Ludwig-Maximilians-Universität München
Mathematisches Institut

Diplomarbeit

The Kakeya Problem

Markus Furtner

Betreuer : Prof. Dr. Laszlo Erdös
Eingereicht am : 2. Juli 2008

Declaration of Authorship

I certify that I have authored this Diploma Thesis independently and without use of others than the indicated resources. It has not been submitted for a degree or examination at any other university.

München, 2. Juli 2008

Abstract

The original Kakeya problem, posed by S. Kakeya in 1917, is to find a planar domain with the smallest area so that a unit line segment (a "needle") can be rotated by 180 degrees in this domain. It turned out that such sets may have arbitrary small measure, as shown by A. S. Besicovitch in 1928, whereas he also showed that for $n \geq 2$ there are subsets of $\mathbb{R}^n$ of measure zero which contain a unit line segment in each direction. Such sets are called Besicovitch sets or Kakeya sets. The Kakeya conjecture is that Kakeya sets must have (Hausdorff or Minkowski-) dimension at least n. It is already proved for $n = 2$ but is still open in higher dimensions. For a more quantitative approach the problem will be translated into bounds for Kakeya maximal functions. This paper deals about such a bound in 3 dimensions, according to Bourgain [8], which implies that the dimension of Kakeya sets is at least $\frac{7}{3}$. The proof goes through a *restricted weak type (p,q)* bound, which involves mostly geometrical considerations, namely the so called "bush" argument. Interpolation with the 2-dimensional bound yield the asserted bound for Kakeya maximal functions.

Contents

1 Kakeya sets

A Kakeya set is a compact set $E \subset \mathbb{R}^n$ which contains a unit line segment in each direction, i.e.

$$\forall \xi \in S^{n-1} \exists x \in \mathbb{R}^n : x + t\xi \in E \quad \forall t \in [-\frac{1}{2}, \frac{1}{2}]. \tag{1.1}$$

For example, a disc of diameter one is a Kakeya set, which has area $\frac{\pi}{4}$ for $n = 2$. The issue is to find Kakeya sets with small Lebesgue measure. It turned out that there are Kakeya sets of measure zero, which was shown first by Besicovitch (1919) [2].

1.1 Besicovitch sets

In 1917, Abram Samoilovitch Besicovitch (1891-1970) was working on problems about Riemann integration when the following question arose:
If f is a Riemann integrable function defined on the plane, is it always possible to find orthogonal coordinate axis such that with respect to these coordinates the integral $\int f(x, y)dx$ exists as a Riemann integral for all y where the resulting function of y is also Riemann integrable?
Besicovitch noticed that if he constructed a compact set F of plane Lebesgue measure zero containing a unit line segment in every direction he could answer his question by showing a counter example: Fix a pair of axes such that each line segment in F which is parallel to one of the axis has no rational distance to it. Let F_0 be the set of all points in F with at least one rational coordinate. As F contains a unit line segment in every direction on which both F_0 and its complement are dense, there is a segment in each direction on which $f := \chi_{F_0}$ is not Riemann integrable. However, f is Riemann integrable over the plane.
Besicovitch actually succeeded in constructing such a set, which is known as *Besicovitch set*. His original construction (1919 [2], 1928 [3]) has been simplified by Perron (1928)[37], Rademacher (1962)[38], Schoenberg (1962)[40][41], Besicovitch (1963)[4][5] himself and Fisher (1971)[22]. We will describe such a simplified construction, which can also be found in [20] and [43]. The basic idea of all these constructions is to form a 'Perron tree' (see figure 3 on page 7). Obviously such a figure contains a unit line segment at each direction between the directions of the left and the right side of the original (equilateral) triangle. By taking two more copies of such a set and rotate

them through 60 and 120 degrees we will obtain a Kakeya set. We show that these Perron trees may have arbitrary small measure.
In this section $|A|$ denotes the area of a set A, i.e. the 2-dimensional Lebesgue measure of A.

Lemma 1.1. *Consider a triangle T with base on a line L. Divide the base of T into two equal segments and join the points of division to the opposite vertex to form two adjacent triangles T_1 and T_2 with base lengths b and heights h (as shown in the left side of figure 1). Let $\frac{1}{2} < \alpha < 1$. If T_2 is slid the distance $2(1-\alpha)b$ along L to overlap T_1 the resulting figure S consists of a triangle T' similar to triangle T with $|T'| = \alpha^2|T|$ and two auxiliary triangles A_1, A_2 (right side of figure 1). For this construction the area of S is given by*

$$|S| = (\alpha^2 + 2(1-\alpha)^2)|T| \tag{1.2}$$

Proof. We will calculate the area of the triangles A_1, A_2 and T', where $|S| = |A_1| + |A_2| + |T'|$.
T' is similar to T and the base of T' has length $2b - 2(1-\alpha)b = 2\alpha b$, therefore $|T'| = \alpha^2|T|$.
For calculating the area of A_1 and A_2 we draw a line parallel to the base line of L that passes through the intersection point of A_1 and A_2 (see figure 2). Hence we get four triangles $A_{1u}, A_{1l}, A_{2u}, A_{2l}$. By elementary geometric consideration A_{1u} is similar to T_1 with ratio $1-\alpha$ and A_{2u} is similar to T_2 also with ratio $1-\alpha$. Furthermore, A_{2l} is congruent to A_{1u}, and A_{1l} is congruent to A_{2u}. Consequently all four triangles have base length $(1-\alpha)b$ and height $(1-\alpha)h$, hence

$$|A_{u1}| = |A_{l1}| = |A_{u2}| = |A_{l2}| = \frac{1}{2}(1-\alpha)^2|T| \tag{1.3}$$

and together

$$|S| = |T'| + |A_1| + |A_2| = (\alpha^2 + 2(1-\alpha)^2) \cdot |T|. \tag{1.4}$$

□

This elementary process is one main ingredient to this construction of a Besicovitch set. We will call triangles like T' the “heart” and triangles like A_1, A_2 the “arms”. The “monster” we create will have a tiny heart and many arms.

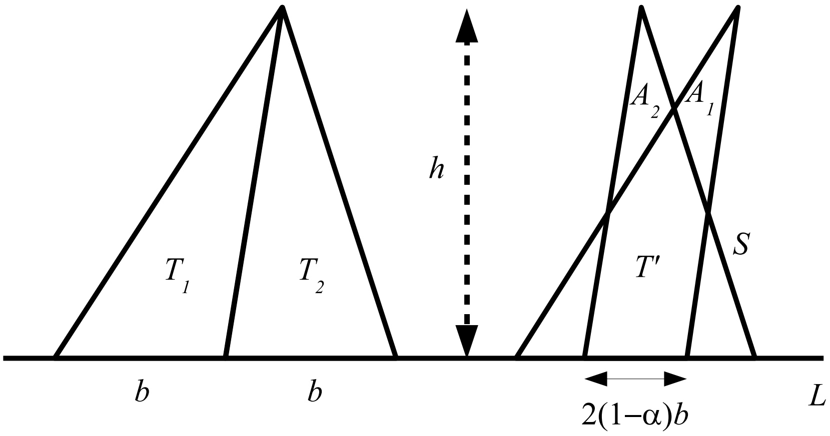

Figure 1: Bisection of the triangle T and the overlapping figure S

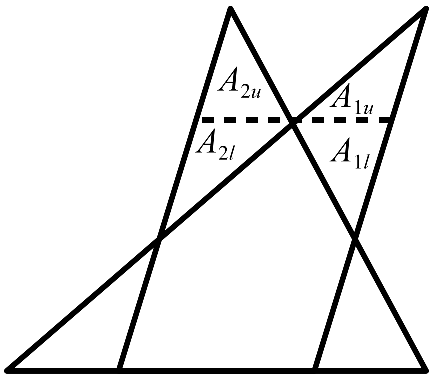

Figure 2: The triangles A_{1u}, A_{1l}, A_{2u} and A_{2l} have the same area

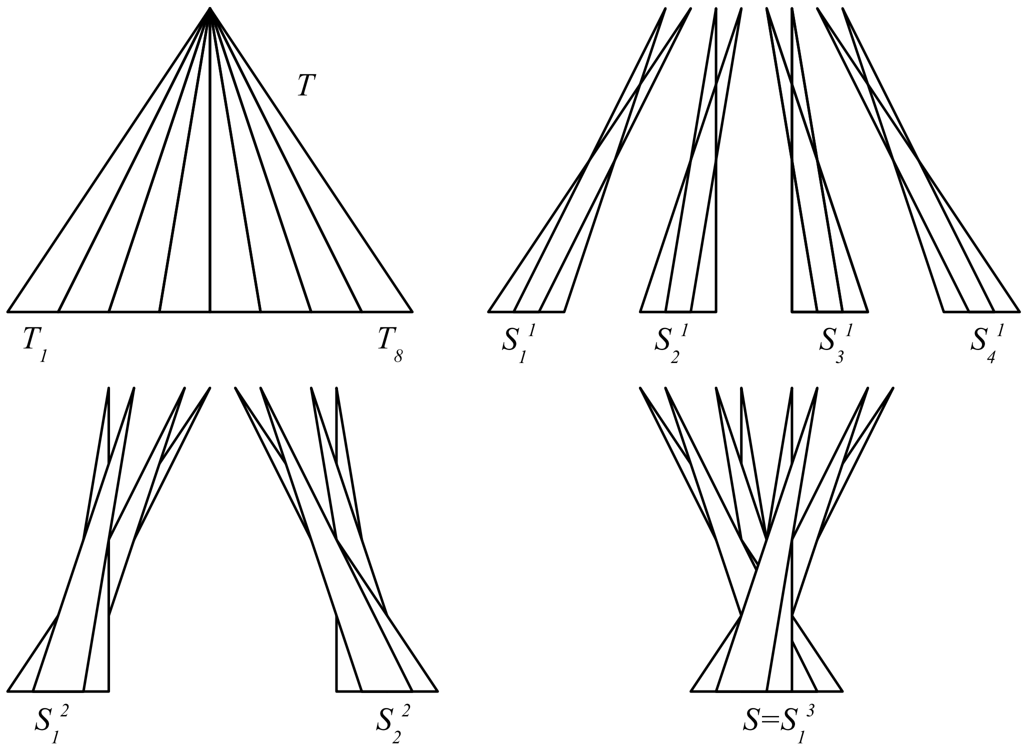

Figure 3: The stages of construction in the case of $k = 3$.

