it is enough for the analysis of tangent vectors. We can calculate

$$|\mathcal{S}(x) - \mathcal{S}(y)| = \sum_{i=1}^{n+1} (\mathcal{S}(x)_i - \mathcal{S}(y)_i)^2 = \frac{4}{(1 + |x|^2)(1 + |y|^2)} |x - y|^2 \quad (2.10)$$

and

$$|\dot{u} - \dot{v}|(0) = \lim_{t \rightarrow 0} \frac{1}{t} \frac{2}{\sqrt{(1 + |x|^2)(1 + |y|^2)}} |x(t) - y(t)| = \frac{2}{1 + |z|^2} |\dot{x} - \dot{y}|(0),$$

so we again get the $|\dot{u}| = \text{const.} |\dot{x}|$ case.

The conformality of stereographic projection is useful in a sense, that it is possible to lift functions on \mathbb{R}^d to the sphere \mathbb{S}^d , perform some conformal transformation on the sphere and then go back to \mathbb{R}^d , getting this way a conformal transformation on \mathbb{R}^d . Rotations, reflections and inversion are easier to visualise on \mathbb{S}^d . Inversion, performed on \mathbb{R}^d , induces reflection of \mathbb{S}^d along the \mathbb{R}^d - coordinate hyperplane in \mathbb{R}^{d+1} . We will prove this fact, i.e that $SIS^{-1}(s) = (s_1, \dots, s_d, -s_{d+1})$:

$$\begin{aligned} |\mathcal{S}^{-1}(s)|^2 &= \sum_{i=1}^d \frac{s_i^2}{(1 + s_{d+1})^2} = \frac{1 - s_{d+1}^2}{(1 + s_{d+1})^2}, \\ IS^{-1}(s) &= \left(\frac{\mathcal{S}^{-1}(s)_1}{|\mathcal{S}^{-1}(s)|^2}, \dots, \frac{\mathcal{S}^{-1}(s)_d}{|\mathcal{S}^{-1}(s)|^2} \right) = \left(\frac{s_1(1 + s_{d+1})}{1 - s_{d+1}^2}, \dots, \frac{s_d(1 + s_{d+1})}{1 - s_{d+1}^2} \right), \\ |IS^{-1}(s)|^2 &= \sum_{i=1}^d \frac{s_i^2 (1 + s_{d+1})^2}{(1 - s_{d+1}^2)^2} = \frac{(1 + s_{d+1})^2}{(1 - s_{d+1}^2)} \end{aligned}$$

and finally

$$[SIS^{-1}(s)]_i = \begin{cases} \left(\frac{2s_i(1+s_{d+1})}{1-s_{d+1}^2} \right) \left(1 + \frac{(1+s_{d+1})^2}{(1-s_{d+1}^2)} \right)^{-1} = s_i, & i = 1, \dots, d \\ \left(1 - \frac{(1+s_{d+1})^2}{(1-s_{d+1}^2)} \right) \left(1 + \frac{(1+s_{d+1})^2}{(1-s_{d+1}^2)} \right)^{-1} = -s_{d+1}, & i = d + 1 \end{cases}.$$

2.3.5 Invariance under stereographic projection, inversion and the whole conformal group

We have already seen that in order to get invariance of the HLS functional under scaling it is not enough to apply transformation to the argument of the function. In general it is the substitution rule that instructs, how to perform a reasonable transformation. As by scaling the idea is to multiply function with a constant, which is exactly the determinant term from the substitution rule. This rule works for functions with compact support, so we will use approximations for transforms of L^p -functions.

Let $f^k \in L^p(\mathbb{R}^d)$ with support on B_k for $k = 1, 2, \dots$ be a sequence of functions converging to f in $L^p(\mathbb{R}^d)$. We define an action of some conformal transformation γ on f^k as

$$(\gamma^* f^k)(x) := |\det \mathcal{J}_{\gamma^{-1}}(x)|^{1/p} f^k(\gamma^{-1}x), \quad (2.11)$$

where \mathcal{J} is the Jacobian. The substitution rule ensures, that $\gamma^* f^k$ stays in $L^p(\mathbb{R}^d)$: we simply have $\|\gamma^* f^k\|_p = \|f^k\|_p$.

Now γ^* is linear, so $\|\gamma^* f^k - \gamma^* f\|_p = \|f^k - f\|_p \rightarrow 0$ as $k \rightarrow \infty$ and γ^* can be extended to the whole $L^p(\mathbb{R}^d)$.

In the case of stereographic projection we set $\gamma := \mathcal{S}$ and define

$$(\mathcal{S}^* f^k)(s) := |\det \mathcal{J}_{\mathcal{S}^{-1}}(s)|^{1/p} f^k(\mathcal{S}^{-1}s). \quad (2.12)$$

The Jacobi determinant can be computed from (2.9):

$$\frac{dx_i}{ds_j} = \frac{1}{1 + s_{d+1}} \delta_{ij} = \frac{1 + |x|^2}{2} \delta_{ij},$$

so $|\det \mathcal{J}_{\mathcal{S}^{-1}}| = |1 + s_{d+1}|^{-d}$. We will also need the inverse

$$ds = \left(\frac{2}{1 + |x|^2} \right)^d dx \quad (2.13)$$

to be ready to prove the invariance of the HLS functional under stereographic projection.

Proposition 2.2. *Let $0 < \lambda < d$, $p = 2d/(2d - \lambda)$, $f \in L^p(\mathbb{R}^d)$ and $F(s) = \mathcal{S}^* f(s)$. Then*

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x) |x - y|^{-\lambda} f(y) dx dy = \int_{\mathbb{S}^d} \int_{\mathbb{S}^d} F(s) |s - t|^{-\lambda} F(t) ds dt \quad (2.14)$$

and $\|F\|_p = \|f\|_p$.

Proof. With (2.12), (2.13) and (2.10) we get (2.14):

$$\begin{aligned} r.h.s. &= \int_{\mathbb{S}^d} \int_{\mathbb{S}^d} |\det \mathcal{J}_{\mathcal{S}^{-1}}(s)|^{1/p} f(\mathcal{S}^{-1}s) |s - t|^{-\lambda} |\det \mathcal{J}_{\mathcal{S}^{-1}}(t)|^{1/p} f(\mathcal{S}^{-1}t) ds dt \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left(\frac{1 + |x|^2}{2} \right)^{d/p} f(x) \left(\frac{4}{(1 + |x|^2)(1 + |y|^2)} |x - y|^2 \right)^{-\lambda/2} \\ &\quad \times \left(\frac{1 + |y|^2}{2} \right)^{d/p} f(y) \left(\frac{2}{1 + |x|^2} \right)^d dx \left(\frac{2}{1 + |y|^2} \right)^d dy \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left(\frac{(1 + |x|^2)(1 + |y|^2)}{4} \right)^{-d+d/p+\lambda/2} f(x) |x - y|^{-\lambda} f(y) dx dy = l.h.s. \end{aligned}$$

Further

$$\|F\|_p^p = \int_{\mathbb{S}^d} |\det \mathcal{J}_{\mathcal{S}^{-1}}(s)| f^p(\mathcal{S}^{-1}s) ds = \int_{\mathbb{R}^d} f^p(x) dx = \|f\|_p^p$$

□

With this invariance we actually get invariance under all isometries of \mathbb{S}^d , because the r.h.s. of (2.14) does not change if we apply some isometry ι : the Jacobi determinant of ι is ± 1 and $|\iota(s) - \iota(t)| = |s - t|$. In particular, the invariance of the HLS functional under inversion $I = \mathcal{S}^{-1} \circ (s_1, \dots, s_d, -s_{d+1}) \circ \mathcal{S}$ can be shown by applying one by one the Proposition 2.2, the isometric transformation and the Proposition 2.2 in the opposite direction.

We have now proven the invariance of the HLS functional under Euclidean transformations, scalings and inversion. It is a theorem, that any conformal transformation is a combination of these. See, e.g., [Dubrovin-Fomenko-Novikov, Th.15.2].

2.4 Symmetric-decreasing rearrangement and its consequence for $H(f)$

Another transformation will now be introduced. Most results appear without proofs, which can be found in [Lieb-Loss, Ch.3].

For every nonnegative measurable function $f \in \mathbb{R}^d$ there exists a layer cake representation, i.e. an expression in terms of the level sets:

$$f(x) = \int_0^\infty \chi_{\{f>t\}}(x) dt, \quad (2.15)$$

where χ is the characteristic function. This representation uses level sets $\{x : f(x) > t\}$ to define the function. The level sets can be rearranged in the following way.

Define the symmetric rearrangement of the set $X \subset \mathbb{R}^d$ as the open ball of the same volume with centre at the origin, i.e.

$$X^* := \{x : |x| < r\} \text{ with } (|\mathbb{S}^{d-1}|/d)r^d = \mathcal{L}^d(X), \quad (2.16)$$

where $|\mathbb{S}^{d-1}|$ is the surface area of a unit ball in \mathbb{R}^d . X has to be measurable with finite Lebesgue measure.

Now define the symmetric-decreasing rearrangement of a measurable function $f : \mathbb{R}^d \rightarrow \mathbb{R}^d$, that is vanishing at infinity, i.e. such, that all level sets of $|f| - \{x : |f(x)| > t\}$ - have finite measure:

$$\mathcal{R}f(x) := \int_0^\infty \chi_{\{|f|>t\}^*}(x) dt, \quad (2.17)$$

Note, that the level sets of $|f|$, a nonnegative function are rearranged.

What we get after rearrangement is a nonnegative function, which is radially symmetric and nonincreasing (call it symmetric-decreasing), i.e.

$$\begin{aligned} |x| = |y| &\Rightarrow \mathcal{R}f(x) = \mathcal{R}f(y) \\ |x| \geq |y| &\Rightarrow \mathcal{R}f(x) \geq \mathcal{R}f(y). \end{aligned}$$

This can be seen, if we rewrite

$$\chi_{\{|f|>t\}^*}(x) = \begin{cases} 1, & x \in \{z : |f(z)| > t\}^* \\ 0, & \text{otherwise} \end{cases} = \begin{cases} 1, & |x| < r(\mathcal{L}^d(z : |f(z)| > t)) \\ 0, & \text{otherwise} \end{cases},$$

where the function r for the radius has the property $r'_t(\mathcal{L}^d(z : |f(z)| > t)) > 0$.

We can now state

Proposition 2.3. *For any nonnegative $f \in L^p(\mathbb{R}^d)$*

$$H(f) \leq H(\mathcal{R}f). \quad (2.18)$$

To prove it we will need

Theorem 2.4 (Riesz's rearrangement inequality). *If for three nonnegative functions $f, g, h \in \mathbb{R}^d$*

$$I(f, g, h) := \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x)g(x-y)h(y)dx dy < \infty,$$

then $I(f, g, h) \leq I(\mathcal{R}f, \mathcal{R}g, \mathcal{R}h)$.

If g is strictly symmetric-decreasing, then there is equality only if $f(x) = \mathcal{R}f(x-a)$ and $h(x) = \mathcal{R}h(x-a)$ for some $a \in \mathbb{R}^d$.

Proof of the Proposition 2.3. The integral in the HLS inequality does not decrease under rearrangement due to the Theorem 2.4 and the equality $\mathcal{R}|x|^{-\lambda} = |x|^{-\lambda}$ for $\lambda > 0$. If we show, that the rearrangement preserves the norm of an L^p -function, (2.18) will be proved.

First notice, that the level sets of $\mathcal{R}f$ are the rearrangements of the level sets of f :

$$\{x : |f(x)| > t\} = \{x : \mathcal{R}f(x) > t\},$$

so $\mathcal{R}f$ and f are equimeasurable:

$$\mathcal{L}^d(\{x : |f(x)| > t\}) = \mathcal{L}^d(\{x : \mathcal{R}f(x) > t\}). \quad (2.19)$$

Now rewrite

$$\begin{aligned} \int_{\mathbb{R}^d} |f(x)|^p dx &= \int_{\mathbb{R}^d} \int_0^\infty \chi_{\{f^p>t\}}(x) dt dx \\ &= \int_0^\infty \mathcal{L}^d(\{x : f^p(x) > t\}) dt \\ &= \int_0^\infty \mathcal{L}^d(\{x : f(x) > t\}) pt^{p-1} dt = \dots = \\ &= \int_{\mathbb{R}^d} [\mathcal{R}f(x)]^p dx, \end{aligned}$$

where in the first step the layer cake representation, in the second one Fubini's theorem and in the last one (2.19) were used. \square

2.5 The proof of the Theorem 1.2

To summarise, it was shown, that the HLS functional is invariant under conformal transformations and nondecreasing under symmetric-decreasing rearrangement. It is therefore enough to look for the upper bound in the class of symmetric-decreasing functions. On the other hand this class being not closed under all conformal transformations is also too broad. We need such a symmetric-decreasing function, that its HLS functional does not change after further conformal transformations and rearrangements. [Carlen-Loss] proposed a transformation which helps to find such a function. We will use this transformation in the Theorem 2.6, which is essential for the Proposition 2.7.