Theorem 1.2. *Consider a triangle T with base on a line L. Divide the base of T into 2^k equal segments and join the points of division to the opposite vertex to form 2^k elementary triangles $T_1, \dots, T_{2^k}$. If k is large enough there is a translation along L for each T_1 $(1 \leq i \leq 2^k)$ such that the area of the resulting (closed) figure S, which is the union of the translated T_i, is as small as desired. Furthermore, for an open set V containing T, this translation can be done such that $S \subset V$ (with larger k if necessary).*

Proof. In the first step we consider consecutive pairs of elementary triangles T_{2i-1}, T_{2i}, $(1 \leq i \leq 2^{k-1})$. Move T_{2i} along L relative to T_{2i-1} as described in lemma 1.1 to obtain a figure S_i^1 consisting of a heart triangle T_i^1 (similar to $T_{2i} \cup T_{2i-1}$) and two arms A_{2i}^1, A_{2i-1}^1. The coefficient α is fixed throughout the construction and will be specified later.

According to lemma 1.1 we have

$$|S_i^1| = (\alpha^2 + 2(1-\alpha)^2)(|T_{2i-1} \cup T_{2i}|).$$

Note that for each i one side of the triangle T_{2i-1}^1 is parallel and equal to the opposite side of T_{2i}^1. Therefore we can translate the S_i^1 such that the 2^{n-1} hearts T_i^1 form one composed heart, which is similar to the original triangle T.
During the construction there might occur some overlap among the arms of S_i^1 although we will not take advantage of it.
In the second step we work with consecutive S_i^1. For $1 \leq i \leq 2^{k-2}$ translate S_{2i} relative to S_{2i-1} to obtain 2^{k-2} figures S_i^2. The total area of its hearts is $\alpha^2 \cdot \alpha^2 \cdot |T|$ and the area of the additional arms is not greater than $2(\alpha-1)^2\alpha^2 \cdot |T|$. Thus

$$\sum_{i=1}^{2^{k-2}} |S_i^2| \leq (\alpha^4 + 2(1-\alpha)^2 + 2(\alpha-1)^2\alpha^2))|T|. \tag{1.5}$$

In the r-th step $(2 \leq r \leq k)$ we translate the figures $S_i^{r-1}(1 \leq i \leq 2^{k-r+1})$ as described in the second step to obtain 2^{k-r} figures $S_1^{r+1}, \ldots, S_{2^{k-r}}^{r+1}$. In each step the area of the hearts will be multiplied by α^2 and the area of the additional arms in the r-th step will not be greater than $2(1-\alpha)^2\alpha^{2r-2} \cdot |T|$. We finally end up with a single figure $S_1^k =: S$, for which holds

$$|S| \leq \big(\alpha^{2k} + 2(1-\alpha)^2 \sum_{i=0}^{n-1} \alpha^{2i}\big)|T|$$

where

$$2(1-\alpha)^2 \sum_{i=0}^{n-1} \alpha^{2i} \leq 2(1-\alpha)^2 \sum_{i=0}^{\infty} \alpha^{2i} = \frac{2(1-\alpha)^2}{1-\alpha^2} = 2\frac{1-\alpha}{1+\alpha} < 2(1-\alpha)$$

thus

$$|S| \leq (\alpha^{2k} + 2(1-\alpha))|T| \tag{1.6}$$

By taking α close enough to 1 and then choosing k large we can make the factor $\alpha^{2k} + 2(1-\alpha)$ arbitrary small.
Now let V be a given open set with $T \subset V$.

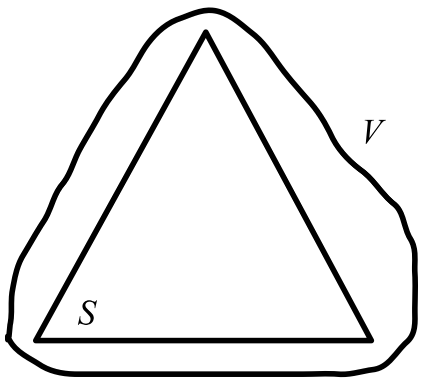

Figure 4: An open set V which contains S

Observe that by fixing the position of one elementary triangle T_1 throughout this construction, no elementary triangle will be moved more than the length of the base of T relative to T_1.
Divide T into subtriangles of base length at most ϵ and apply the described construction on these subtriangles, i.e. we construct about $\frac{1}{\epsilon}$ Perron trees which together form a set S' of arbitrary small area, where for every point in S' the distance to T will be less than ϵ. If ϵ is chosen small enough then $S' \subset V$ □

Theorem 1.3. *For $n \geq 2$, there is a set $F \subset \mathbb{R}^n$ with n-dimensional Lebesgue measure zero which contains a unit line segment in every direction.*

Proof. For $n = 2$, we construct a set F_1 of measure zero containing a unit line segment in every direction within a sector of 60 degrees. By rotating copies of F_1 and taking their union we obtain the desired set.
Let S_0 be an equilateral triangle with unit height. Let V_0 be an open set containing S_0 such that its closure $\overline{V_0}$ is not too large, say $|\overline{V_0}| \leq 2|S_0|$. Apply the construction described in theorem 1.2 on S_0 to obtain a figure $S_1 \subset V_0$ with $|S_1| \leq 2^{-2}$. Since S_1 is a finite union of triangles there is an open set V_1 which satisfies $S_1 \subset V_1 \subset V_0$ and $|\overline{V_1}| \leq 2|S_1|$. Analogously we reapply theorem 1.2 on S_{i-1} to obtain a figure S_i which satisfies

(i)
$$|S_i| \leq 2^{-i-1}$$

(ii) there is an open set V_i such that
$$S_i \subset V_i \subset V_{i-1} \text{ and } |\overline{V_i}| \leq 2|S_i|.$$

Let

$$F_1 := \bigcap_{i=0}^{\infty} \overline{V_i}$$

We show that this closed set has the required properties. Obviously $|F_1| = 0$. On the other hand we know that each S_i and thus each $\overline{V}_i$ contains a unit line segment in every direction θ which makes an angle of at least 60 degrees with the base line. Let M_i be a unit line segment with such a direction θ and $M_i \subset \overline{V}_i$. For a fixed j note that $M_i \subset \overline{V}_j$ for $i \geq j$ and that $\overline{V}_j$ is compact. Hence the sequence $\{M_i\}$ has a subsequence which converges to a unit line segment M with direction θ. Since for each j we have $M_i \subset \overline{V}_j$ if $i \geq j$, we also have $M \subset \overline{V}_j$ for each j. Thus

$$M \subset \bigcap_{i=0}^{\infty} \overline{V}_i = F_1$$

For $n > 2$ take a 2-dimensional Besicovitch set F_2. Then the set $F := F_2 \times \mathbb{R}^{n-2}$ obviously has n-dimensional Lebesgue measure zero and contains a unit line segment in every direction. □

Another way to construct Besicovitch sets is the *dual approach* [20]. The basic idea is to parametrize lines by points of so called *irregular 1-sets*, for example the *Cantor dust* (see figure 5). The iteration of the Cantor dust E starts with a square of side length 2 centered in the origin. In each step every square A_k will be replaced by four smaller squares with side length $\frac{1}{4}d$, where d is the side length of the A_k. Relative to the lower left corner of A_k the position of the lower left corners of the small squares is $(0, \frac{1}{2}d), (\frac{1}{4}d, 0), (\frac{1}{2}d, \frac{3}{4}d)$ and $(\frac{3}{4}d, \frac{1}{4}d)$ respectively.
Observe that both the projection to the x-axis and the projection to the y-axis contain the whole interval $[-1, 1]$ in each step of iteration. Hence we can parametrize a set of lines $L(E) := \{y = ax + b,\ (a, b) \in E\}$ with all slopes within $[-1, 1]$. Again, by taking another rotated copie of $L(E)$ we get a set with an (infinite) line in every direction, a Kakeya set. It turns out that $L(E)$ indeed has measure zero . One has to show that every projection of E to an axis which is neither parallel to the x-axis nor to the y-axis has measure zero. Sets with this property are called irregular 1-sets. Then one applies the following statement according to [20], [21]:
If E is an irregular 1-set then $L(E)$ has measure zero.
Thus, there is a strengthened version of theorem 1.3.

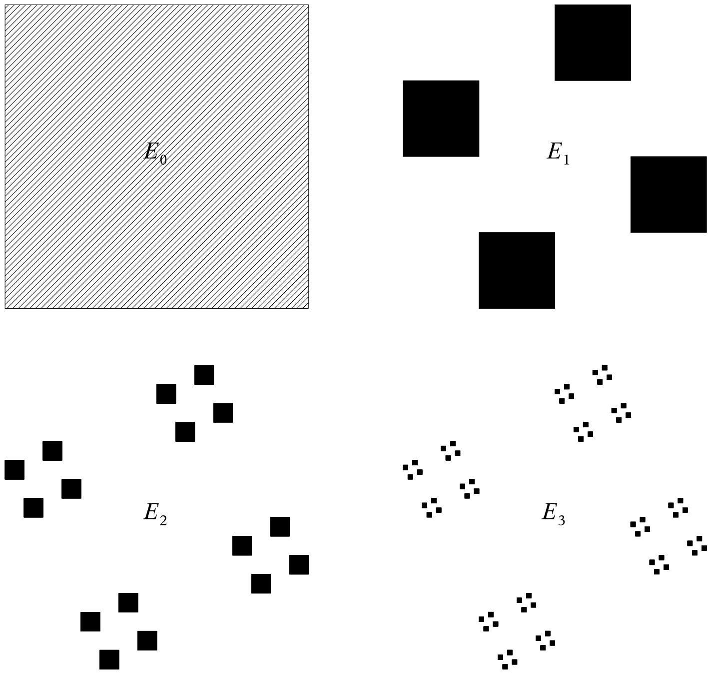

Figure 5: The first three iterations of the Cantor dust

Theorem 1.4. *For $n > 2$ there is a set $F \subset \mathbb{R}^n$ of n-dimensional Lebesgue measure zero which contains a line in every direction.*

1.2 The Kakeya Needle Problem

Around the same time as Besicovitch, in 1917, the Japanese mathematician Soichi Kakeya asked a similar question as Besicovitch:
What is the smallest area of a planar region within which a unit line segment (a "needle") can be rotated continuously through 180 degrees, returning to its original position but with reversed orientation?
Kakeya [24] and Fujiwara-Kakeya [23] conjectured that the smallest *convex* set which satisfies this property was an equilateral triangle of unit height, which was proved only a few years later by Julius Pàl [36]. For the case without the convexity assumption they apparently thought that it was a three-pointed deltoid, which has area $\frac{\pi}{8} \approx 0.39$, whereas the area of the triangle is $\frac{\sqrt{3}}{3} \approx 0.58$. However, the problem remained unsolved. Due to

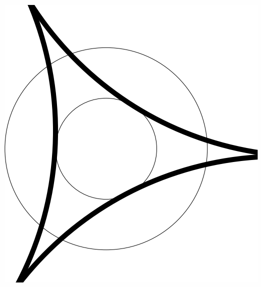

Figure 6: A three pointed deltoid, within which a needle can be rotated

isolation of Russia from the Western world Besicovitch did not notice about the Kakeya needle problem until he left Russia. After the problem finally arrived to him, he realized that a modification of his original construction together with a trick from Pàl will provide the unexpected answer:
For any $\epsilon > 0$, there is a planar region of area less than ϵ within which a needle can be rotated through 180 degrees.
This solution was published in 1928 in *Mathematische Zeitschrift* [3] where also Besicovitch's construction from 1919 finally drew the attention.
The trick of Pàl, also known as *Pàl joins*, yields the following lemma:

Lemma 1.5. *Given two parallel lines L_1, L_2. Then any unit line segment can be moved continuously from L_1 to L_2 on a set of arbitrary small measure.*

Proof. Given $\epsilon > 0$ and a unit line segment $M \subset L_1$ centered at x_M. Rotate M around x_M such that a positive area less than $\frac{\epsilon}{2}$ is taken, which allows an angle about ϵ. Then translate M along its new direction until $x_M \in L_2$ and rotate it back, so the total area of the set is less than ϵ (see figure 7). □

Together with theorem 1.2, the solution of the Kakeya needle problem is immediate.

Theorem 1.6. *Given $\epsilon > 0$ and a unit line segment M_0 with end points $a_{t=0}$ and $b_{t=0}$. There is a set E with $|E| < \epsilon$ and a continuous moving $\{M_t\}_{t\in[0,1]}$ such that $M_t \subset E$ for all $t \in [0,1]$ and $a_0 = b_1$, $b_0 = a_1$.*

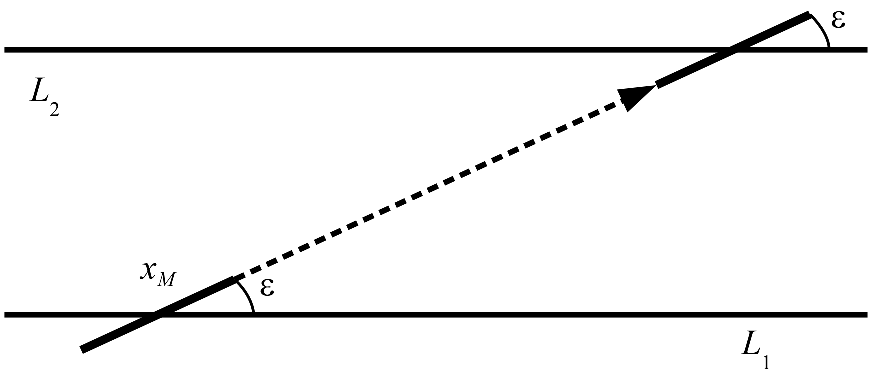

Figure 7: Translation of a needle from L_1 to L_2

Proof. Let T be an equilateral triangle with unit height. We use theorem 1.2 to construct from T a figure S_k with $|S_k| < \frac{\epsilon}{6}$, which is a union of $m = 2^k$ elementary triangles. S_k still contains a unit line segment in all directions within a sector of 60 degrees. Again, taking three appropriately rotated copies of S_k we obtain a Kakeya set E_0 with $|E_0| < \frac{\epsilon}{2}$.
E_0 is a union of $3m$ elementary triangles T_i, $1 \leq i \leq 3m$. For each i one side of T_i is parallel to the opposite side of T_{i+1}. By lemma 1.5 we can add a set N_i with $|N_i| \leq \frac{\epsilon}{6m}$ to E_0 such that a unit line segment can be moved from T_i to T_{i+1}. We eventually obtain the set $E := \bigcup_{i=1}^{3m-1} N_i \cup E_0$, which has the desired properties. □

Let a *Kakeya needle set* be a set as described above, i.e. a set within which a unit line segment can be rotated continuously through 180 degrees, returning to its original position with reversed orientation. One can ask for specific Kakeya needle sets, for example convex, star-shaped, simply connected or bounded sets. We already mentioned the rather strong restriction to convex sets, where the lower bound is $\frac{\sqrt{3}}{3}$ [36]. Note that due to the Pàl joins, the Kakeya needle set constructed above is highly multiply connected and may have large diameter. Van Alphen (1941)[1] showed that there are arbitrary small Kakeya needle sets inside a circle with radius $2 + \epsilon$ (arbitrary $\epsilon > 0$). Simply connected Kakeya needle sets with smaller area than the deltoid were found in 1965. Melvin Bloom and I.J. Schoenberg [15] independently presented Kakeya needle sets with areas approaching to $(5 - 2\sqrt{2})\pi/24 \approx 0.9\pi$,

the *Bloom-Schoenberg number.* While Schoenberg conjectured that this number is the lower bound for the area of simply connected Kakeya needle sets, again, an unexpected answer due to Cunningham (1971)[13] settled the problem:
Given $\epsilon > 0$, there is a simply connected Kakeya needle set of area less than ϵ contained in a circle of radius 1.
Note that the radius of the circle cannot become smaller, also if the assumption of simply connectedness is dropped: Let $r < 1$ and consider the discs $B(0, r)$ and $B(0, 1-r)$. For $x \in B(0, 1-r)$, the set $B(0, r) \setminus \{x\}$ obviously is not a Kakeya needle set (note that it is necessary for the needle to return to its starting point with reversed direction). Hence the whole disc $B(0, 1-r)$ must be contained in a Kakeya needle set within $B(0, r)$, which means the set must have area at least $(1-r)^2\pi$.
In Cunningham's construction there is a small fixed convex set in the center of the circle (the "nucleus") and from this center a large number of thin triangles and joins radiates. To these triangles a construction resembling to the construction of Perron trees will be applied. The number of elementary triangles and joins required to obtain area less than, for example, the Bloom-Schoenberg number, is approximately 10^{43}.
A class between simply connected sets and convex sets are the star-shaped sets. A set K is called star-shaped with respect to z, if for any point $x \in K$ the line joining z and x is contained in K. Cunningham [13] showed that every star-shaped Kakeya needle set has area at least $\frac{\pi}{108}$. The smallest star-shaped sets known are those from Bloom-Schoenberg. Whether there are star-shaped Kakeya needle sets with area between $\frac{\pi}{108}$ and the Bloom-Schoenberg number is still not known.

1.3 Analogues and Generalizations of the Kakeya problem

There are several problems which are closely related to the original Kakeya problem. The first three problems mentioned are the "Three Kakeya problems" discussed by Cunningham (1974)[14].

1. *The spherical Kakeya problem.*
   Instead of a plane, the rotation takes place on the surface of a unit sphere and an arc of great circle plays the role of the needle. Knowing the answer to the Kakeya needle problem one expects that the value

of the greatest lower bound k_a is zero for small arc lengths a. This has been shown by Wilker [47] for $a < \frac{\pi}{2}$. On the other hand, if $a = 2\pi$ it follows from topological arguments that the area always has to be the whole sphere, i.e. $k_{2\pi} = 4\pi$. Any proper subset of a sphere is topologically equivalent to a subset of the plane, hence the problem is topologically equivalent to a one-to-one reversal of a circle in the plane, which is impossible. Following this argument will also give an answer to the case $a = \pi$. Consider a great semicircle and its antipodal image, which form together one whole great circle. As one semicircle gets reversed in a set K the antipodal semicircle gets reversed in a set K', hence a whole great circle gets reversed in $K \cup K'$, which means that $K \cup K'$ is the unit sphere. Since K and K' are congruent and, in particular, have equal area, consequently $|K| = |K'| \geq 2\pi$. Obviously, this minimum can be attained by a simple rotation, thus $k_\pi = 2\pi$.
Cunningham again gives the — meanwhile maybe less unexpected — answer to the spherical Kakeya problem.
With k_a as defined above for $0 < a \leq 2\pi$, we have

$$k_a = \begin{cases} 0 & \textit{for } a < \pi \\ 2\pi & \textit{for } \pi \leq a < 2\pi \\ 4\pi & \textit{for } a = 2\pi. \end{cases}$$

The lower bound is achieved when $a = \pi$ or 2π, not otherwise.