We take $f \in L^p(\mathbb{R}^d)$, lift it to the sphere \mathbb{S}^d , rotate the sphere by 90° , mapping north pole $e_{d+1} = (0, \dots, 0, 1)$ into $e_d = (0, \dots, 0, 1, 0)$, and then push the function back to \mathbb{R}^d . The rotation is defined as

$$D : \mathbb{S}^d \rightarrow \mathbb{S}^d, \quad D(s) := (s_1, \dots, s_{d-1}, s_{d+1}, -s_d), \quad (2.20)$$

and the new function $\mathcal{S}^{*-1}D^*\mathcal{S}^*f$ can be computed in the following steps.

First define $F(s) := \mathcal{S}^*f$. From (2.12) we know, that

$$F(s) = \left(\frac{1 + |x|^2}{2} \right)^{d/p} f(x), \quad (2.21)$$

s and x being related as in (2.9). Now with (2.11) and (2.20)

$$(D^*F)(s) = |\det \mathcal{J}_{D^{-1}}(s)|^{1/p} F(D^{-1}s) = F(D^{-1}s). \quad (2.22)$$

We will need several steps to find $F(D^{-1}s)$:

$$\begin{aligned} D^{-1}s &= D^{-1}\mathcal{S}(x) = (s_1, \dots, s_{d-1}, -s_{d+1}, s_d) = \\ &= \left(\frac{2x_1}{1 + |x|^2}, \dots, \frac{2x_{d-1}}{1 + |x|^2}, \frac{|x|^2 - 1}{1 + |x|^2}, \frac{2x_d}{1 + |x|^2} \right) \\ [\mathcal{S}^{-1}D^{-1}\mathcal{S}(x)]_i &= \begin{cases} \frac{\frac{2x_i}{1 + |x|^2}}{1 + \frac{2x_d}{1 + |x|^2}} = \frac{2x_i}{1 + 2x_d + |x|^2} = \frac{2x_i}{|x + e_d|^2}, & i = 1, \dots, d - 1 \\ \frac{\frac{|x|^2 - 1}{1 + |x|^2}}{1 + \frac{2x_d}{1 + |x|^2}} = \frac{|x|^2 - 1}{|x + e_d|^2}, & i = d \end{cases}, \end{aligned}$$

where e_d is now the unit vector $(0, \dots, 0, 1)$ in d dimensions. Going forth,

$$\begin{aligned} |\mathcal{S}^{-1}D^{-1}\mathcal{S}(x)|^2 &= \frac{4 \sum_{i=1}^{d-1} x_i^2 + |x|^4 - 2|x|^2 + 1}{|x + e_d|^4} = \frac{|x|^4 + 2|x|^2 + 1 - 4x_d^2}{|x + e_d|^4} \\ &= \frac{(|x|^2 + 2x_d + 1)(|x|^2 - 2x_d + 1)}{|x + e_d|^4} = \frac{|x|^2 - 2x_d + 1}{|x + e_d|^2}, \\ \frac{1 + |\mathcal{S}^{-1}D^{-1}\mathcal{S}(x)|^2}{2} &= \frac{|x|^2 + 1}{|x + e_d|^2} \end{aligned} \quad (2.23)$$

and finally

$$F(D^{-1}s) = \left(\frac{|x|^2 + 1}{|x + e_d|^2} \right)^{d/p} f \left(\frac{2x_1}{|x + e_d|^2}, \dots, \frac{2x_{d-1}}{|x + e_d|^2}, \frac{|x|^2 - 1}{|x + e_d|^2} \right).$$

The last step is pushing back to \mathbb{R}^d . We will again make use of (2.12) or directly (2.21):

$$(\mathcal{S}^{*-1}D^*\mathcal{S}^*f)(x) = \left(\frac{2}{|x + e_d|^2} \right)^{d/p} f \left(\frac{2x_1}{|x + e_d|^2}, \dots, \frac{2x_{d-1}}{|x + e_d|^2}, \frac{|x|^2 - 1}{|x + e_d|^2} \right). \quad (2.24)$$

For simplicity we will call $\mathcal{S}^{*-1}D^*\mathcal{S}^*f$ just $\mathcal{D}f$.

We now ask, whether $\mathcal{D}f$ stays symmetric-decreasing, if f is such. The following is stated for radially symmetric functions (see the definition in section 2.4) and gives us the necessary condition.

Proposition 2.5. *Let $f \in L^p(\mathbb{R}^d)$ be radially symmetric. Then $\mathcal{D}f$ defined by (2.24) is radially symmetric if and only if*

$$f(x) = C(1 + |x|^2)^{-d/p}, \quad (2.25)$$

for some $C \in \mathbb{R}$.

Proof. Consider $F(s) = \mathcal{S}^*f(x)$. As $f(x)$ is radially symmetric, $F(s)$ is invariant under any rotation, which keeps the north pole axis e_{d+1} fixed, so it is a function of s_{d+1} only, i.e.

$$F(s) = \phi(s_{d+1}) \quad \text{for some } \phi : \mathbb{S}^d \rightarrow \mathbb{R}.$$

$F(D^{-1}s)$ is then invariant under any rotation, which keeps the e_d -axis fixed. If we want $F(D^{-1}s)$ still to be invariant under rotations keeping the north pole axis e_{d+1} fixed (which would imply that $\mathcal{D}f$ is radially symmetric), then the following should hold:

$$F(D^{-1}s) = \psi(s_{d+1}) \quad \text{for some } \psi : \mathbb{S}^d \rightarrow \mathbb{R}.$$

Together it would imply

$$\phi(s_{d+1}) = F(s) = \psi((Ds)_{d+1}) = \psi(-s_d) \quad \text{for all } s \in \mathbb{S}^d,$$

which is only possible, if ϕ , ψ and then also F are constant on \mathbb{S}^d . From (2.21) we get, that $\mathcal{D}f$ stays radially symmetric only for functions of the form (2.25). \square

2.5.1 Convergence of $f \in L^p$ under \mathcal{RD}

We now want to find out, what happens with an arbitrary function f after applying to it the transformation \mathcal{D} and the rearrangement \mathcal{R} many times. It turns out, that $(\mathcal{RD})^k f$ always converges, and the HLS functional, computed for the limit function, gives us the sharp bound. We start with the convergence.

Theorem 2.6 (Competing symmetries). *For $1 < p < \infty$ let $f \in L^p(\mathbb{R}^d)$ be any nonnegative function. Then the sequence $f^k := (\mathcal{RD})^k f$ converges in L^p as $k \rightarrow \infty$ to the function $h_f := \|f\|_p h$, where*

$$h(x) = |\mathbb{S}^d|^{-1/p} \left(\frac{2}{1 + |x|^2} \right)^{d/p}, \quad (2.26)$$

and $|\mathbb{S}^d|$ is the area of the sphere with radius 1 in \mathbb{R}^{d+1} .

Proof. We will prove the theorem for the bounded functions that vanish outside a bounded set. These are dense in L^p and it suffices to show the statement for a dense set.

The density can be shown using dominated convergence theorem. Take some $f \in L^p(\mathbb{R}^d)$. Define $f_n := f^p \chi_{f(x) \leq n} \chi_{B(o,n)}(x)$. f_n converges pointwise towards f^p and lies in L^1 , as $\|f_n\|_1 \leq \|f^p\|_1 < \infty$; f_n is dominated by an L^1 function, namely f^p itself. It follows, that $\|f^p - f_n\|_1 \rightarrow 0$ and also $\left\|f - f_n^{1/p}\right\|_p \rightarrow 0$, where $f_n^{1/p}$ is still bounded and vanishing outside a bounded set.

Now suppose, that we have already proven the theorem for some dense set of functions in $L^p(\mathbb{R}^d)$. Fix $\varepsilon > 0$ and for an arbitrary $f \in L^p(\mathbb{R}^d)$ find $g \in L^p(\mathbb{R}^d)$ from the dense set, such that $\|f - g\|_p < \varepsilon/3$. It follows, that

$$\|h_f - h_g\|_p = \left|(\|f\|_p - \|g\|_p)\right| \|h\|_p < \varepsilon/3,$$

because

$$\|h\|_p = 1,$$

and

$$\left\|g^k - f^k\right\|_p \leq \|g - f\|_p < \varepsilon/3$$

for every $k \in \mathbb{N}$, because of the nonexpansivity of rearrangement $\|\mathcal{R}f - \mathcal{R}g\|_p \leq \|f - g\|_p$ (see [Lieb-Loss, Th.3.5]) and the equality $\|\mathcal{D}f - \mathcal{D}g\|_p = \|f - g\|_p$ due to the linearity and norm-preserving property of the conformal transformation (see 2.3.5). Further for sufficiently large k it holds $\|h_g - g^k\|_p < \varepsilon/3$ and with the triangle inequality we get

$$\left\|h_f - f^k\right\|_p \leq \|h_f - h_g\|_p + \|h_g - g^k\|_p + \left\|g^k - f^k\right\|_p < \varepsilon.$$

So let $f \in L^p(\mathbb{R}^d)$ be bounded and vanishing outside a bounded set. Then there exists such a constant C , that

$$f(x) \leq Ch_f(x) \quad \text{for a. e. } x \in \mathbb{R}^d,$$

because $h_f > 0$. The same relation with the same constant C holds for every f^k :

$$f^k(x) \leq Ch_f(x) \quad \text{for a. e. } x \in \mathbb{R}^d. \quad (2.27)$$

To prove this, notice that both \mathcal{D} and \mathcal{R} are order-preserving. If $f(x) \leq g(x)$, then the level sets of f are contained in the level sets of g , which implies that the level sets of $\mathcal{R}f$ are contained in the level sets of $\mathcal{R}g$, i.e. $\mathcal{R}f(x) \leq \mathcal{R}g(x)$. Also for $f(x) \leq g(x)$ for every x it holds $F(D^{-1}s) \leq G(D^{-1}s)$ for every s (with $G(D^{-1}s) := D^*S^*g(x)$) and $(\mathcal{D}f)(x) \leq (\mathcal{D}g)(x)$. 2.27 follows now from the fact, that h_f is invariant under \mathcal{D} : with (2.23)

$$(\mathcal{D}h_f)(x) = \left(\frac{2}{|x + e_d|^2}\right)^{d/p} |\mathbb{S}^d|^{-1/p} \left(\frac{|x + e_d|^2}{|x|^2 + 1}\right)^{d/p} = h_f(x)$$

and under rearrangement as a symmetric-decreasing function.

The f^k 's are therefore uniformly bounded. Then there exists a subsequence f^{k_l} , that converges pointwise a.e. to some function g . This can be found by the Helly's selection principle as follows:

Take first all the rational points on the support of f^k 's - there are countably many. Find a subsequence, which converges in the first rational point, then a subsubsequence, which converges in the first two rational points and so on. The resulting f^{k_l} will converge in all rational points; we have to prove that it will also converge in almost all irrational points. Let y be some irrational point, in which all f^{k_l} 's are continuous. Since f^{k_l} 's are symmetric-decreasing, there exist two sequences x_{m_l} and x_{n_l} of rational points such that $f^{k_l}(x_{m_l}) \leq f^{k_l}(y) \leq f^{k_l}(x_{n_l})$ holds for every l . Now for $l \rightarrow \infty$ both $f^{k_l}(x_{m_l})$ and $f^{k_l}(x_{n_l})$ converge, so by sandwich lemma $f^{k_l}(y)$ converges as well. Finally, the set of all points, where f^{k_l} 's are discontinuous, has zero measure. To see this consider first one symmetric-decreasing function f^{k_i} and the discontinuity points of the form $x_a := (a, 0, \dots, 0)$ with $a \in \mathbb{R}^n$. There are at most countably many, because for every such x_a there exists a rational number $r(x_a)$ with $\lim_{x \rightarrow x_a^-} f^{k_i}(x) < r(x_a) < \lim_{x \rightarrow x_a^+} f^{k_i}(x)$ (the argument stems from [Rudin, Th.4.30]). It follows that the measure of all discontinuity points of f^{k_i} is zero and since there are countably many f^{k_l} 's, all their discontinuity points have zero measure.