2. *Shrinking a circular arc.*
It is known that there are "thin sets of circles" — sets of measure zero which contain circles of every radius [6],[29], [17]. Besicovitch called the Kakeya needle problem the twin to his problem about sets which contain a unit line in every direction. Analogously a twin to the thin set of circles would be the following: Given an arc A of $a < 2\pi$ radians, and shrink A continuously to a single point while the radius is variable but the radian measure has to be constant. What is the greates lower bound of area within which that can be done? Note, again for topological reasons that for $a = 2\pi$ the area cannot be less than the area of the disc encircled by the (degenerated) arc. Cunningham applies his solution to the spherical Kakeya problem, so again the situation changes dramatically when $a < 2\pi$:
Given $a < 2\pi$ and $\epsilon > 0$, there is a plane set K of area less than ϵ

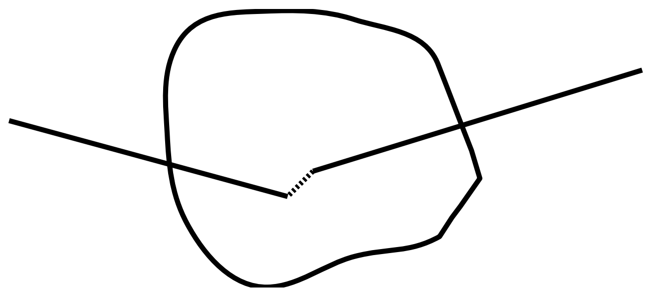

Figure 8: The body (dashed line) of the "bird" has to sweep over a given set

within which a circular arc of a radians and unit radius can move continuously, with variable radius but fixed radian measure, so as to shrink to a point.
However, Wolff [48] [49] proved that sets containing a sphere of every radius must have full dimension by proving bounds on a circular maximal function analogous to the *Kakeya maximal function* in section 2.4. This statement holds in any dimension.
It was also conjectured that there existed sets containing a sphere around every point of measure zero. Stein [42] proved that such sets must have positive measure when $n \geq 3$. Marstrand [33] and Bourgain [7] independently proved that this is also true for $n = 2$.

3. *Alternative moving figure.*
Instead of a straight line segment consider a broken line segment consisting of three segments (see figure 8). We call such a figure a "bird", where the central segment represents the body and the end segments the wings . Now, for a given set move the bird continuously such that the body touches every point of that set but the wings should sweep only over a set of small measure. Again, for any given length for the (small) body and the (large) wings, this can be done such that the wings stay in a set of arbitrary small measure.
If the claim for a continuous moving is dropped, the body may be just one point while the wings are infinitely long. Then the positions of this point may cover the whole plane and the wings stay in a set of measure zero.

4. *Nikodym sets.*
A Nikodym set in the unit square $K := [0, 1]^2$ in the plane is a subset

$N \subset K$ with area 1 such that for every point $x \in N$ there is a straight line L through x such that $L \cap N = \{x\}$.
The existence of such sets was first proved in 1927 by O. M. Nikodym [35]. While for Kakeya sets the issue is to find estimates on the *Kakeya maximal function* in order to obtain Hausdorff bounds, the issue for Nikodym sets is to find estimates of the closely related *Nikodym maximal function*. One often considers estimates for both of them at the same time, e.g. [8],[50]. The connection between Hausdorff dimension and Kakeya maximal functions will be discussed in section 2.4.

5. *(n,k)-sets.*
A (n,k)-set is a compact set $E \subset \mathbb{R}^n$ which contains a translate of every k-dimensional subspace, or alternatively an essential portion of that subspace like a disc. According to that definition Kakeya sets are $(n,1)$-sets and we say that (n,k)-Besicovitch sets are (n,k)-sets of measure zero.
One conjectures that there are no (n,k)-Besicovitch sets for $k > 1$. Marstrand (1979)[34] proved that there are no $(3,2)$-Besicovitch sets and Falconer (1980)[19] proved that there were no (n,k)-Besicovitch sets for $k > \frac{n}{2}$. The best bound to date is due to Bourgain [8][10], who proved that no (n,k)-Besicovitch sets exist when $2^{k-1} + k \geq n$. In particular, there are no $(4,2)$-Besicovitch sets. Whether there are (n,k)-Besicovitch sets for $n \geq 5$ is still open.

2 Concepts in the Research of Kakeya sets

2.1 Fractal Dimension

We have seen that (surprisingly) there are Kakeya sets of Lebesgue measure zero. However, it turns out that many problems in analysis require a more detailed information about the size of Kakeya sets, namely in terms of *fractal dimension* (or just *dimension*). There are several concepts of fractal dimensions; in the research of Kakeya sets the *Minkowski dimension* and the *Hausdorff dimension* are of substantial interest. For sets like lines, squares or cubes both the Minkowski and the Hausdorff dimension coincide with the usual concept of dimension. However, the fractal dimension of a set, as the name suggests, need not be a natural number.

Minkowski dimension

The Minkowski dimension, or *box dimension* is defined for compact sets $E \subset \mathbb{R}^n$. Define the δ-neighborhood of E

$$E_\delta := \{x \in \mathbb{R}^n : \operatorname{dist}(x, E) < \delta\} \tag{2.1}$$

and consider the n-dimensional volume of E_δ as $\delta \to 0$. The *upper Minkowski dimension* α_u and *lower Minkowski dimension* α_l are given by

$$\alpha_u = \overline{\dim}_M(E) := \inf\{\alpha \in [0, n] : |E_\delta| \leq C_\alpha \delta^{n-\alpha}\} \tag{2.2}$$

$$\alpha_l = \underline{\dim}_M(E) = \sup\{\alpha \in [0, n] : |E_\delta| \geq c_\alpha \delta^{n-\alpha}\} \tag{2.3}$$

where the constants C_α and c_α are uniform for $\delta \leq 1$. If $\alpha_u = \alpha_l = \beta$ we will say that E has Minkowski dimension β.
Considering Kakeya sets, the goal is to find lower bounds for the Minkowski dimension of Kakeya sets, which gives us one formulation of the *Kakeya conjecture*

Conjecture 1. *Kakeya sets in $\mathbb{R}^n$ must have Minkowski dimension n.*

Hausdorff dimension

The definition of the Hausdorff dimension goes via *Hausdorff measures.* Fix $\alpha > 0$ and let $E \subset \mathbb{R}^n$. For $1 \geq \epsilon > 0$ define

$$H_\alpha^\epsilon(E) = \inf_{\mathcal{K}_\epsilon} \big(\sum_{i=1}^{\infty} r_i^\alpha \big) \tag{2.4}$$

where $\mathcal{K}_\epsilon$ is the set of countable coverings of E by discs $D(x_i, r_i)$ with radius $r_i < \epsilon$, i.e.

$$\mathcal{K}_\epsilon := \{K \supset E | K = \bigcup_{\text{countable}} D(x_i, r_i), \quad r_i < \epsilon\}$$

Obviously $\mathcal{K}_{\epsilon_1} \subset \mathcal{K}_{\epsilon_2}$ for $\epsilon_1 < \epsilon_2$ and therefore

$$H_\alpha^{\epsilon_2} \leq H_\alpha^{\epsilon_1}. \tag{2.5}$$

The *Hausdorff measure* of E with parameter α is given by

$$H_\alpha(E) := \lim_{\epsilon \to 0} H_\alpha^\epsilon(E) \tag{2.6}$$

which may also be infinity.

Proposition 2.1. *1.*

$$H_\alpha^\epsilon(E) \leq H_\beta^\epsilon(E) \tag{2.7}$$

for $\alpha > \beta$ and $\epsilon < 1$, i.e. $H_\alpha(E)$ is a nonincreasing function of α.

2. If $H_\alpha^1(E) = 0$, then $H_\alpha(E) = 0$

3. If $\alpha > n$, then $H_\alpha(E) = 0$ for all $E \subset \mathbb{R}^n$

Proof. 1. Follows directly from the definition since $r_i^\alpha \leq r_i^\beta$ for $r_i < 1$.

2. $H_\alpha^1(E) < \delta$ implies that there is a covering which consists of discs of radius $< \delta^{\frac{1}{\alpha}}$, hence also $H_\alpha^{\delta^{\frac{1}{\alpha}}}(E) < \delta$ for all $\delta > 0$. Thus $H_\alpha^{\delta'}(E) < \delta'^\alpha$ and $H_\alpha(E) = \lim_{\delta' \to 0} H_\alpha^{\delta'}(E) \leq \lim_{\delta' \to 0} \delta'^\alpha = 0$.

3. If $\alpha > n$ we can cover $\mathbb{R}^n$ by discs $D(x_i, r_i)$ with $\sum_i r_i^\alpha$ arbitrary small: Given $\epsilon > 0$. Each unit disc in $\mathbb{R}^n$ can be covered by $C2^{kn}$ discs of radius $r_i = 2^{-k}$ for any $k > 0$ and some constant C. Thus for a unit disc there is some k such that $\sum_i r_i^\alpha = C2^{kn - \alpha k} < \epsilon$. Now the assertion follows since there is a countable covering of $\mathbb{R}^n$ by unit discs.