So there exists a symmetric-decreasing function g with $g(x) = \lim_{l \rightarrow \infty} f^{k_l}(x)$ for a.e. x . With the dominated convergence theorem it follows from $(f^k)^p(x) \leq C^p h_f^p(x)$ and $(f^k)^p, h_f^p \in L^1(\mathbb{R}^d)$, that $g \in L^p(\mathbb{R}^d)$. Now we want to show, that $g = h_f$.

Using once again $\|\mathcal{R}f - \mathcal{R}g\|_p \leq \|f - g\|_p$, $\|\mathcal{D}f - \mathcal{D}g\|_p = \|f - g\|_p$ and the invariance of h_f under both \mathcal{R} and \mathcal{D} we may state:

$$\begin{aligned} \inf_k \|h_f - f^k\|_p &= \lim_{k \rightarrow \infty} \|h_f - f^k\|_p = \lim_{l \rightarrow \infty} \|h_f - f^{k_l}\|_p \\ &= \lim_{l \rightarrow \infty} \|h_f - f^{k_l+1}\|_p = \|h_f - \mathcal{R}\mathcal{D}g\|_p \\ &= \|\mathcal{R}\mathcal{D}h_f - \mathcal{R}\mathcal{D}g\|_p \leq \|\mathcal{D}h_f - \mathcal{D}g\|_p \\ &= \|h_f - g\|_p = \lim_{l \rightarrow \infty} \|h_f - f^{k_l}\|_p \\ &= \inf_k \|h_f - f^k\|_p, \end{aligned} \tag{2.28}$$

where the trick comes up in the second line - $\mathcal{R}\mathcal{D}$ is applied once to g .

It is clear that an equality should hold in (2.28). In particular

$$\|h_f - \mathcal{R}\mathcal{D}g\|_p = \|\mathcal{D}h_f - \mathcal{D}g\|_p = \|h_f - \mathcal{D}g\|_p,$$

from which follows, that $\mathcal{R}\mathcal{D}g = \mathcal{D}g$ (this is the equality case in the nonexpansivity of rearrangement theorem, see [Lieb-Loss, Th.3.5]), i.e. $\mathcal{D}g$ is symmetric-decreasing. So both g and $\mathcal{D}g$ are symmetric-decreasing, and as we know from Proposition 2.5 this is only possible, if $g = \mathcal{C}h$. Since

$$\|g\|_p = \lim_{k \rightarrow \infty} \|f^k\|_p = \|f\|_p,$$

we have $C = \|f\|_p$ and $g = h_f$. Now, extracting from (2.28),

$$\lim_{k \rightarrow \infty} \|h_f - f^k\|_p = \|h_f - g\|_p = 0,$$

i.e. f^k converges in L^p to h_f as $k \rightarrow \infty$. \square

2.5.2 Sharp HLS inequality and the optimisers

The sharp HLS inequality can be seen as a corollary of the Theorem 2.6 or a direct consequence of the

Proposition 2.7. *For every $f \in L^p(\mathbb{R}^d)$ the sequence $H((\mathcal{RD})^k f)$ converges to $H(h_f)$ from below.*

Proof. It follows from our definition of rearrangement, that $\mathcal{RD}f$ is nonnegative for any $f \in L^p(\mathbb{R}^d)$. Then with Theorem 2.6 the sequence $f^k = (\mathcal{RD}f)^{k-1}$ converges in $L^p(\mathbb{R}^d)$ as $k \rightarrow \infty$.

Define $f_m(x) := \min\{f(x), mh_f(x)\}$. For a fixed m , f_m^k converges in $L^p(\mathbb{R}^d)$ to h_f as $k \rightarrow \infty$ and there exists a subsequence $f_m^{k_l}$, that converges to h_f pointwise. The uniform bound $f_m^{k_l} \leq C(1 + |x|^2)^{-d/p}$ from the Theorem 2.6 and the general HLS inequality allow us to use dominated convergence to prove that $H(f_m^{k_l})$ converges to $H(h_f)$ (since the norm in the denominator of H stays unchanged under \mathcal{RD}).

Now f_m 's converge monotonically pointwise to $f(x)$ as $m \rightarrow \infty$. By monotone convergence theorem $H(f_m) \rightarrow H(f)$ and the upper bound of $H(f_m)$ is also the upper bound of $H(f)$, i.e.

$$H(f) \leq H(h_f) \text{ for every } f \in L^p(\mathbb{R}^d).$$

\square

One gets (1.2) after computing $H(h_f)$.

Finally we return to the very beginning, to the $p = q$ case of the general HLS inequality, to determine the case of equality. From Proposition 2.1 it is known that $g = \text{const.} \cdot f$ should hold. For optimising f the necessary condition is $H(f) = H(h)$, so any conformal transformation of h - and h itself - is an optimiser. Are there any other?

The answer is no. If f is an optimiser and the rearrangement of f does not improve the value of the functional, then by the Riesz rearrangement inequality f must be a translate of some symmetric-decreasing function and \mathcal{R} - doing nothing but translating f to the origin - act as a conformal transformation. So after a (infinite) series of conformal transformations $\gamma := \mathcal{RD}$ any optimiser f converges to $\text{const.} \cdot h_f$, from which should follow that the optimiser itself is a conformal transformation of h_f .

Proposition 2.8. *For $p = q = 2d/(2d - \lambda)$ there is equality in (1.1) with the constant (1.2) if and only if $g(x) = cf(x)$ for some $c \in \mathbb{R} \setminus \{0\}$ and*

$$f(x) = \left| \mathbb{S}^d \right|^{-1/p} \|f\|_p \lambda^{d/p} \left(\frac{2}{\lambda^2 + |x - a|^2} \right)^{d/p} \quad (2.29)$$

for some $\lambda \neq 0$ and $a \in \mathbb{R}^d$.

Proof. First we prove, that for any conformal γ there exist such $\lambda \neq 0$ and $a \in \mathbb{R}^d$, that $(\gamma^* h_f)(x)$ is equal to the r.h.s. of (2.29). Clearly it holds for h_f itself. Now assume, that it holds for some $(\gamma^* h_f)(x)$. It is enough to prove, that it will stay true after we apply to $(\gamma^* h_f)(x)$ some Euclidean transformation, scaling or inversion.

This is obvious for scaling. For the Euclidean transformation E we get with (2.11)

$$\begin{aligned} (E^*(\gamma^*h_f))(x) &= 1 \cdot |\mathbb{S}^d|^{-1/p} \|f\|_p \lambda^{d/p} \left(\frac{2}{\lambda^2 + |R^{-1}x - a_1 - a_2|^2} \right)^{d/p} \\ &= |\mathbb{S}^d|^{-1/p} \|f\|_p \lambda^{d/p} \left(\frac{2}{\lambda^2 + |R^{-1}(x - R(a_1 + a_2))|^2} \right)^{d/p} \\ &= |\mathbb{S}^d|^{-1/p} \|f\|_p \lambda^{d/p} \left(\frac{2}{\lambda^2 + |x - R(a_1 + a_2)|^2} \right)^{d/p}, \end{aligned}$$

since $|R^{-1}x| = |x|$. For inversion

$$\begin{aligned} (I^*(\gamma^*h_f))(x) &= |x|^{-2d/p} |\mathbb{S}^d|^{-1/p} \|f\|_p \lambda^{d/p} \left(\frac{2}{\lambda^2 + \left| \frac{x}{|x|^2} - a \right|^2} \right)^{d/p} \\ &= |\mathbb{S}^d|^{-1/p} \|f\|_p \lambda^{d/p} \left(\frac{2}{\lambda^2 |x|^2 + |1 - 2bx + b^2|x|^2|} \right)^{d/p} \\ &= |\mathbb{S}^d|^{-1/p} \|f\|_p \left(\frac{\lambda}{\lambda^2 + b^2} \right)^{d/p} \left(\frac{2}{\left[\frac{\lambda}{\lambda^2 + b^2} \right]^2 + \left| x - \frac{b}{\lambda^2 + b^2} \right|^2} \right)^{d/p}. \end{aligned}$$

Now from the previous considerations $\gamma := \mathcal{RD}$ acts on optimisers as a conformal transformation. In particular it is an isometry and since $f^k := \gamma^k f$ converges strongly to h_f , it holds

$$\lim_{k \rightarrow \infty} \left\| f - (\gamma^{-1})^k h_f \right\|_p = 0$$

for any optimiser f . With

$$(\gamma^{-1})^k h_f = |\mathbb{S}^d|^{-1/p} \|f\|_p \lambda_k^{d/p} \left(\frac{2}{\lambda_k^2 + |x - a_k|^2} \right)^{d/p}.$$

it follows from the strong convergence that λ_k and a_k converge as $k \rightarrow \infty$ to some $\lambda \neq 0$ and $a \in \mathbb{R}^d$ respectively and f satisfies (2.29). \square

Simplifying (2.29), we get the equality condition, stated in the introduction.

Chapter 3

Fast diffusion equation proof

[Carlen-Carrillo-Loss] consider the $\lambda = d - 2$, $d \geq 3$, case of the sharp HLS inequality (Theorem 1.2), because it expresses the L^p smoothing properties of $(-\Delta)^{-1}$ - the solution of the Poisson equation: the HLS inequality can be rewritten as $F(f) \geq 0$, where

$$F(f) := C_S \|f\|_{2d/(d+2)}^2 - \int_{\mathbb{R}^d} f(x)[(-\Delta)^{-1}f](x)dx \quad (3.1)$$

with

$$C_S := \frac{4}{d(d-2)} |\mathbb{S}^d|^{-2/d} \quad (3.2)$$

and as before due to the positive definiteness of the integral we consider only the $g = f$ case of the HLS inequality (see section 2.2). We will now call F the HLS functional.

It turns out, that the optimisers of the sharp HLS inequality are attracting steady states for a flow, generated by a certain fast diffusion equation (FDE), i.e. the equation

$$\frac{\partial u}{\partial t}(x, t) = \Delta u^m(x, t)$$

for some $0 < m < 1$ and some suitable initial conditions.

More than that, the HLS functional F monotonically decreases along this flow. As shown in section 3.2, the time derivative of F can be rewritten in a form, which makes clear, that its nonpositivity follows from a certain sharp Gagliardo-Nirenberg-Sobolev (GNS) inequality - an inequality similar to the HLS. From that form an expression using GNS can be found for F itself. The sharp HLS inequality with $\lambda = d - 2$ follows then from the sharp GNS inequality.

However the sharp GNS inequality is only an intermediate step in the proof. In section 3.3 we will show, how it can be proved using another FDE. We will take a functional, which is monotonically decreasing along the gradient flow of this new FDE, compute its second time derivative and apply the Cauchy-Schwarz inequality to the resulting expression. After some transformations it will be equivalent to the sharp GNS inequality.

As in the first proof we assume, that all functions are real-valued, and make use of the general HLS inequality.

3.1 Preliminaries

3.1.1 The $\lambda = d - 2$ case of the sharp HLS inequality

The solution of the Poisson equation $-\Delta u = f$ has the form (see [Evans, p.23])

$$u(x) = [(-\Delta)^{-1}f](x) = \frac{1}{(d-2)|\mathbb{S}^{d-1}|} \int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^{d-2}} dy.$$

Thus the HLS inequality (1.1) for $f = g$ can be rewritten as

$$(d-2) \left| \mathbb{S}^{d-1} \right| \int_{\mathbb{R}^d} f(x)[(-\Delta)^{-1}f](x) dx \leq C(d, d-2) \|f\|_{2d/(d+2)}^2, \quad (3.3)$$

where $C(d, d-2)$ is defined by (1.2). We now show the equivalence of (3.3) and $F(f) \geq 0$, i.e. we prove

$$\begin{aligned} \frac{C(d, d-2)}{(d-2)|\mathbb{S}^{d-1}|} &= \frac{\pi^{(d-2)/2} \frac{\Gamma(1)}{\Gamma((d+2)/2)} \left[\frac{\Gamma(d/2)}{\Gamma(d)} \right]^{-2/d}}{(d-2) \frac{2\pi^{d/2}}{\Gamma(d/2)}} \\ &= \frac{1}{2\pi(d-2)} \frac{\Gamma(d/2)}{\Gamma(d/2)(d/2)} \left[\frac{\Gamma(d)}{\Gamma(d/2)} \right]^{2/d} \\ &= \frac{1}{\pi d(d-2)} \left[\frac{\Gamma(d/2)}{\Gamma(d)} \right]^{-2/d} = \frac{1}{\pi d(d-2)} \left[\frac{2^{1-d}\sqrt{\pi}}{\Gamma((d+1)/2)} \right]^{-2/d} \\ &= \frac{4}{d(d-2)} \left[\frac{2\pi^{(d+1)/2}}{\Gamma((d+1)/2)} \right]^{-2/d} = C_S, \end{aligned}$$

where in the third line the multiplication theorem for Gamma functions (see, e.g., [Handbook, p.256]) was used.

Note, that for $\lambda = d - 2$ it holds $p = 2d/(d+2)$ and the power in the optimiser is $(2d - \lambda)/2 = (d+2)/2$.

We now turn us to the objects, which we will use throughout the proof.

3.1.2 The sharp Gagliardo-Nirenberg-Sobolev inequality

This inequality can be found in [Del Pino-Dolbeault, Th.1] and it states, that for every $d \geq 3$, $1 < p \leq d/(d+2)$ and for every locally integrable function f on \mathbb{R}^d with $\nabla f \in L^2(\mathbb{R}^d)$ it holds

$$\|f\|_{2p} \leq C_{GNS}(d, p) \|\nabla f\|_2^\theta \|f\|_{p+1}^{1-\theta}, \quad (3.4)$$

where

$$C_{GNS}(d, p) = \left(\frac{\zeta(p-1)^2}{2\pi d} \right)^{\theta/2} \left(\frac{2\zeta - d}{2\zeta} \right)^{1/2p} \left(\frac{\Gamma(\zeta)}{\Gamma(\zeta - d/2)} \right)^{\theta/d} \quad (3.5)$$

with

$$\zeta = \frac{p+1}{p-1}, \quad \theta = \frac{d(p-1)}{p(d+2 - (d-2)p)}$$

and the equality in (3.4) occurs if and only if $f(x) = c_1 h_1(x/\mu^2 - a)$, where

$$h_1(x) = \left(\frac{1}{1 + |x|^2} \right)^{1/(p-1)} \quad (3.6)$$

with $a \in \mathbb{R}^d$, $c_1, \mu \in \mathbb{R} \setminus \{0\}$.

Note, that the optimisers of the GNS inequality are some powers of the optimisers of the HLS inequality. The reason for it is that both inequalities are somehow related to the fast diffusion equation, which we will now discuss.

3.1.3 The fast diffusion equation

The equation

$$\frac{\partial u}{\partial t}(x, t) = \Delta u^m(x, t) \quad (3.7)$$

with $x \in \mathbb{R}^d$ is called the FDE if $0 < m < 1$. In case $m = 1$ it is the well-known heat equation, in case $m > 1$ it is called the porous medium equation (PME). The PME was thoroughly studied and the results were well structured by [Vázquez]. Results about FDE are distributed among many articles. We will mention those relevant for the proof.

Proposition 3.1. *A function $u(x, t)$ is the solution of (3.7) if and only if*

$$v(x, t) := e^{td} u(e^t x, e^{\beta t}) \quad (3.8)$$

with $\beta = 2 - d(1 - m)$ satisfies the equation

$$\frac{\partial v}{\partial t}(x, t) = \beta \Delta v^m(x, t) + \nabla \cdot [xv(x, t)]. \quad (3.9)$$

Proof. We compute both sides of (3.9) using (3.8):

$$\frac{\partial v}{\partial t}(x, t) = de^{td} u(e^t x, e^{\beta t}) + e^{td} \left[\nabla u(e^t x, e^{\beta t}) \cdot e^t x + \frac{\partial u}{\partial t}(e^t x, e^{\beta t}) \beta e^{\beta t} \right]$$

and

$$\begin{aligned} r.h.s. &= \beta e^{(md+2)t} \Delta u^m(e^t x, e^{\beta t}) + \sum_{i=1}^d \frac{\partial [x_i v(x, t)]}{\partial x_i} \\ &= \beta e^{(d+\beta)t} \Delta u^m(e^t x, e^{\beta t}) + de^{td} u(e^t x, e^{\beta t}) + e^{td} \sum_{i=1}^d x_i \frac{\partial u}{\partial x_i}(e^t x, e^{\beta t}) \\ &= de^{td} u(e^t x, e^{\beta t}) + e^{t(d+1)} x \cdot \nabla u(e^t x, e^{\beta t}) + \beta e^{(d+\beta)t} \Delta u^m(e^t x, e^{\beta t}). \end{aligned}$$

It follows, that (3.9) holds under (3.8) if and only if $u(x, t)$ is the solution of (3.7). □

As stated in [Carlen-Carrillo-Loss], for all $1 - 2/d < m < 1$ equation (3.8) has integrable stationary solutions of the form

$$v_{\infty, M}(x) := \left(\frac{1}{D(M) + \frac{1-m}{2\beta m} |x|^2} \right)^{1/(1-m)}, \quad (3.10)$$

where $D(M)$ is a parameter, depending on the mass

$$M := \int_{\mathbb{R}^d} v_{\infty, M}(x) dx. \quad (3.11)$$

Its precise value is now irrelevant and will be computed later, see (3.82).

The solution $v_{\infty, M}$ is called the Barenblatt profile (after G.I. Barenblatt, who proposed it in [Barenblatt]). The solution of the FDE (3.7) can be then computed from the Barenblatt profile by the change of variables (3.8):

$$u(x, e^{\beta t}) = \frac{1}{e^{td}} \left(\frac{B}{\frac{|x|^2}{e^{2t}} + BD(M)} \right)^{1/(1-m)},$$

where $B = \frac{2\beta m}{1-m}$ and

$$\begin{aligned} u(x, t) &= \frac{1}{t^{d/\beta}} \left(\frac{Bt^{2/\beta}}{|x|^2 + BD(M)t^{2/\beta}} \right)^{1/(1-m)} \\ &= \left(\frac{Bt}{|x|^2 + BD(M)t^{2/\beta}} \right)^{1/(1-m)}, \end{aligned} \quad (3.12)$$

since $\frac{2}{\beta} - \frac{d}{\beta}(1-m) = 1$. The expression (3.12) is known as the Barenblatt solution.

In course of the proof we will need some estimates. Let again $v(x, t)$ be the (3.8) transformation of the FDE solution $u(x, t)$. It is known that

$$\lim_{t \rightarrow \infty} \|v(\cdot, t) - v_{\infty, M}\|_1 = 0. \quad (3.13)$$

In [Blanchet et al., Th.4] the uniform C^k regularity of the quotient $\frac{v(x, t)}{v_{\infty, M}}$ is shown:

$$\sup_{t > t_*} \left\| \frac{v(x, t)}{v_{\infty, M}} \right\|_{C^k(\mathbb{R}^d)} < \infty \quad (3.14)$$

for every $t_* > 0$ and all $k \in \mathbb{N}$.

Further, basing on [Bonforte-Vázquez, Th.1.2], we will prove the following

Proposition 3.2. *Let the nonnegative initial data $f \in L^1(\mathbb{R}^d)$ of mass M satisfy*

$$\sup_{|x| > R} f(x) |x|^{2/(1-m)} < \infty \quad (3.15)$$

for some $R > 0$. Then for any $t_* > 0$ there exists a constant $C_{t_*} > 0$ such that for any $(t, x) \in (t_*, \infty) \times \mathbb{R}^d$

$$\frac{1}{C_{t_*}} \leq \frac{v(x, t)}{v_{\infty, M}} \leq C_{t_*}. \quad (3.16)$$

Proof. It is only the lower bound to be proved, since we have the upper bound from (3.14).

In [Bonforte-Vázquez, Th.1.2] it was shown, that under the conditions of this proposition for any $t_* > 0$ and $(t, x) \in (t_*, \infty) \times \mathbb{R}^d$

$$\mathcal{B}(x, t - \tau, M_1) \leq u(x, t), \quad (3.17)$$

where $\mathcal{B}(x, t, M)$ is the Barenblatt solution for the initial mass M and τ, M_1 are some constants with

$$\tau = \begin{cases} \lambda t_c & t_* \geq t_c \\ \lambda t_* & t_* < t_c \end{cases},$$

$\lambda \in (0, 1)$ and $t_c = C(m, d)M_{R_0}^{1-m}R_0^{1/\vartheta}$ with the constants $C(m, d)$, ϑ , R_0 and $M_{R_0} = \int_{B_{R_0}} f(x)dx$. We will derive the lower bound in (3.16) from this inequality.

First we change the variables in (3.17) to get $v(x, t)$. From (3.8) we have

$$u(x, t) = t^{-d/\beta} v\left(\frac{x}{t^{1/\beta}}, \frac{\ln t}{\beta}\right), \quad (3.18)$$

so (3.17) becomes

$$e^{td}\mathcal{B}(e^t x, e^{t\beta} - \tau, M_1) \leq v(x, t), \quad (3.19)$$

where the l.h.s. can be rewritten as

$$\begin{aligned} & e^{td} \left(\frac{B(e^{t\beta} - \tau)}{|e^t x|^2 + BD(M_1)(e^{t\beta} - \tau)^{2/\beta}} \right)^{1/(1-m)} \\ &= \left(\frac{B(1 - \tau e^{-t\beta})}{|x|^2 + BD(M_1)(1 - \tau e^{-t\beta})^{2/\beta}} \right)^{1/(1-m)}, \end{aligned} \quad (3.20)$$

since $d + \frac{\beta-2}{1-m} = 0$.

Now note that, since M_{R_0} is bounded by the mass M , one can choose such R_0 , that $t_c < 1$. For such t_c it holds

$$0 < 1 - \tau < (1 - \tau e^{-t\beta}) < 1,$$

because $0 < \tau < 1$ is then true for any t_* . With $c_{t_*} := 1 - \tau$ it follows from (3.19) and (3.20), that

$$\begin{aligned} \frac{v(x, t)}{v_{\infty, M}} &\geq \left((1 - \tau e^{-t\beta}) \frac{|x|^2 + BD(M)}{|x|^2 + BD(M_1)(1 - \tau e^{-t\beta})^{2/\beta}} \right)^{1/(1-m)} \\ &\geq \left(c_{t_*} \frac{|x|^2 + BD(M)}{|x|^2 + BD(M_1)} \right)^{1/(1-m)} \geq \left(c_{t_*} \min \left\{ 1, \frac{D(M)}{D(M_1)} \right\} \right)^{1/(1-m)} \end{aligned}$$

and we get the lower bound on the quotient. The double inequality (3.16) can be derived by choosing C_{t_*} as a maximum of the upper bound and the reciprocal of the lower bound. \square

3.2 Expressing HLS via GNS with the help of FDE

Recall from the invariance proof, that in $\lambda = d - 2$ case of the sharp HLS inequality an optimiser is any function of the form

$$c \left(\frac{1}{\mu^2 + |x - a|^2} \right)^{(d+2)/2} \quad (3.21)$$

with $a, c, \mu \in \mathbb{R}, c, \mu \neq 0$. It follows, that for $m = d/(d + 2)$ the Barenblatt profile $v_{\infty, M}$ is also an optimiser - the minimiser of the HLS functional. It may then be the case, that the HLS functional monotonically decreases along the fast diffusion flow for this m . It is indeed so, as states the following

Theorem 3.3. *Let $f \in L^{2d/(d+2)}(\mathbb{R}^d)$, $d \geq 3$, be a nonnegative function, satisfying (3.15) for $m = d/(d + 2)$ and some $R > 0$ and such that*

$$\int_{\mathbb{R}^d} f(x) dx = \int_{\mathbb{R}^d} h(x) dx = M_*, \quad (3.22)$$

where $h(x)$ is defined in (1.3) with $\lambda = d - 2$. Let $u(x, t)$ be the solution of the FDE

$$\frac{\partial u}{\partial t}(x, t) = \Delta u^{d/(d+2)}(x, t) \quad (3.23)$$

with the initial condition $u(x, 1) = f(x)$.

Then for every $t > 1$

$$\frac{d}{dt} F[u(\cdot, t)] = -2G[u^{(d-1)/(d+2)}(\cdot, t)] \leq 0, \quad (3.24)$$

where

$$G[g] := C_S \frac{d(d-2)}{(d-1)^2} \|g\|_{2d/(d-1)}^{4/(d-1)} \|\nabla g\|_2^2 - \|g\|_{2(d+1)/(d-1)}^{2(d+1)/(d-1)}. \quad (3.25)$$

Note, that the assumption (3.22) will be used only in the next theorem, but is also made here, since both theorems are related and this one can be considered as the preparation for the next one.