□

Lemma 2.2. *(Hausdorff dimension)*
Let

$$\alpha_0 := \inf\{\alpha : H_\alpha(E) = 0\}$$
$$\alpha_1 := \sup\{\alpha : H_\alpha(E) = \infty\}.$$

Then $\alpha_0 = \alpha_1$*. This number is called the* Hausdorff dimension *of* E*.*

Proof. By (2.7) we have $H_\alpha(E) = \infty$ if $\alpha < \alpha_1$. Now let $\alpha > \alpha_1$ and $\beta \in (\alpha_1, \alpha)$. Define $M := 1 + H_\beta(E) < \infty$. Then there is a covering of E by discs with radius $r_i < \epsilon$ and $H_\beta^\epsilon(E) \leq \sum_i r_i^\beta \leq M$ for $\epsilon > 0$ according to (2.5). Hence

$$H_\alpha^\epsilon(E) \leq \sum_i r_i^\alpha \leq \epsilon^{\alpha-\beta} \sum_i r_i^\beta \leq \epsilon^{\alpha-\beta} M$$

which goes to 0 as $\epsilon \to 0$. Thus $H_\alpha(E) = 0$. □

Analogously to conjecture 1 we can state the Kakeya conjecture for the Hausdorff dimension, which is the most common formulation:

Conjecture 2. *Kakeya sets in* $\mathbb{R}^n$ *must have Hausdorff dimension* n*.*

For $n = 2$ this was first proved by Davies (1971) [16].
The Minkowski dimension can also be defined via Hausdorff measures restricted to coverings by discs of the same size. Therefore the Minkowski dimension of a set cannot be smaller than its Hausdorff dimension, i.e.

$$\dim_H(E) \leq \underline{\dim}_M(E) \leq \overline{\dim}_M(E) \tag{2.8}$$

Usually Hausdorff dimension and Minkowski dimension are equal but it is also possible that the Minkowski dimension is strictly greater than the Hausdorff dimension as we will see in example 2.
This means that Conjecture 1 is slightly weaker than Conjecture 2.
Examples.

1. *Cantor set.*
   The usual $\frac{1}{3}$-Cantor set S can be covered by 2^k intervals of length 3^{-k}, thus the Hausdorff measure is

   $$H_\alpha(S) = \lim_{k\to\infty} H_\alpha^{3^{-k}}(S) = \lim_{k\to\infty} \left(\frac{2}{3^\alpha}\right)^n.$$

   Hence $H_\alpha(S) = 0$ for $\alpha < \frac{\log 2}{\log 3}$ and $H_\alpha(S) = \infty$ for $\alpha > \frac{\log 2}{\log 3}$, i.e. the Hausdorff dimension (and also the Minkowski dimension) of the $\frac{1}{3}$-Cantor set is $\frac{\log 2}{\log 3}$.

2. Consider the set $K := \bigcup_{k=1}^{\infty}\{\frac{1}{k}\} \cup \{0\}$. Since K is countable it obviously has Hausdorff dimension 0. However, K has Minkowski dimension $\frac{1}{2}$: Let

$$K' := \bigcup_{k=1}^{m-1}\{\frac{1}{k}\}$$

and

$$K'' := \bigcup_{k=m}^{\infty}\{\frac{1}{k}\} \cup \{0\}$$

where $\sqrt{2}\delta^{-\frac{1}{2}} \leq m < \sqrt{2}\delta^{-\frac{1}{2}}+1$. The δ-neighborhood K'_δ of K' consists of disjoint discs since $2\delta \leq \frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)} < \frac{1}{n^2}$. Thus

$$|K_\delta| \geq |K'_\delta| = c\delta^{n-\frac{1}{2}}$$

i.e. K has lower Minkowski dimension $\frac{1}{2}$. On the other hand the δ-neighborhood of K'' is not greater than the δ-neighborhood of a line of length m. Hence

$$|K_\delta| \leq |K'_\delta| + |K''_\delta| \leq c\delta^{n-\frac{1}{2}} + (m+2\delta)\delta^{n-1} \leq C\delta^{n-\frac{1}{2}}$$

i.e. K also has upper Minkowski dimension $\frac{1}{2}$.

3. *δ-separated subsets.*
Let $E \subset \mathbb{R}^n$. A set $S \subset E$ is called a *δ-separated* subset, if any two distinct points $x, y \in S$ satisfy

$$|x - y| \geq \delta.$$

We show that the maximal cardinality of a δ-separated ("δ-entropy") subset of E is comparable to the minimum number of δ-discs required to cover E. Define

$$D_\delta(E) := \min\left\{k : \exists x_1, x_2, ...x_k, \text{ s.t. } E \subset \bigcup_{j=1}^{k} D(x_j, \delta)\right\} \tag{2.9}$$

and the δ-entropy on E

$$\mathcal{E}_\delta(E) := \max\left\{\#S \middle| \forall x, y \in S : |x-y| > \delta \vee x = y\right\} \tag{2.10}$$

Then on the one hand

$$D_\delta(E) \leq C_n \mathcal{E}_\delta(E) \tag{2.11}$$

holds.
To show (2.11), assume S satisfies $\#S = \mathcal{E}_\delta(E)$. Let each $x \in S$ be the center of a δ-disc $D(x,\delta)$. If $\bigcup_{x\in S} D(x,\delta)$ does not cover E, then we have points $y \in E$ which satisfy $|x-y| \geq \delta$ for all $x \in S$. Due to the assumption the ineqality cannot be strict, i.e. $|x-y| = \delta$ and the only possibility for any y is to be located at the boundary of a disc. However, it takes only a fixed number of δ-discs to cover the boundary of another δ-disc.
On the other hand we have

$$D_\delta(E) \geq c_n \mathcal{E}_\delta(E) \tag{2.12}$$

since the maximal number of δ-separated points one δ-disc can cover is bounded.
Now, observe that $D_\delta(E) \leq C_\alpha \delta^{-\alpha}$ and $|E_\delta| \leq C'_\alpha \delta^{n-\alpha}$ are equivalent. These inequalities are according to (2.11) and (2.12) equivalent to $\log \mathcal{E}_\delta(E) \leq C_\alpha \alpha \log \frac{1}{\delta}$. Thus the upper Minkowski dimension of E can also be defined as

$$\alpha_l = \liminf_{\delta\to 0} \frac{\log \mathcal{E}_\delta(E)}{\log \frac{1}{\delta}} \tag{2.13}$$

and analogously the lower Minkowski dimension

$$\alpha_u = \limsup_{\delta\to 0} \frac{\log \mathcal{E}_\delta(E)}{\log \frac{1}{\delta}} \tag{2.14}$$

2.2 Fourier Transform

Let $f \in L^1(\mathbb{R}^n)$. Then define the Fourier transform as

$$\hat{f}(\lambda) := \frac{1}{(2\pi)^{n/2}} \int e^{-i\langle\lambda,x\rangle} f(x)dx$$

For details and proofs of the following statements, see for example textbooks like [32] or [39].

Proposition 2.3. *Let $f, g \in L^1(\mathbb{R}^n)$, $\alpha \in \mathbb{R}$, $\epsilon > 0$. Then the following properties hold.*

- *Convolution:*
$$\widehat{(f * g)} = (2\pi)^{n/2}\hat{f}\hat{g} \tag{2.15}$$
- *Translation:*
$$\widehat{f(x-\alpha)} = e^{-i\lambda\alpha}\hat{f}(\lambda) \tag{2.16}$$
- *Scaling:*
$$\widehat{f(\epsilon^{-1}x)} = \epsilon^n \hat{f}(\epsilon\lambda) \tag{2.17}$$
- *Inversion:*
$$\check{\hat{f}} = f. \tag{2.18}$$
where the inverse Fourier transform is defined as
$$\check{f}(\lambda) := \frac{1}{(2\pi)^{n/2}} \int e^{i\langle\lambda,x\rangle} f(x)dx.$$

Theorem 2.4. *(Plancherel).*
Let $f, g \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, then
$$\langle \hat{f}, \hat{g} \rangle = \langle f, g \rangle, \tag{2.19}$$
in particular
$$\|\hat{f}\|_2 = \|f\|_2. \tag{2.20}$$

Another useful property is the invariance of the *Gaussian* $e^{-x^2/2}$ under Fourier transformation:

$$\begin{aligned}
\hat{f}(\lambda) &= \frac{1}{(2\pi)^{n/2}} \int e^{-i\lambda x} e^{-x^2/2} dx \\
&= \frac{1}{(2\pi)^{n/2}} \int e^{-\frac{1}{2}(x+i\lambda)^2 - \frac{\lambda^2}{2}} dx \\
&= \frac{1}{(2\pi)^{n/2}} e^{\lambda^2/2} \int e^{-z^2/2} dz \\
&= e^{\lambda^2/2}.
\end{aligned} \tag{2.21}$$

There is an interesting relation between Fourier transform and dimension [51]. Consider a compactly supported probability measure μ, i.e. $\int d\mu = 1$, and its Fourier transformation $\hat{\mu}$, for which the definition is the same as above. Intuitively, if supp μ is only a set of finite discrete points the measure is some finite linear combination of Dirac δ-functions and therefore there will be no decay of $\hat{\mu}$, i.e. $\hat{\mu}(\lambda) \nrightarrow 0$ as $|\lambda| \to \infty$. On the other hand for a set E with positive measure, there is a probability measure such that $\hat{\mu}$ decays with some exponent, i.e. $|\hat{\mu}(\lambda)| \leq c|\lambda|^{-\alpha}$ for some $\alpha > 0$.
Wolff's result is that if there is a probability measure μ supported on a compact set E with $|\hat{\mu}(\lambda)| \leq C|\lambda|^{-\beta}$ then the dimension of E is at least 2β.
One can ask for an analogous relation for the opposite direction:
Let E be a compact set with dimension α. Does a measure with support on a E and

$$|\hat{\mu}(\lambda)| \leq C_\epsilon(1 + |\lambda|)^{-\frac{\alpha}{2}+\epsilon}$$

for all $\epsilon > 0$ always exist?
The answer is emphatically no:
For the line segment $E = [0, 1] \times \{0\} \subset \mathbb{R}^2$. E has dimension 1, but for any measure supported on E its Fourier transform $\hat{\mu}(\lambda)$ depends on λ_1 only, hence $\hat{\mu}(\lambda)$ cannot vanish at ∞.
Also for $n = 1$ there is a counterexample: The standard $\frac{1}{3}$ Cantor set does not support a measure such that $\hat{\mu}(\lambda) \to 0$ as $|\lambda| \to \infty$.