Proof. We first show, that $\int_{\mathbb{R}^d} f(x) dx < \infty$ and (3.22) makes sense. With $L := \sup_{|x|>R} f(x) |x|^{d+2} < \infty$ and $f_1(x) := \max\{f(x), 1\}$ we get

$$\begin{aligned} \int_{\mathbb{R}^d} f(x) dx &= \int_{|x|<R} f(x) dx + \int_{|x|>R} f(x) dx \\ &\leq \int_{|x|<R} f_1^{2d/(d+2)}(x) dx + \int_{|x|>R} L |x|^{d+2} dx \\ &= \int_{|x|<R} f_1^{2d/(d+2)}(x) dx + d\omega_d \int_R^\infty r^{-3} dr < \infty, \end{aligned}$$

where ω_d is the volume of a d -dimensional ball.

Now we compute the time derivative of F to get the equality in (3.24):

$$\begin{aligned}
\frac{d}{dt}F[u] &= \frac{d}{dt} \left[C_S \left(\int_{\mathbb{R}^d} |u|^{2d/(d+2)} \right)^{(d+2)/d} - \int_{\mathbb{R}^d} u [(-\Delta)^{-1}u] \right] \\
&= C_S \frac{d+2}{d} \left(\int_{\mathbb{R}^d} u^{2d/(d+2)} \right)^{2/d} \frac{2d}{d+2} \int_{\mathbb{R}^d} u^{(d-2)/(d+2)} \frac{d}{dt}u \\
&\quad - \left(\int_{\mathbb{R}^d} \frac{d}{dt}u [(-\Delta)^{-1}u] + \int_{\mathbb{R}^d} u [(-\Delta)^{-1} \frac{d}{dt}u] \right) \\
&= 2C_S \left(\int_{\mathbb{R}^d} u^{2d/(d+2)} \right)^{2/d} \int_{\mathbb{R}^d} u^{(d-2)/(d+2)} \Delta u^{d/(d+2)} \\
&\quad - \left(\int_{\mathbb{R}^d} \Delta u^{d/(d+2)} [(-\Delta)^{-1}u] + \int_{\mathbb{R}^d} u [(-\Delta)^{-1} \Delta u^{d/(d+2)}] \right) \\
&= -2C_S \left(\int_{\mathbb{R}^d} u^{2d/(d+2)} \right)^{2/d} \int_{\mathbb{R}^d} \nabla u^{(d-2)/(d+2)} \nabla u^{d/(d+2)} - 2 \int_{\mathbb{R}^d} \Delta u^{d/(d+2)} [(-\Delta)^{-1}u] \\
&= -2C_S \frac{d(d-2)}{(d+2)^2} \left(\int_{\mathbb{R}^d} u^{2d/(d+2)} \right)^{2/d} \int_{\mathbb{R}^d} u^{-6/(d+2)} |\nabla u|^2 + 2 \int_{\mathbb{R}^d} u^{(2d+2)/(d+2)}.
\end{aligned}$$

Here in the second equality we exploit the fact that FDE leaves nonnegative initial f always nonnegative, i.e. $|u| = u$. In the fourth and the fifth equality integration-by-parts is used, justified by the uniform bounds (3.14):

$$u(x, \cdot) = v(Ux) = \frac{v(Ux, \cdot)}{v_{\infty, M}} v_{\infty, M} \in C^k,$$

since $v_{\infty, M} \in C^k$ and U , the transformation, defined in (3.8), is in C^∞ .

Further define $g = u^{(d-1)/(d+2)}$. It holds

$$\int_{\mathbb{R}^d} u^{-6/(d+2)} |\nabla u|^2 = \left(\frac{d+2}{d-1} \right)^2 \int_{\mathbb{R}^d} |\nabla g|^2$$

and

$$\begin{aligned}
\frac{d}{dt}F[u] &= -2C_S \left(\int_{\mathbb{R}^d} g^{2d/(d-1)} \right)^{2/d} \frac{d(d-2)}{(d-1)^2} \int_{\mathbb{R}^d} |\nabla g|^2 + 2 \int_{\mathbb{R}^d} g^{(2d+2)/(d-1)} \\
&= -2G[g].
\end{aligned}$$

It stays to prove, that G is a nonnegative functional. We show, that it follows from the GNS inequality (3.4) for $p = (d+1)/(d-1)$.

With this p

$$\begin{aligned}
2p &= \frac{2d+2}{d-1}, \quad p+1 = \frac{2d}{d-1} \quad \text{and} \\
\theta &= \frac{d \left(\frac{d+1}{d-1} - 1 \right)}{\frac{d+1}{d-1} \left(\frac{(d+2)(d-1) - (d-2)(d+1)}{d-1} \right)} = \frac{2d}{(d+1) \frac{2d}{d-1}} = \frac{d-1}{d+1},
\end{aligned}$$

so, taken to the power $2p$, the GNS inequality can be written in the form

$$C_{GNS} \|g\|_2^2 \|g\|_{2d/(d-1)}^{4/(d-1)} \geq \|g\|_{2(d+1)/(d-1)}^{2(d+1)/(d-1)}. \quad (3.26)$$

Using (3.5) and (3.2), one can calculate, that

$$C_{GNS} = C_S \frac{d(d-2)}{(d-1)^2} \quad (3.27)$$

which proves the nonnegativity of G .

There exists a finer way to prove this last equality. If the initial data of the FDE is h itself, $u(x, t)$ is always in the steady state and does not depend on t . By what we just proved it means, that $G[h^{(d-1)/(d+2)}] = 0$. Now

$$h^{(d-1)/(d+2)}(x) = \left(\frac{1}{1+|x|^2} \right)^{(d-1)/2} = \left(\frac{1}{1+|x|^2} \right)^{1/(p-1)}$$

is the optimiser of our GNS inequality. So for a GNS optimiser g

$$C_S \frac{d(d-2)}{(d-1)^2} \|g\|_2^2 \|g\|_{2d/(d-1)}^{4/(d-1)} = \|g\|_{2(d+1)/(d-1)}^{2(d+1)/(d-1)}$$

and since C_{GNS} is a sharp constant, (3.27) is proved. \square

So we managed to show a relationship between the time derivative of the HLS functional and the GNS inequality. Now we will do the same for the HLS functional itself, proving the corollary of the previous theorem.

Theorem 3.4. *For $f \in L^{2d/(d+2)}(\mathbb{R}^d)$, $d \geq 3$, and $u(x, t)$, satisfying the conditions of the Theorem 3.3,*

$$F[f] = \frac{8}{d+2} \int_0^\infty e^{\beta t} G[u^{(d-1)/(d+2)}(\cdot, e^{\beta t})] dt \geq 0 \quad (3.28)$$

and $F[f] = 0$ if and only if $f(x) = ch(x/\mu^2 - a)$ for some $a \in \mathbb{R}^d$, $c, \mu \in \mathbb{R} \setminus \{0\}$ and h given by (1.3) with $\lambda = d - 2$.

Proof. First of all, as in the Theorem 3.3 the assumption (3.15) together with the fact that $f \in L^{2d/(d+2)}(\mathbb{R}^d)$ implies the integrability of f . Now let $v(x, t)$ be the solution of (3.9) with $v(x, 0) = f(x)$, i.e. the solution corresponding to $u(x, t)$ via (3.8) for $\beta = 4/(d+2)$. Then

$$\begin{aligned} F[v(x, t)] &= C_S \left(\int_{\mathbb{R}^d} [e^{td} u(e^t x, e^{\beta t})]^{\frac{2d}{d+2}} dx \right)^{\frac{d+2}{d}} - \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{e^{2td} u(e^t x, e^{\beta t}) u(e^t y, e^{\beta t})}{|x-y|^{d-2}} dx dy \\ &= C_S \left(\int_{\mathbb{R}^d} [e^{td} u(x, e^{\beta t})]^{\frac{2d}{d+2}} \frac{1}{e^{td}} dx \right)^{\frac{d+2}{d}} - \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{2td} \frac{u(x, e^{\beta t}) u(y, e^{\beta t})}{\left| \frac{x}{e^t} - \frac{y}{e^t} \right|^{d-2}} \frac{1}{e^{2td}} dx dy \\ &= C_S e^{td \left(\frac{2d}{d+2} - 1 \right) \frac{d+2}{d}} \left(\int_{\mathbb{R}^d} [u(x, e^{\beta t})]^{\frac{2d}{d+2}} dx \right)^{\frac{d+2}{d}} - e^{t(d-2)} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{u(x, e^{\beta t}) u(y, e^{\beta t})}{|x-y|^{d-2}} dx dy \\ &= e^{t(d-2)} F[u(x, e^{\beta t})] \end{aligned}$$

and the Theorem 3.3 implies that, for all $t > 0$

$$\frac{d}{dt} \left(e^{-t(d-2)} F[v(\cdot, t)] \right) = \frac{8}{d+2} e^{\beta t} G[u^{(d-1)/(d+2)}(\cdot, e^{\beta t})]. \quad (3.29)$$

In order to integrate the both sides of (3.29), we prove that

$$\lim_{t \rightarrow 0} F[v(\cdot, t)] = F[f] \quad \text{and} \quad \lim_{t \rightarrow \infty} F[v(\cdot, t)] = 0. \quad (3.30)$$

We start with the latter fact and aim to apply the dominated convergence theorem to show that $F[v(\cdot, t)] \rightarrow F[h]$ as $t \rightarrow \infty$. Because of the assumption (3.22), $v(x, t) \rightarrow h(x)$ as $t \rightarrow \infty$ for every $x \in \mathbb{R}^d$. Due to the upper bound in (3.16) for every $t > t_*$ (fix $t_* = 1$)

$$\begin{aligned} \int_{\mathbb{R}^d} v(x, t)^{\frac{2d}{d+2}} dx &\leq \int_{\mathbb{R}^d} (C_{t_*} v_{\infty, M}(x))^{\frac{2d}{d+2}} dx \quad \text{and} \\ \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{v(x, t)v(y, t)}{|x-y|^{d-2}} dx dy &\leq C_{t_*}^2 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{v_{\infty, M}(x)v_{\infty, M}(y)}{|x-y|^{d-2}} dx dy \end{aligned} \quad (3.31)$$

Now both terms on the r.h.s. are bounded, because

$$\begin{aligned} \int_{\mathbb{R}^d} (v_{\infty, M}(x))^{\frac{2d}{d+2}} dx &= \int_{\mathbb{R}^d} \left(\frac{1}{D(M) + \frac{1-m}{2\beta m} |x|^2} \right)^d dx \\ &\leq \int_{|x| < R_1} \frac{1}{D(M)^d} dx + \int_{|x| > R_1} v_{\infty, M}(x) dx < \infty \end{aligned}$$

for some $R_1 > 0$ and due to the general HLS inequality (for the second term). So the dominated convergence argument is justified and $\lim_{t \rightarrow \infty} F[v(\cdot, t)] = F[h] = 0$ since h is the optimiser of the HLS inequality (what we actually do not know in this proof yet, but could have computed).

The proof of the former fact is based on the $L^{2d/(d+2)}$ convergence

$$\lim_{t \rightarrow 0} \|v(\cdot, t) - f\|_{2d/(d+2)} \rightarrow 0. \quad (3.32)$$

and the continuity of the HLS functional F on $L^{2d/(d+2)}$.

The convergence (3.32) can be proved, starting with some facts from [Vázquez]. In [Vázquez, Def.9.1] the strong solution $u(x, t)$ of the FDE (3.7) with the integrable initial condition $u_0(x)$ was defined in such a way, that

$$\|u(\cdot, t) - u_0(\cdot)\|_1 \rightarrow 0 \quad \text{as} \quad t \rightarrow 0. \quad (3.33)$$

In [Vázquez, Th.9.3] it was shown for $u_0(x) \in L^p(\mathbb{R}^d)$, $p \geq 1$, that

$$\|u(t)\|_p \leq \|u_0\|_p. \quad (3.34)$$

From (3.33) and (3.34) it can be followed with dominated convergence, that for every $p \geq 1$

$$\|u(\cdot, t) - u_0(\cdot)\|_p \rightarrow 0 \quad \text{as} \quad t \rightarrow 0. \quad (3.35)$$

In the same way we can claim, that for every $p \geq 1$

$$\|u(\cdot, t) - u(x, 1)\|_p \rightarrow 0 \quad \text{as } t \rightarrow 1, \quad (3.36)$$

since by the definition [Vázquez, Def.9.1] the solution u is continuous for all $t \geq 0$: $u \in C([0, \infty) : L^1(\mathbb{R}^d))$, so both (3.33) and (3.34) should hold for $u(x, 1), t \rightarrow 1$ instead of $u_0, t \rightarrow 0$. Finally, (3.32) follows from (3.36) by the change of variables according to (3.18):

$$\begin{aligned} \lim_{t \rightarrow 1} \|u(\cdot, t) - u(x, 1)\|_p &= \lim_{t \rightarrow 1} \left(\int_{\mathbb{R}^d} \left| t^{-d/\beta} v \left(\frac{x}{t^{1/\beta}}, \frac{\ln t}{\beta} \right) - v(x, 0) \right|^p dx \right)^{1/p} \\ &= \lim_{t \rightarrow 0} \left(\int_{\mathbb{R}^d} \left| e^{-td} v(x, t) - v(x, 0) \right|^p e^{td} dx \right)^{1/p} \\ &= \lim_{t \rightarrow 0} \|v(\cdot, t) - v(x, 0)\|_p \end{aligned} \quad (3.37)$$

for $p = 2d/(d+2)$.

As for the continuity of F , the norm is always continuous and for the term with the Laplacian it holds

$$\begin{aligned} H'(f, f) &= H'(f - f_k + f_k, f) = H'(f - f_k, f) + H'(f_k, f) \\ &= H'(f - f_k, f) + H'(f_k, f - f_k) + H(f_k, f_k), \end{aligned}$$

where we use notation from the Proposition 2.1. From the general HLS inequality it follows, that

$$|F[f] - F[f_k]| \rightarrow 0 \quad \text{if } \|f - f_k\|_{2d/(d+2)} \rightarrow 0,$$

so $\lim_{t \rightarrow 0} F[v(\cdot, t)] = F[f]$.

Now integrating (3.29) over $[0, \infty)$ and using (3.30) we get the identity (3.28). It follows that $F[f] = 0$ if and only if $G[u^{(d-1)/(d+2)}(\cdot, e^{\beta t})] = 0$ for a.e. t , i.e. $u^{(d-1)/(d+2)}(\cdot, e^{\beta t})$ is an optimiser of the GNS inequality for $p = (d+1)/(d-1)$:

$$\begin{aligned} u(x, e^{\beta t}) &= c\mu^{-2d}(t) \left[h_1 \left(\frac{x}{\mu^2(t)} - a(t) \right) \right]^{(d+2)/(d-1)}, \\ &= c\mu^{-2d}(t) \left[\frac{1}{1 + \left| \frac{x}{\mu^2(t)} - a(t) \right|^2} \right]^{(d+2)/2} \end{aligned} \quad (3.38)$$

where $\mu^{-2d}(t)$ is the Jacobian of the transformation $x \mapsto \frac{x}{\mu^2(t)} - a(t)$ and h_1 stems from (3.6).

Notice, that $u(x, e^{\beta t})$ has at each $t > 0$ a form of a Barenblatt profile for $m = (d+2)/2$. Since the Cauchy problem for the FDE is unique, $u(x, e^{\beta t})$ has to be the Barenblatt solution of the FDE. It follows from $\lim_{t \rightarrow 0} \|u(x, e^{\beta t}) - f(x)\|_1 = 0$, that $f(x)$ itself is a Barenblatt profile, i.e. $f(x) = ch(x/\mu^2 - a)$ for some $a \in \mathbb{R}^d, c, \mu \in \mathbb{R} \setminus \{0\}$. □

To prove the sharp HLS inequality it remains to drop the numerous assumptions on f .

Theorem 3.5. *Let $f \in L^{2d/(d+2)}(\mathbb{R}^d)$, $d \geq 3$ be a nonnegative function. Then $F[f] \geq 0$, and $F[f] = 0$ if and only if $f(x) = ch(x/\mu^2 - a)$ for some $a \in \mathbb{R}^d$, $c, \mu \in \mathbb{R} \setminus \{0\}$ and h given by (1.3) with $\lambda = d - 2$.*

Proof. In view of the Theorems 3.3 and 3.4 there are two things to show: the irrelevance of fixing the integral of f and the possibility of avoiding the condition (3.15), which guaranteed the integrability of f .

The first one is easy, since $F[\lambda f] = \lambda^2 F[f]$, so one can multiply an arbitrary function with a constant, which would not affect the nonnegativity of F .

To prove the second one, take a nonnegative $f \in L^{2d/(d+2)}(\mathbb{R}^d)$, that does not necessarily satisfy (3.15). Define $f_n := \min\{f, nh\}$. Then $f_n \rightarrow f$ monotonically pointwise from below and both integrals in $F[f_n]$ are controlled by the respective integrals in $F[f]$, which are finite, as the general HLS inequality holds and $f \in L^{2d/(d+2)}(\mathbb{R}^d)$. With the monotone convergence theorem $F[f_n] \rightarrow F[f]$ as $n \rightarrow \infty$ and, since for every f_n the Theorem 3.4 holds, $F[f] \geq 0$.

For the case of equality we will need results from the previous chapter. Recall the Riesz rearrangement inequality, from which follows, that any optimiser $f \in L^{2d/(d+2)}(\mathbb{R}^d)$ of the HLS inequality should be a translate of a symmetric-decreasing function (see also the explanation, preceding the Proposition 2.8). For simplicity we assume, that f is itself symmetric-decreasing. Since the HLS functional is invariant under any conformal transformation, e.g. inversion on the unit sphere, we can construct another optimiser $\tilde{f} := |x|^{-(d+2)} f(x/|x|^2)$. For any unit vector x_0 the function f is uniformly bounded on the unit ball centred at $2x_0$, because it is symmetric-decreasing, so $|x|^{-(d+2)} f(x/|x|^2 - 2x_0)$ satisfies (3.15) for $R = 1$. We can apply to this function the Theorem 3.4 to show, that it must be a Barenblatt profile, i.e.

$$|x|^{-(d+2)} f(x/|x|^2 - 2x_0) = \left(\tilde{D} + \frac{1-m}{2\beta m} |x|^2 \right)^{-1/(1-m)}$$

for some \tilde{D} and $m = d/(d+2)$. It follows, that

$$\begin{aligned} f(x) &= \left| \frac{x - 2x_0}{|x - 2x_0|^2} \right|^{d+2} \left(\tilde{D} + \frac{1-m}{2\beta m} \left| \frac{x - 2x_0}{|x - 2x_0|^2} \right|^2 \right)^{-(d+2)/2} \\ &= \left(\tilde{D} |x - 2x_0|^2 + \frac{1-m}{2\beta m} \right)^{-(d+2)/2} \end{aligned}$$

is also a Barenblatt profile (because translations are irrelevant by integrating on the whole \mathbb{R}^d), i.e. $f(x) = ch(x/\mu^2 - a)$ for some $a \in \mathbb{R}^d$, $c, \mu \in \mathbb{R} \setminus \{0\}$. \square

The sharp HLS inequality has thus been put down to the sharp GNS inequality via a certain fast diffusion flow. In the next section we will show, that with another FDE the sharp GNS inequality follows from the Cauchy-Schwarz inequality.

3.3 Proving GNS with the help of FDE

We consider the functional

$$L[v(x, t)] := \int_{\mathbb{R}^d} \left[\frac{|x|^2}{2} v(x, t) + \frac{\beta}{m-1} v^m(x, t) \right] dx. \quad (3.39)$$

As stated in [Carlen-Carrillo-Loss, p.8], for $1 - 2/d < m < 1$ its gradient flow (with respect to the Euclidean Wasserstein distance, but we do not use this fact) is the transformed FDE (3.9). We aim to show, that taking the second time derivative of L with $m = (p + 1)/2p$ and applying the Cauchy-Schwarz inequality, one can get the sharp GNS inequality.

3.3.1 The second time derivative of L

In this section we will extensively use integration-by-parts, which is justified by the uniform bounds (3.14). Writing v for $v(x, t)$, we compute

$$\begin{aligned}
\frac{d}{dt}L[v] &= \int_{\mathbb{R}^d} \left[\frac{|x|^2}{2} + \frac{\beta m}{m-1} v^{m-1} \right] \frac{\partial v}{\partial t} dx \\
&= \int_{\mathbb{R}^d} \left[\frac{|x|^2}{2} + \frac{\beta m}{m-1} v^{m-1} \right] \nabla \cdot (\beta \nabla v^m + xv) dx \\
&= \int_{\mathbb{R}^d} \left[\frac{|x|^2}{2} + \frac{\beta m}{m-1} v^{m-1} \right] \nabla \cdot \left(\beta m \frac{m-1}{m-1} v^{m-1} \nabla v + xv \right) dx \\
&= \int_{\mathbb{R}^d} \left[\frac{|x|^2}{2} + \frac{\beta m}{m-1} v^{m-1} \right] \nabla \cdot \left(v \left[\frac{m\beta}{m-1} \nabla v^{m-1} + x \right] \right) dx \\
&= - \int_{\mathbb{R}^d} \left| \frac{m\beta}{m-1} \nabla v^{m-1} + x \right|^2 v dx.
\end{aligned} \tag{3.40}$$

As expected, the first time derivative of L is negative. We denote $I[v] := -\frac{d}{dt}L[v]$.

The further computations are based on [Carrillo-Toscani, pp. 124-127]. It holds

$$\begin{aligned}
\frac{d}{dt}I[v] &= \int_{\mathbb{R}^d} \frac{\partial v}{\partial t} \left| \frac{m\beta}{m-1} \nabla v^{m-1} + x \right|^2 dx \\
&\quad + 2 \int_{\mathbb{R}^d} v \left(\frac{m\beta}{m-1} \nabla v^{m-1} + x \right) \cdot \frac{\partial}{\partial t} \left[\frac{m\beta}{m-1} \nabla v^{m-1} \right] dx =: I_1 + I_2
\end{aligned}$$

and we compute both integrals separately.

Define

$$y = y(x) := x + \frac{m\beta}{m-1} \nabla v^{m-1}. \tag{3.41}$$

Then

$$vy = vx + \beta \nabla v^m \tag{3.42}$$

and

$$\begin{aligned}
I_1 &= \int_{\mathbb{R}^d} \nabla \cdot (\beta \nabla v^m + xv) \left| \frac{m\beta}{m-1} \nabla v^{m-1} + x \right|^2 dx \\
&= - \int_{\mathbb{R}^d} (\beta \nabla v^m + xv) \nabla \left| \frac{m\beta}{m-1} \nabla v^{m-1} + x \right|^2 dx \\
&= -2 \int_{\mathbb{R}^d} vy \cdot \text{Jacobi}(y) \cdot y^T dx \\
&= -2 \int_{\mathbb{R}^d} vy \cdot \left[I_d + \frac{m\beta}{m-1} \text{Hess}(v^{m-1}) \right] \cdot y^T dx \\
&= -2 \int_{\mathbb{R}^d} v |y|^2 dx - 2 \frac{m\beta}{m-1} \int_{\mathbb{R}^d} vy \cdot \text{Hess}(v^{m-1}) \cdot y^T dx, \tag{3.43}
\end{aligned}$$

where in the first line we use (3.9) and I_d is the identity matrix. Denoting the integral in the second term of (3.43) as I_3 we compute:

$$\begin{aligned}
I_3 &= \sum_{i,j=1}^d \int_{\mathbb{R}^d} vy_i y_j \frac{\partial^2 v^{m-1}}{\partial x_i \partial x_j} dx = - \sum_{i,j=1}^d \int_{\mathbb{R}^d} \frac{\partial v^{m-1}}{\partial x_i} \frac{\partial}{\partial x_j} \left[\frac{1}{v} vy_i vy_j \right] dx \\
&= - \sum_{i,j=1}^d (m-1) \int_{\mathbb{R}^d} v^{m-2} \frac{\partial v}{\partial x_i} \left[-\frac{1}{v^2} vy_i vy_j \frac{\partial v}{\partial x_j} + \frac{1}{v} \frac{\partial [vy_i vy_j]}{\partial x_j} \right] dx \\
&= \sum_{i,j=1}^d (m-1) \left[\int_{\mathbb{R}^d} v^{m-2} y_i y_j \frac{\partial v}{\partial x_i} \frac{\partial v}{\partial x_j} dx - \int_{\mathbb{R}^d} v^{m-2} \frac{\partial v}{\partial x_i} \left(y_i \frac{\partial [vy_j]}{\partial x_j} + y_j \frac{\partial [vy_i]}{\partial x_j} \right) dx \right]
\end{aligned}$$