2.3 Interpolation

To get a more quantitative approach to the Kakeya conjecture it turned out that it is useful to reformulate the problem in terms of *Harmonic analysis*. In Harmonic analysis one typically considers functions[1] $f : \mathbb{R}^n \to \mathbb{R}$ and linear or sublinear operators T on these functions. The issue is to find estimates of norms between f and Tf, i.e.

$$\|Tf\|_p \leq C_{T,p,q,n}\|f\|_q \tag{2.22}$$

or

$$\|Tf\|_p \geq c_{T,p,q,n}\|f\|_q \tag{2.23}$$

[1]Harmonic analysis may also deal with more general objects like measures, distributions etc. and the domains of these objects also can be more general.

where the constants $C_{T,p,q,n}$ and $c_{T,p,q,n}$ do not depend on f, but may depend on T, p, q and n.

Estimates of the form $X \leq CY$ occur quite often in harmonic analysis and the constant C usually is of minor interest or too complicated to calculate. Therefore a modified *Vinogradov notation* will be used:
Denote $X \leq CY$ by $X \lesssim Y$ (read: X is less than or comparable to Y). We also use $X \approx Y$ to denote $X \lesssim Y \lesssim X$. The dependence on parameters can be expressed by subscripts, thus $X \lesssim_k Y$ is synonymous with $X \leq C_k Y$. Note that this notation is transitive if it is used only finitely many times.

Define the quantity

$$\|f\|_{p,\infty} := \sup_{\lambda>0} \lambda |\{|f| > \lambda\}|^{\frac{1}{p}}. \tag{2.24}$$

The set of functions s.t. $\|f\|_{p,\infty} < \infty$ is called the *weak L^p space*. Note that (2.24) is not a norm, but it is equivalent to the norm given by

$$\|f\|^*_{p,\infty} := \sup_A |A|^{-\frac{1}{q}} \int_A |f(x)| dx, \tag{2.25}$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and A denotes an arbitrary set of measure $|A| < \infty$.
Observe that

$$\|f\|_p^p = \int |f|^p \geq \int \lambda^p \chi_{|f| \geq \lambda} = \lambda^p |\{|f| \geq \lambda\}| \tag{2.26}$$

for any $\lambda \geq 0$. Therefore *Chebyshev's inequality*

$$\|f\|_{p,\infty} \leq \|f\|_p \tag{2.27}$$

holds.
An operator T with $Tf : D \to \mathbb{R}$ is said to be *sublinear* if for any constant c

$$|T(cf)| = |c||Tf| \tag{2.28}$$

and the pointwise estimate

$$|T(f+g)| \leq |Tf + Tg| \tag{2.29}$$

holds.
Let $0 < p, q \leq \infty$. There are different concepts for sublinear operators T:

- T is *strong-type* (p, q) with constant C if there is a bound

$$\|Tf\|_q \lesssim_{p,q} C\|f\|_p \tag{2.30}$$

for all $f \in L^p$ or in a dense subclass thereof. Note that in the latter case there is a unique extension to all $f \in L^p$.

- T is *weak-type* (p, q) $(0 < q < \infty)$ with constant C if there is a bound

$$\|Tf\|_{q,\infty} \lesssim_{p,q} C\|f\|_p \tag{2.31}$$

- T is *restricted strong-type* (p, q) with constant C if there is a bound

$$\|T\chi_A\|_q \lesssim_{p,q} C|A|^{\frac{1}{p}} \tag{2.32}$$

- T is *restricted weak-type* (p, q) $(0 < q < \infty)$ with constant C if there is a bound

$$\|T\chi_A\|_{q,\infty} \lesssim_{p,q} C|A|^{\frac{1}{p}} \tag{2.33}$$

For fixed (p, q) strong-type implies weak-type and restricted strong-type whereas either of them imply restricted weak-type. By simple functions we mean finite linear combinations of characteristic functions. The set of simple functions is a dense subclass of L^p, thus for strong-type inequalities it suffices to consider simple functions.

Theorem 2.5. *(Marcinkiewicz interpolation theorem). Let T be a sublinear operator. Suppose that T is restricted weak type (p_1, q_1) with constant $C_1 > 0$ and restricted weak-type (p_2, q_2) with constant $C_2 > 0$ for some $0 < p_1, p_2, q_1, q_2 \leq \infty$ with $p_1 \neq p_2$ and $q_1 \neq q_2$. Define for $0 \leq \theta \leq 1$*

$$\frac{1}{p_\theta} := \frac{1-\theta}{p_1} + \frac{\theta}{p_2}; \quad \frac{1}{q_\theta} := \frac{1-\theta}{q_1} + \frac{\theta}{q_2}; \quad C_\theta := C_1^{1-\theta} C_2^{\theta} \tag{2.34}$$

Then for any $0 < \theta < 1$ with $q_\theta > 1$ we have

$$\|Tf\|_{q_\theta,\infty} \lesssim_{p_1,p_2,q_1,q_2,\theta} C_\theta \|f\|_{p_\theta,\infty} \tag{2.35}$$

for all simple functions with $|suppf| < \infty$.
In particular, if $q_\theta \geq p_\theta$, then T is strong-type (p_θ, q_θ), i.e.

$$\|Tf\|_{q_\theta} \lesssim C_\theta \|f\|_{p_\theta} \tag{2.36}$$

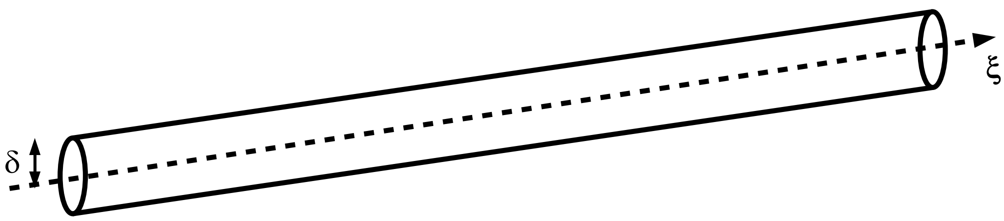

Figure 9: A δ-tube

Remark. *The second part is a stronger version of the Riesz-Thorin interpolation theorem, which makes the same statement for interpolation between strong type operators.*

The proof of the Marcinkiewicz interpolation theorem can be found in [45], [44].

2.4 Kakeya Maximal Functions

Define a δ-tube $\tau_\xi^\delta(a)$ to a direction $\xi \in S^{n-1}$, centered at a and radius $\delta > 0$

$$\tau_\xi^\delta(a) := \{x \in \mathbb{R}^n : |(x-a)\cdot\xi| \le \frac{1}{2}, |(x-a)^\perp| \le \delta\}, \tag{2.37}$$

where $x^\perp = x - (x\cdot\xi)\cdot\xi$.
The Kakeya maximal function of $f \in L^1_{loc}(\mathbb{R}^n)$ is the function $f_\delta^* : S^{n-1} \longrightarrow \mathbb{R}$

$$f_\delta^*(\xi) := \sup_{a\in\mathbb{R}^n} \frac{1}{|\tau_\xi^\delta(a)|}\int_{\tau_\xi^\delta(a)} |f| \tag{2.38}$$

There are various other maximal functions like the closely related Nikodym maximal function $f_\delta^{**} : \mathbb{R}^n \longrightarrow \mathbb{R}$, which is defined as

$$f_\delta^{**}(x) := \sup_{\xi\in S^{n-1}} \frac{1}{|\tau_\xi^\delta(x)|}\int_{\tau_\xi^\delta(x)} |f| \tag{2.39}$$

or the Hardy-Littlewood maximal function $Mf : \mathbb{R}^n \longrightarrow \mathbb{R}$

$$Mf(x) := \sup_{r>0} \frac{1}{|B(x,r)|}\int_{B(x,r)} |f|, \tag{2.40}$$

where $B(x, r)$ is a ball of radius r centered in x. While for nice functions the average value of the Hardy-Littlewood maximal function will be approximately the value of f, this can be quite different for the other two cases. The goal is to get bounds[2] of the form

$$\|f_\delta^*\|_{L^q(S^{n-1})} \leq K_n(\delta, p, q)\|f\|_{L^p(\mathbb{R}^n)} \tag{2.41}$$

in particular

$$\forall \epsilon > 0 \, \exists C_\epsilon : \quad \|f_\delta^*\|_{L^p(S^{n-1})} \leq C_\epsilon \delta^{-\epsilon}\|f\|_{L^p} \tag{2.42}$$

for some $p < \infty$.

Proposition 2.6. *1.*

$$(f + g)_\delta^* \leq f_\delta^* + g_\delta^* \tag{2.43}$$

i.e. the operation $f \to f_\delta^*$ is sublinear.

2.

$$\|f_\delta^*\|_\infty \leq \|f\|_\infty \tag{2.44}$$

$$\|f_\delta^*\|_\infty \leq \delta^{-(n-1)}\|f\|_1 \tag{2.45}$$

3. For $n \geq 2$ and $p < \infty$, the coefficient $K_n(\delta, p, q)$ in inequality (2.41) *cannot be independent of δ.*

4. Inequality (2.42) *cannot hold for $p < n$.*

5. If inequality (2.41) *holds when f is supported on the unit ball then there is a constant C which does not depend on δ such that*

$$\|f_\delta^*\|_{L^q(S^{n-1})} \leq CK_n(\delta, p, q)\|f\|_{L^p(\mathbb{R}^n)} \tag{2.46}$$

holds for functions with arbitrary support. This means that (2.41) *is a local problem.*

Proof. 1. Follows from the definition and the inequality

$$\sup_x(|F(x)| + G(x)|) \leq \sup_x |F(x)| + \sup_y |G(y)|.$$

[2]Most bounds hold both for the Kakeya and for the Nikodym maximal function, however, we will state them for Kakeya maximal functions only

2. Follows directly from the definition.

3. Consider a Besicovitch set E. Let E_δ be its δ-neighborhood and

$$f := \chi_{E_\delta}$$

.

Then we have $f(\xi) = 1$ for all directions $\xi \in S^{n-1}$, which implies $\|f_\delta^*\|_q = |S^{n-1}| \approx_n 1$. On the other hand, $\lim_{\delta\to 0} |E_\delta| = 0$, hence $\lim_{\delta\to 0} \|\chi_{E_\delta}\|_p = 0$ for any $p < \infty$.

4. Let $f := \chi_{B(0,\delta)}$, i.e. f is the characteristic function of a ball with radius δ. Then we have $B(0,\delta) \subset T_\xi^\delta(0)$ and $f_\delta^*(\xi) = \frac{|B(0,\delta)|}{|T_\xi^\delta(0)|} \gtrsim_n \delta$ for all directions $\xi \in S^{n-1}$. Therefore $\|f_\delta^*\|_p \approx_n \delta$. On the other hand $\|f\|_p \approx_n \delta^{\frac{n}{p}}$. Hence (2.42) requires that for any $\epsilon > 0$

$$\delta \lesssim_{n,\epsilon} \delta^{\frac{n}{p}-\epsilon}$$

holds for any $\delta > 0$.
This is impossible if $p < n$.

5. If inequality (2.41) holds for functions supported on a unit ball, clearly (2.46) will hold for functions supported on the smallest ball which contains a unit cube, where the constant C depends on n.
Now, given a function f with arbitrary support. Divide $\mathbb{R}^n$ into disjoint unit cubes $\{\Omega_i\}_{\{i\in\mathbb{Z}^n\}}$ and let $f_i := f \cdot \chi_{\Omega_i}$, i.e. $f := \sum_i f_i$. For all f_i, we may assume that inequality (2.41) holds.
According to the definition of Kakeya Maximal Functions and the disjointness of the cubes Ω_j we have

$$\begin{aligned}
\|f_\delta^*\|_p^p = \|(\sum_i f_i)_\delta^*\|_p^p &= \int_{S^{n-1}} d\xi \Big(\max_a \frac{1}{|\tau_\xi(a)|} \int_{\tau_\xi(a)} \Big|\sum_i f_i\Big|\Big)^p \\
&= \int_{S^{n-1}} d\xi \Big(\sum_j \chi_{\{a(\xi)\in\Omega_j\}} \frac{1}{|\tau_\xi(a(\xi))|} \int_{\tau_\xi(a(\xi))} \Big|\sum_i f_i\Big|\Big)^p \\
&= \int_{S^{n-1}} d\xi \sum_j \chi_{\{a(\xi)\in\Omega_j\}} \Big(\frac{1}{|\tau_\xi(a(\xi))|} \int_{\tau_\xi(a(\xi))} \Big|\sum_i f_i\Big|\Big)^p \\
&= \sum_j \int_{\{\xi:a(\xi)\in\Omega_j\}} d\xi \Big(\frac{1}{|\tau_\xi(a(\xi))|} \int_{\tau_\xi(a(\xi))} \Big|\sum_i f_i\Big|\Big)^p
\end{aligned} \tag{2.47}$$

where $a(\xi)$ is a point in $\mathbb{R}^n$ such that for a given $\xi \in S^{n-1}$

$$\frac{1}{|\tau_\xi(a(\xi))|} \int_{\tau_\xi(a(\xi))} \Big| \sum_i f_i \Big| = f_\delta^*(\xi).$$

Note that for $a(\xi) \in \Omega_j$ the tube $\tau_\xi(a(\xi))$ will only overlap with cubes in the neighborhood of Ω_j, i.e. with the cubes $\{\Omega_{j+e}\}_{\{e\in\{-1,0,1\}^n\}}$ and hence

$$\begin{aligned}\left(\frac{1}{|\tau_\xi(a(\xi))|} \int_{\tau_\xi(a(\xi))} |\sum_i f_i|\right)^p &= \left(\frac{1}{|\tau_\xi(a(\xi))|} \int_{\tau_\xi(a(\xi))} \sum_e |f_{j+e}|\right)^p \\ &\approx_{n,p} \sum_e \left(\frac{1}{|\tau_\xi(a(\xi))|} \int_{\tau_\xi(a(\xi))} |f_{j+e}|\right)^p \\ &\leq \sum_e ((f_{j+e})_\delta^*(\xi))^p.\end{aligned} \tag{2.48}$$

Consequently

$$\begin{aligned}\|f_\delta^*\|_p^p &\lesssim_{n,p} \sum_j \int_{\{\xi: a(\xi)\in\Omega_j\}} d\xi \sum_e ((f_{j+e})_\delta^*(\xi))^p \\ &\leq \sum_e \sum_j \int_{S^{n-1}} d\xi ((f_{j+e})_\delta^*(\xi))^p \\ &\lesssim_n \sum_j \int_{S^{n-1}} d\xi |(f_j)_\delta^*(\xi)|^p \\ &= \sum_j \|(f_j)_\delta^*\|_p^p \\ &\leq K_n^p(\delta,p) \sum_j \|f_j\|_p^p \\ &= K_n^p(\delta,p) \|\sum_j f_j\|_p^p \\ &= K_n^p(\delta,p)\|f\|_p^p\end{aligned} \tag{2.49}$$

□

Proposition 2.7. *If* (2.42) *is true for some $p < \infty$, then Kakeya sets in $\mathbb{R}^n$ have Hausdorff dimension n.*

Remark. *For the Minkowski dimension this is immediate by following the same argument as in the second part of the previous proposition. Consider the characteristic function* χ_{E_δ} *of the* δ*-neighborhood* E_δ *of a Kakeya set* E*. Then* $(\chi_{E_\delta})^*_\delta = 1$ *and therefore with* (2.42)

$$1 \leq C_\epsilon \delta^{-\epsilon} |E_\delta|^{1/p}$$

equivalent to

$$|E_\delta| \geq c_{\epsilon p} \delta^{\epsilon p}$$

which means that the Minkowski dimension of E *is* n*.*

Proof. Given a Kakeya set E. Note that if E did not have Hausdorff dimension n there would be some $\alpha < n$ such that the Hausdorff measure $H_\alpha(E)$ is zero. Recall that $H^\delta_\alpha(E) \leq H_\alpha(E)$ for any $\delta > 0$.
Thus it suffices to prove for some $\delta > 0$

$$\forall \alpha < n : \quad H^\delta_\alpha(E) > 0 \tag{2.50}$$

where

$$H^\delta_\alpha(E) := \inf \left\{ \sum_{j=1}^{\infty} r_j^\alpha \middle| \exists x_1, x_2, ... \text{ s.t. } E \subset \bigcup_{j=1}^{\infty} D(x_j, r_j) \text{ and } r_j < \delta \right\} \tag{2.51}$$

Now let $\delta := \frac{1}{100}$ and fix a covering of E by discs $D_j := D(x_j, r_j)$ with $r_j < \frac{1}{100}$ for all j.
Define

$$J_k := \{ j : \ 2^{-k} \leq r_j \leq 2^{-(k-1)} \} \tag{2.52}$$

so we get sets $F_k^0 := \bigcup_{j \in J_k} D_j$ which consist of discs with comparable size for each k.
For a direction $\xi \in S^{n-1}$, let I_ξ be a unit line segment parallel to ξ, which is contained in the Kakeya set E. Let

$$S_k := \{ \xi \in S^{n-1} : \ |I_\xi \cap \bigcup_{j \in J_k} D_j| \geq \frac{1}{100k^2} \} \tag{2.53}$$

Note that $\sum_k \frac{1}{100k^2} < 1$ and $\sum_k |I_\xi \cap \bigcup_{j \in J_k} D_j| \leq |I_\xi| = 1$, hence for each direction ξ there is a k such that $\xi \in S_k$, i.e.

$$\bigcup_{k=1}^{\infty} S_k = S^{n-1}.$$

Define

$$f := \chi_{F_k}, \quad F_k := \bigcup_{j \in J_k} D(x_j, 10r_j).$$

If $F_k^{0\prime}$ is a translation of F_k^0 along a distance shorter than 2^{-k} then still $F_k^{0\prime} \subset F_k$. Let $I_\xi{}'$ be the corresponding translation of I_ξ, such that the set $I_\xi{}' \cap F_k^{0\prime}$ is a translation of $I_\xi \cap F_k^0$. Consequently for $\xi \in S_k$

$$|I_\xi{}' \cap F_k| \geq |I_\xi \cap F_k^0|$$

hence

$$|T_\xi^{2^{-k}}(a_\xi) \cap F_k| \geq \frac{1}{100k^2}|T_\xi^{2^{-k}}(a_\xi)|, \tag{2.54}$$

where a_ξ is the midpoint of I_ξ such that $T_\xi^{2^{-k}}(a_\xi)$ is a tube around I_ξ of radius 2^{-k}.

We can write equivalent to (2.54)

$$f_{2^{-k}}^*(\xi) \geq \frac{1}{|T_\xi^{2^{-k}}(a_\xi)|}\int_{T_\xi^{2^{-k}}(a_\xi)} \chi_{F_k} \geq \frac{1}{100k^2}, \tag{2.55}$$

thus

$$\|f_{2^{-k}}^*\|_p \geq \left(\int_{S_k} (f_{2^{-k}}^*)^p\right)^{\frac{1}{p}} \geq \frac{1}{100k^2}\sigma(S_k)^{\frac{1}{p}}, \tag{2.56}$$

where $\sigma(S_k)$ is the surface measure of S_k.