The last expression can be rewritten once more:

$$\begin{aligned}
I_3 &= (m-1) \left(\sum_{i,j=1}^d \int_{\mathbb{R}^d} v^{m-2} y_i y_j \frac{\partial v}{\partial x_i} \frac{\partial v}{\partial x_j} dx - 2 \sum_{i,j=1}^d \int_{\mathbb{R}^d} v^{m-2} y_i \frac{\partial v}{\partial x_i} \frac{\partial [vy_j]}{\partial x_j} \right) \\
&\quad + (m-1) \sum_{i,j=1}^d \int_{\mathbb{R}^d} v^{m-2} \frac{\partial v}{\partial x_i} \left(y_i \frac{\partial [vy_j]}{\partial x_j} - y_j \frac{\partial [vy_i]}{\partial x_j} \right) dx, \tag{3.44}
\end{aligned}$$

where, due to the fact, that $\partial y_i / \partial x_j = \partial y_j / \partial x_i$,

$$\begin{aligned}
\sum_{i,j=1}^d \frac{\partial v}{\partial x_i} \left(y_i \frac{\partial [vy_j]}{\partial x_j} - y_j \frac{\partial [vy_i]}{\partial x_j} \right) &= v \sum_{i,j=1}^d \left(y_i \frac{\partial v}{\partial x_i} \frac{\partial y_j}{\partial x_j} - y_j \frac{\partial v}{\partial x_i} \frac{\partial y_i}{\partial x_j} \right) \\
&= v \sum_{i,j=1}^d \left(y_i \frac{\partial v}{\partial x_i} \frac{\partial y_j}{\partial x_j} - \frac{1}{2} \frac{\partial v}{\partial x_i} \frac{\partial y_j^2}{\partial x_i} \right) \\
&= v \left([y \cdot \nabla v] \nabla \cdot y - \frac{1}{2} \nabla |y|^2 \cdot \nabla v \right). \tag{3.45}
\end{aligned}$$

Further

$$\begin{aligned}
I_2 &= 2 \int_{\mathbb{R}^d} v y \frac{\partial}{\partial t} \left[\frac{m\beta}{m-1} \nabla v^{m-1} \right] dx = 2 \frac{m\beta}{m-1} \int_{\mathbb{R}^d} v y \nabla \frac{\partial}{\partial t} v^{m-1} dx \\
&= -2 \frac{m\beta}{m-1} \int_{\mathbb{R}^d} \nabla \cdot [v y] \frac{\partial}{\partial t} v^{m-1} dx \\
&= -2m\beta \int_{\mathbb{R}^d} (\nabla \cdot [v y]) v^{m-2} \nabla \cdot (\beta \nabla v^m + x v) dx \\
&= -2m\beta \int_{\mathbb{R}^d} v^{m-2} (\nabla \cdot [v y])^2 dx, \tag{3.46}
\end{aligned}$$

where in the last line (3.42) was used. Collecting all terms, we get

$$\begin{aligned}
\frac{d}{dt} I[v] &= -2 \int_{\mathbb{R}^d} v |y|^2 dx - 2m\beta \left(\sum_{i,j=1}^d \int_{\mathbb{R}^d} v^{m-2} y_i y_j \frac{\partial v}{\partial x_i} \frac{\partial v}{\partial x_j} dx - 2 \sum_{i,j=1}^d \int_{\mathbb{R}^d} v^{m-2} y_i \frac{\partial v}{\partial x_i} \frac{\partial [v y_j]}{\partial x_j} \right) \\
&\quad - 2m\beta \int_{\mathbb{R}^d} v^{m-1} \left([y \cdot \nabla v] \nabla \cdot y - \frac{1}{2} \nabla |y|^2 \cdot \nabla v \right) dx - 2m\beta \int_{\mathbb{R}^d} v^{m-2} (\nabla \cdot [v y])^2 dx \\
&= -2I[v] - 2m\beta \int_{\mathbb{R}^d} v^{m-1} \left([y \cdot \nabla v] \nabla \cdot y - \frac{1}{2} \nabla |y|^2 \cdot \nabla v \right) dx \\
&\quad - 2m\beta \int_{\mathbb{R}^d} v^{m-2} \left([\nabla v \cdot y + v \nabla \cdot y]^2 + \sum_{i,j=1}^d y_i y_j \frac{\partial v}{\partial x_i} \frac{\partial v}{\partial x_j} - 2 \sum_{i,j=1}^d v y_i \frac{\partial v}{\partial x_i} \frac{\partial [v y_j]}{\partial x_j} \right) dx \\
&= -2I[v] - 2\beta \int_{\mathbb{R}^d} \left([y \cdot \nabla v^m] \nabla \cdot y - \frac{1}{2} \nabla |y|^2 \cdot \nabla v^m \right) dx \\
&\quad - 2m\beta \int_{\mathbb{R}^d} v^{m-2} ([\nabla v \cdot y + v \nabla \cdot y]^2 + [\nabla v \cdot y]^2 - 2(v \nabla \cdot y) \nabla v \cdot y) dx \\
&= -2I[v] - 2\beta \int_{\mathbb{R}^d} \left([y \cdot \nabla v^m] \nabla \cdot y - \frac{1}{2} \nabla |y|^2 \cdot \nabla v^m \right) dx - 2m\beta \int_{\mathbb{R}^d} v^m [\nabla \cdot y]^2 dx.
\end{aligned}$$

Since

$$\nabla \cdot [y \nabla \cdot y] = [\nabla \cdot y]^2 + y \cdot \nabla [\nabla \cdot y],$$

it follows by the integration-by parts, that

$$\begin{aligned}
\frac{d}{dt} I[v] &= -2I[v] - 2\beta \int_{\mathbb{R}^d} v^m \left(\frac{1}{2} \Delta |y|^2 - [\nabla \cdot y]^2 - y \cdot \nabla [\nabla \cdot y] \right) dx - 2m\beta \int_{\mathbb{R}^d} v^m [\nabla \cdot y]^2 dx \\
&= -2I[v] - 2(m-1)\beta \int_{\mathbb{R}^d} v^m [\nabla \cdot y]^2 dx - 2\beta \int_{\mathbb{R}^d} v^m \left(\frac{1}{2} \Delta |y|^2 - y \cdot \nabla [\nabla \cdot y] \right) dx \\
&= -2I[v] - 2(m-1)\beta \int_{\mathbb{R}^d} v^m [\nabla \cdot y]^2 dx - 2\beta \int_{\mathbb{R}^d} v^m \left[\sum_{i,j=1}^d \left(\frac{\partial y_i}{\partial x_j} \right)^2 \right] dx,
\end{aligned}$$

where the last step can be checked by expanding the matrices and simplifying.

Finally, defining

$$\xi := \frac{|x|^2}{2} + \frac{m\beta}{m-1} v^{m-1} \tag{3.47}$$

and noting, that $y = \nabla \xi$, we get the expression stated in [Carlen-Carrillo-Loss, p.8]:

$$\frac{d}{dt} I[v] = -2I[v] - 2(m-1)\beta \int_{\mathbb{R}^d} v^m [\Delta \xi]^2 dx - 2\beta \int_{\mathbb{R}^d} v^m \left[\sum_{i,j=1}^d \left(\frac{\partial^2 \xi}{\partial x_i \partial x_j} \right)^2 \right] dx. \quad (3.48)$$

3.3.2 Applying Cauchy-Schwarz to derive GNS

The last integral on the r.h.s. of (3.48) can be estimated from below:

$$\int_{\mathbb{R}^d} v^m \left[\sum_{i,j=1}^d \left(\frac{\partial^2 \xi}{\partial x_i \partial x_j} \right)^2 \right] dx \geq \int_{\mathbb{R}^d} v^m \left[\sum_{i=1}^d \left(\frac{\partial^2 \xi}{\partial x_i^2} \right)^2 \right] dx \geq \frac{1}{d} \int_{\mathbb{R}^d} v^m [\Delta \xi]^2 dx$$

where in the last step the Cauchy-Schwarz inequality was used. Define

$$R[v] := \int_{\mathbb{R}^d} v^m [\Delta \xi]^2 dx. \quad (3.49)$$

It follows from (3.48), that

$$\frac{d}{dt} I[v] \leq -2I[v] - 2\beta(m-1 + \frac{1}{d})R[v]. \quad (3.50)$$

Recall, that we defined $I[v]$ as $\frac{d}{dt} L[v]$, so

$$\frac{d}{dt} L[v] \geq \frac{1}{2} \frac{d}{dt} I[v] + \beta(m-1 + \frac{1}{d})R[v]. \quad (3.51)$$

Integrating this inequality in t from 0 to ∞ , one gets

$$L[v_{\infty, M}] - L[v(\cdot, 0)] \geq \frac{1}{2} (I[v_{\infty, M}] - I[v(\cdot, 0)]) + \int_0^\infty \beta(m-1 + \frac{1}{d})R[v(\cdot, t)] dt. \quad (3.52)$$

Since

$$I[v_{\infty, M}] = \int_{\mathbb{R}^d} \left| x + \frac{m\beta}{m-1} \nabla \left(D(M) + \frac{1-m}{2\beta m} |x|^2 \right) \right|^2 = 0$$

and $\int_0^\infty \beta(m-1 + \frac{1}{d})R[v(\cdot, t)] dt$ is positive as long as $(d-1)/d < m < 1$, we can state

$$\frac{1}{2} I[v(\cdot, 0)] \geq L[v(\cdot, 0)] - L[v_{\infty, M}]. \quad (3.53)$$

We now show, that the inequality (3.53) is equivalent to the sharp GNS inequalities.

First we rewrite (3.53), using the definitions of I and L and expanding squares in I (here $v := v(\cdot, 0)$):

$$\begin{aligned} \frac{1}{2} \int_{\mathbb{R}^d} \left(|x|^2 v + 2 \frac{m\beta}{m-1} x \cdot \nabla v^{m-1} v + \left(\frac{m\beta}{m-1} \right)^2 |\nabla v^{m-1}|^2 v \right) dx \\ \geq \int_{\mathbb{R}^d} \left(\frac{|x|^2}{2} v + \frac{\beta}{m-1} v^m \right) dx - L[v_{\infty, M}]. \end{aligned} \quad (3.54)$$

With

$$\frac{m}{m-1}x \cdot \nabla v^{m-1}v = mxv^{m-1} \cdot \nabla v = x \cdot \nabla v^m \quad (3.55)$$

and

$$|\nabla v^{m-1}|^2 = (m-1)^2 v^{2m-4} |\nabla v|^2, \quad (3.56)$$

(3.54) is equivalent to

$$\beta \int_{\mathbb{R}^d} x \cdot \nabla v^m dx + \frac{1}{2} \int_{\mathbb{R}^d} (m\beta)^2 v^{2m-3} |\nabla v|^2 dx \geq \int_{\mathbb{R}^d} \frac{\beta}{m-1} v^m dx - L[v_{\infty, M}]. \quad (3.57)$$

After the integration-by-parts in the first integral (justified by (3.16)) one gets

$$\beta \left(\frac{1}{1-m} - d \right) \int_{\mathbb{R}^d} v^m dx + \frac{1}{2} \int_{\mathbb{R}^d} (m\beta)^2 v^{2m-3} |\nabla v|^2 dx \geq -L[v_{\infty, M}]. \quad (3.58)$$

Now we change the dependent variable $f(x)^{2p} := v(x, 0)$ and consider the special case $m = (p+1)/(2p)$. With

$$2m-3 = \frac{2-4p}{2p}, \quad 1-m = \frac{p-1}{2p} \quad (3.59)$$

(3.58) becomes

$$\beta \left(\frac{2p}{p-1} - d \right) \int_{\mathbb{R}^d} f^{1+p} dx + \frac{1}{2} \int_{\mathbb{R}^d} \left(\beta \frac{p+1}{2p} \right)^2 f^{2-4p} |\nabla f^{2p}|^2 dx \geq -L[v_{\infty, M}] \quad (3.60)$$

and with

$$|\nabla f^{2p}|^2 = 4p^2 f^{4p-2} |\nabla f|^2 \quad (3.61)$$

$$\beta \left(\frac{2p}{p-1} - d \right) \int_{\mathbb{R}^d} f^{1+p} dx + \frac{1}{2} \beta^2 (1+p)^2 \int_{\mathbb{R}^d} |\nabla f|^2 dx \geq -L[v_{\infty, M}], \quad (3.62)$$

where $\beta = 2 - d \frac{p-1}{2p}$ (see Proposition 3.1).

In the next step we apply the standard scaling argument to the inequality (3.62). For the transformation

$$f(x) \rightarrow \lambda^{\frac{d}{2p}} f(\lambda x) \quad (3.63)$$

with $\lambda > 0$ we can rewrite

$$\begin{aligned} \int_{\mathbb{R}^d} \left| \nabla_x (\lambda^{\frac{d}{2p}} f(\lambda x)) \right|^2 dx &= \int_{\mathbb{R}^d} \lambda^{\frac{d}{p}+2} |\nabla_{\lambda x} f(\lambda x)|^2 dx = \lambda^{\frac{d}{p}+2-d} \int_{\mathbb{R}^d} |\nabla f(x)|^2 dx, \\ \int_{\mathbb{R}^d} (\lambda^{\frac{d}{2p}} f(\lambda x))^{1+p} dx &= \lambda^{\frac{d(1+p)}{2p}-d} \int_{\mathbb{R}^d} f^{1+p} dx = \lambda^{-\frac{d(p-1)}{2p}} \int_{\mathbb{R}^d} f^{1+p} dx, \end{aligned}$$

so for every $\lambda > 0$ and

$$T_1 := \beta \left(\frac{2p}{p-1} - d \right) \int_{\mathbb{R}^d} f^{1+p} dx, \quad (3.64)$$

$$T_2 := \frac{1}{2} \beta^2 (1+p)^2 \int_{\mathbb{R}^d} |\nabla f|^2 dx \quad (3.65)$$

it holds

$$\lambda^{-\frac{d(p-1)}{2p}} T_1 + \lambda^{\frac{d}{p}+2-d} T_2 \geq -L[v_{\infty, M}]. \quad (3.66)$$

We minimise the l.h.s. of this inequality with respect to λ :

$$-\frac{d(p-1)}{2p} \lambda_*^{-\frac{dp+d-2p}{2p}} T_1 + \left(\frac{d}{p} + 2 - d\right) \lambda_*^{\frac{d}{p}+1-d} T_2 = 0 \quad (3.67)$$

$$\lambda_*^{\frac{d-dp+p}{p}} = \frac{d(p-1)}{2p} \frac{p}{d-dp+2p} \frac{T_1}{T_2} \lambda_*^{\frac{d-dp-2p}{2p}} \quad (3.68)$$

$$\lambda_*^{\frac{2d-2dp+2p-d+dp+2p}{2p}} = \frac{d(p-1)}{2d-2dp+4p} \frac{T_1}{T_2} \quad (3.69)$$

$$\lambda_* = \left[\frac{d(p-1)}{2d-2dp+4p} \frac{T_1}{T_2} \right]^{\frac{2p}{d-dp+4p}}. \quad (3.70)$$

For this minimising λ_* the inequality (3.66) has the form

$$\left[\frac{d(p-1)}{2d-2dp+4p} \frac{T_1}{T_2} \right]^{\frac{d-dp}{d-dp+4p}} T_1 + \left[\frac{d(p-1)}{2d-2dp+4p} \frac{T_1}{T_2} \right]^{\frac{2(d-dp+2p)}{d-dp+4p}} T_2 \geq -L[v_{\infty, M}] \quad (3.71)$$

or

$$C_* \left(\int_{\mathbb{R}^d} f^{1+p} dx \right)^{\frac{2(d-dp+2p)}{d-dp+4p}} \left(\int_{\mathbb{R}^d} |\nabla f|^2 dx \right)^{\frac{dp-d}{d-dp+4p}} \geq -L[v_{\infty, M}], \quad (3.72)$$

where

$$\lambda_{c_*} = \frac{d(p-1)}{2d-2dp+4p} \frac{\frac{2p}{p-1} - d}{\frac{1}{2}\beta(1+p)^2} = \frac{d}{\beta(1+p)^2}, \quad (3.73)$$

$$C_* = \beta \left(\frac{2p}{p-1} - d \right) \lambda_{c_*}^{\frac{d-dp}{d-dp+4p}} + \frac{1}{2} \beta^2 (1+p)^2 \lambda_{c_*}^{\frac{2(d-dp+2p)}{d-dp+4p}}. \quad (3.74)$$

We can rewrite (3.72) as

$$C_* (\|f\|_{1+p}^{1-\theta} \|\nabla f\|_2^\theta)^\delta \geq -L[v_{\infty, M}], \quad (3.75)$$

where

$$\delta = 2p \frac{d+2-(d-2)p}{4p-d(p-1)}, \quad \theta = \frac{d(p-1)}{p(d+2-p(d-2))}. \quad (3.76)$$

For the last step recall, that the mass M is fixed and since we substituted $f^{2p}(x) = v(x, 0)$, the following equality should be fulfilled:

$$\int_{\mathbb{R}^d} |f(x)|^{2p} dx = M. \quad (3.77)$$

We thus may write

$$\|f\|_{1+p}^{1-\theta} \|\nabla f\|_2^\theta \geq \left(\frac{-L[v_{\infty, M}]}{C_*} \right)^{1/\delta} \frac{1}{M^{1/(2p)}} \|f\|_{2p}. \quad (3.78)$$

Since for each $\lambda > 0$

$$\frac{\|\lambda f\|_{1+p}^{1-\theta} \|\nabla(\lambda f)\|_2^\theta}{\|\lambda f\|_{2p}} = \frac{\|f\|_{1+p}^{1-\theta} \|\nabla f\|_2^\theta}{\|f\|_{2p}}, \quad (3.79)$$

the inequality (3.78) holds for every locally integrable function $f \in \mathbb{R}^d$ with $\nabla f \in L^2(\mathbb{R}^d)$. It is the GNS inequality (3.4), for we finally prove the

Proposition 3.6.

$$C_{GNS}(d, p) = M^{1/(2p)} \left(\frac{C_*}{-L[v_{\infty, M}]} \right)^{1/\delta}, \quad (3.80)$$

where $C_{GNS}(d, p)$ is defined in (3.5), C_* and δ in (3.74) and (3.76), $v_{\infty, M}$ and L in (3.10) and (3.39) respectively; $m = (p+1)/2p$ and $\beta = 2 - d(p-1)/2p$.

Proof. For convenience we state the following relations:

$$\frac{1}{1-m} = \frac{2p}{p-1}, \quad \frac{m}{1-m} = \frac{p+1}{p-1}, \quad \frac{1-m}{\beta m} = \frac{p-1}{p+1} \frac{p}{4p-dp+d}$$

Now we compute $L[v_{\infty, M}]$. The first step is to find an explicit expression for $D(M)$ in the definition of $v_{\infty, M}$. Using polar coordinates we rewrite

$$\begin{aligned} \int_{\mathbb{R}^d} v_{\infty, M} dx &= \frac{d\pi^{d/2}}{\Gamma(\frac{d}{2}+1)} \int_0^\infty \left(\frac{1}{D(M) + \frac{1-m}{2\beta m} r^2} \right)^{\frac{1}{1-m}} r^{d-1} dr \\ &= \frac{d\pi^{d/2}}{\Gamma(\frac{d}{2}+1)} \left(\frac{1}{D(M)} \right)^{\frac{1}{1-m}} \left(\frac{2\beta m}{1-m} D(M) \right)^{\frac{d}{2}} \int_0^\infty \left(\frac{1}{1+s^2} \right)^{\frac{1}{1-m}} s^{d-1} ds \\ &= \frac{d\pi^{d/2}}{\Gamma(\frac{d}{2}+1)} \left(\frac{1}{D(M)} \right)^{\frac{1}{1-m}} \left(\frac{2\beta m}{1-m} D(M) \right)^{\frac{d}{2}} \frac{\Gamma(\frac{d}{2}) \Gamma(\frac{2p}{p-1} - \frac{d}{2})}{\Gamma(\frac{2p}{p-1})}, \end{aligned} \quad (3.81)$$

where in the last step we used the formula [Table of integrals, 3.251.2].

With this expression and (3.11) we find

$$D(M) = \left(M \frac{\Gamma(\frac{2p}{p-1})}{\Gamma(\frac{2p}{p-1} - \frac{d}{2})} \left(\frac{1-m}{2\beta m} \right)^{\frac{d}{2}} \frac{1}{\pi^{d/2}} \right)^{\frac{2(p-1)}{dp-d-4p}}, \quad (3.82)$$

since $\Gamma(x+1) = x\Gamma(x)$ for all $x \in \mathbb{R}$.

Using once again the formula [Table of integrals, 3.251.2] we compute

$$\begin{aligned} L[v_{\infty, M}] &= D(M)^{\frac{(d-2)p-(d+2)}{2(p-1)}} \frac{d\pi^{d/2}}{\Gamma(\frac{d}{2}+1)} \left(\frac{2\beta(p+1)}{p-1} \right)^{\frac{d}{2}} \frac{\beta p}{p-1} \Gamma\left(1 - \frac{d}{2} + \frac{2}{p-1}\right) \\ &\quad \cdot \left[\frac{p+1}{2p} \frac{\Gamma(1 + \frac{d}{2})}{\Gamma(\frac{2p}{p-1})} - \frac{\Gamma(\frac{d}{2})}{\Gamma(\frac{1+p}{p-1})} \right] \end{aligned}$$

A long direct computation shows, that

$$\begin{aligned}
M^{1/2p} \left(\frac{C_*}{-L[v_{\infty, M}]} \right)^{1/\delta} &= M^{1/2p} \\
&\cdot \left[\beta \left(\frac{2p}{p-1} - d \right) \left(\frac{d}{\beta(1+p)^2} \right)^{\frac{d-dp}{d-dp+4p}} + \frac{1}{2} \beta^2 (1+p)^2 \left(\frac{d}{\beta(1+p)^2} \right)^{\frac{2(d-dp+2p)}{d-dp+4p}} \right]^{\frac{1}{2p} \frac{4p-dp+d}{d+2-(d-2)p}} \\
&\cdot \left[M \frac{\Gamma\left(\frac{2p}{p-1}\right)}{\Gamma\left(\frac{2p}{p-1} - \frac{d}{2}\right)} \left(\frac{1-m}{2\beta m} \right)^{\frac{d}{2}} \frac{1}{\pi^{d/2}} \right]^{\frac{2(p-1)}{dp-d-4p} \frac{(d-2)p-(d+2)}{2(p-1)} \left(-\frac{1}{2p} \frac{4p-dp+d}{d+2-(d-2)p} \right)} \\
&\cdot \left[\frac{\Gamma\left(\frac{d}{2} + 1\right)}{d\pi^{d/2}} \left(\frac{p-1}{2\beta(p+1)} \right)^{\frac{d}{2}} \frac{p-1}{\beta p} \frac{1}{\Gamma\left(1 - \frac{d}{2} + \frac{2}{p-1}\right)} \right]^{\frac{1}{2p} \frac{4p-dp+d}{d+2-(d-2)p}} \\
&\cdot \left[\frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{1+p}{p-1}\right)} - \frac{p+1}{2p} \frac{\Gamma\left(1 + \frac{d}{2}\right)}{\Gamma\left(\frac{2p}{p-1}\right)} \right]^{-\frac{1}{2p} \frac{4p-dp+d}{d+2-(d-2)p}} \quad (3.83)
\end{aligned}$$

is indeed C_{GNS} . □

It remains to characterise the optimisers of the sharp GNS inequality.

3.3.3 The optimisers of the sharp GNS inequality

We will use the derivations of the previous section, in particular (3.52). The following proof is not complete, since we have found a mistake in [Carlen-Carrillo-Loss].

Theorem 3.7. *Let f be a positive measurable function on \mathbb{R}^d , $d \geq 2$, with a square integrable gradient and satisfying*

$$\sup_{x \in \mathbb{R}^d} \frac{f^{-(p-1)/2p}(x)}{1 + |x|^2} \leq \infty.$$

Then f is an optimiser of the GNS inequality (3.4) if and only if it has the form (3.6) up to translations and dilations.

Proof. Consider $v(x, 0) = f^{2p}(x)$ as the initial data for the FDE (3.9) with $m = (p+1)/2p$. Under these conditions the inequality (3.52) holds and the inequality (3.53) is equivalent to the sharp GNS inequality for f . If f is a GNS optimiser, it must be the case that

$$\int_0^\infty R[v(\cdot, t)] dt = 0.$$

Since $v(\cdot, t)$ is positive for all $t > 0$, we conclude that $\Delta \xi = 0$ for all $t > 0$. We aim to use Liouville's theorem (see, e.g., [Harmonic, Th.2.1]) to show, that ξ is constant. ξ is harmonic, so we only need to prove, that it is globally bounded.

If the global boundedness takes place, ξ is indeed constant. $v(\cdot, t)$ is thus a Barenblatt profile for each t , also for $t = 0$. It follows from

$$\frac{1}{m-1} = \frac{2p}{1-p}$$

that f has the form (3.6) up to translations and dilations. □

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