On the other hand by (2.42) we have

$$\|f_{2^{-k}}^*\|_p \leq C_\epsilon 2^{k\epsilon}\|f\|_p \lesssim_n C_\epsilon 2^{k\epsilon}(|J_k| \cdot 2^{-(k-1)n})^{\frac{1}{p}} \tag{2.57}$$

and together with (2.56)

$$\sigma(S_k) \lesssim_n C_\epsilon^p\, 2^{k\epsilon p-(k-1)n}k^{2p}|J_k| \lesssim_{\epsilon,p,n} 2^{-k(n-2p\epsilon)}|J_k|$$

using the fact that $k^{2p} \leq C_{\epsilon p}2^{k\epsilon p}$. Now fix $\alpha < n$ and choose $\epsilon > 0$ such that $\alpha < n - 2p\epsilon$. Then

$$\sum_j r_j^{n-2p\epsilon} \geq \sum_k 2^{-k(n-2p\epsilon)}|J_k| \geq c_{\epsilon pn}\sum_k \sigma(S_k) \geq c_{\epsilon pn}|S^{n-1}| =: c > 0. \tag{2.58}$$

The constant c does not depend on the covering of E, hence (2.50) holds. $\square$

Now we obtained a third, formally stronger version of the Kakeya conjecture:

Conjecture 3. *(Kakeya maximal function conjecture).*
Given $\epsilon > 0$ *then there is a constant* C_ϵ *such that*

$$\|f_\delta^*\|_{L^n(S^{n-1})} \leq C_\epsilon \delta^{-\epsilon} \|f\|_{L^n} \tag{2.59}$$

holds for all $0 < \delta \ll 1$.

For $n = 2$ this is proved by Bourgain [8] and by Cordoba [12][3]. In that case, inequality (2.59) is only an L^2-estimate, such that the Plancherel theorem 2.4 can be applied. Bourgain's proof will be presented in the next section.
Partial results on Conjecture 3 are obtained by interpolation (according to theorem 2.5) between (2.59) and (2.45), which generates a new family of conjectured inequalities

$$\|f_\delta^*\|_{L^q(S^{n-1})} \leq C_\epsilon \delta^{-\frac{n}{p}+1-\epsilon} \|f\|_{L^p}, \tag{2.60}$$

where q depends on p which in turn is determined by the interpolation variable θ.
The issue is to find the largest p for which the inequality holds. If inequality (2.60) holds for some $p_0 > 1$ then, again, by interpolation (2.60) holds also for all $1 \leq p \leq p_0$.
By the same argument as in the proof of proposition 2.7 it follows that (2.60) implies that Kakeya sets in $\mathbb{R}^n$ have Hausdorff dimension at least p.

[3] see also [51]

3 Estimates on the 3-dimensional Kakeya Maximal Function

In this section we show

Theorem 3.1. *Given $\epsilon > 0$ and let $2 \leq p \leq \frac{7}{3}$. Then there is a constant C_ϵ such that*

$$\|f_\delta^*\|_p \leq C_\epsilon \delta^{-\frac{3}{p}+1-\epsilon} \|f\|_p \quad \textit{for} \tag{3.1}$$

holds uniformly for $0 < \delta \ll 1$.
In particular 3-dimensional Kakeya sets must have Hausdorff dimension at least $\frac{7}{3}$.

For the proof we will follow the arguments from Bourgain [8]. Again, the idea is to interpolate between two inequalities, i.e. one for $p = 2$ and a restricted weak-type inequality for $p = \frac{7}{3} + \epsilon$.

3.1 The Bounds for $p = 2$

Theorem 3.2. *For $n = 2$ there is a constant C such that*

$$\|f_\delta^*\|_2 \leq C(\log \frac{1}{\delta})^{1/2} \|f\|_2 \tag{3.2}$$

holds for any $0 < \delta \ll 1$.
For $n \geq 3$ the bound is

$$\|f_\delta^*\|_2 \leq C \delta^{-\frac{n-2}{2}} \|f\|_2 \tag{3.3}$$

Proof. Assume f nonnegative, i.e. write $f := |f|$.
Consider a Gaussian function $\varphi(x) = ce^{-\frac{x^2}{4}}$ where c is a constant s.t.

$$\varphi(x) \geq 1 \qquad \text{for} \quad |x| \leq 1 \tag{3.4}$$

Note that according to (2.21) and (2.17) we have

$$\hat{\varphi}(\lambda) = c_1 e^{-\lambda^2} \tag{3.5}$$

where $c_1 = \sqrt{2}c$. Let

$$\psi(x) := \varphi(x_1) \prod_{i=2}^{n} \delta^{-1} \varphi(\delta^{-1} x_i) \tag{3.6}$$

and

$$\rho_\delta^\xi(x) := \left(\frac{1}{\delta}\right)^{n-1} \chi_{\tau_\xi^\delta(0)} \tag{3.7}$$

where $\xi \in S^{n-1}$ and τ_ξ^δ defined as in (2.37).
Let $p_\xi \in SO(n)$ a rotation such that $\xi = p_\xi^{-1} e_1$ and

$$\psi_\xi := \psi \circ p_\xi \tag{3.8}$$

Therefore we have

$$\rho_\delta^\xi \leq \psi_\xi. \tag{3.9}$$

and hence

$$\begin{aligned}
|f_\delta^*(\xi)| &= \left| \sup_{a \in \mathbb{R}^n} \frac{1}{|\tau_\xi^\delta(a)|} \int_{\tau_\xi^\delta(a)} f(x)dx \right| \\
&= \left| \sup_{a \in \mathbb{R}^n} (\rho_\delta^\xi \star f)(a) \right| \\
&\overset{(3.9)}{\leq} \|(\psi_\xi \star f)\|_\infty \\
&\overset{(2.18)}{=} \|(\widehat{\psi_\xi \star f})\check{}\|_\infty \\
&= \sup_{a \in \mathbb{R}^n} \left| \frac{1}{(2\pi)^n/2} \int_{\mathbb{R}^n} \widehat{f}(\lambda)\widehat{\psi}_\xi(\lambda) e^{i\langle a,\lambda\rangle} d\lambda \right| \\
&\leq \int_{\mathbb{R}^n} |\widehat{f}(\lambda)\widehat{\psi}_\xi(\lambda)| d\lambda \\
&\leq \left(\int_{\mathbb{R}^n} |\widehat{\psi}_\xi(\lambda)||\widehat{f}(\lambda)|^2(1+|\lambda|)d\lambda \right)^{1/2} \left(\int_{\mathbb{R}^n} \frac{|\widehat{\psi}_\xi(\lambda)|}{1+|\lambda|} d\lambda \right)^{1/2}
\end{aligned} \tag{3.10}$$

where in the last line we apply Hölder's inequality.
Observe that for all $\lambda = (\lambda_1, ..., \lambda_n)$

$$\widehat{\psi}(\lambda) = c_1^n e^{-\lambda_1^2} \prod_{i=2}^n e^{-(\delta\lambda_i)^2} \lesssim 1 \tag{3.11}$$

and

$$\widehat{\psi}_\xi = \widehat{\psi} \circ p_\xi. \tag{3.12}$$

For the estimate of the second term of the r.h.s. of (3.10) we choose coordinates such that $\xi = e_1$ and write $\lambda' := (\lambda_2, ..., \lambda_n)$.

this yields for $n \geq 3$

$$\begin{aligned}\int \frac{|\widehat{\psi}_\xi(\lambda)|}{1+|\lambda|} d\lambda &\lesssim \int_{|\lambda'|<1/\delta} \frac{e^{-\lambda_1^2}}{1+|\lambda|} d^n\lambda + \int_{|\lambda'|\geq 1/\delta} \frac{e^{-\lambda_1^2} \prod_{i=2}^n e^{-(\delta\lambda_i)^2}}{1+|\lambda|} d^n\lambda \\ &\lesssim \int_{\lambda' \leq 1/\delta} \frac{1}{|\lambda'|} d^{n-1}\lambda' + \int_{|\lambda'|\geq 1/\delta} \frac{\prod_{i=2}^n e^{-(\delta\lambda_i)^2}}{1+|\lambda'|} d^{n-1}\lambda' \\ &\lesssim \int_{S^{n-2}} \int_0^{1/\delta} \frac{1}{|\lambda'|} |\lambda'|^{n-2} d|\lambda'| d\sigma + \delta \int \prod_{i=2}^n e^{-(\delta\lambda_i)^2} d^{n-1}\lambda' \\ &\approx \delta^{-(n-2)}\end{aligned} \tag{3.13}$$

and analogously for $n = 2$

$$\begin{aligned}\int_{\mathbb{R}^2} \frac{\widehat{\psi}_\xi(\lambda)}{1+|\lambda|} d\lambda &\lesssim \int_{\mathbb{R}^2} \frac{e^{-\lambda_1^2} e^{-(\delta\lambda_2)^2}}{1+\sqrt{\lambda_1^2+\lambda_2^2}} d\lambda_1 d\lambda_2 \\ &\lesssim \int_0^{1/\delta} \frac{1}{1+\lambda_2} d\lambda_2 + \int_{1/\delta}^\infty \frac{e^{-(\delta\lambda_2)^2}}{\lambda_2} d\lambda_2 \\ &\approx 1 + \int_1^{1/\delta} \frac{1}{\lambda_2} d\lambda_2 + \delta \int e^{k^2} \delta^{-1} dk \\ &\approx \log \frac{1}{\delta}.\end{aligned} \tag{3.14}$$

Write

$$K_n(\delta) := \begin{cases} (\log 1/\delta)^{1/2} & \text{for } n = 2 \\ \delta^{\frac{-(n-2)}{2}} & \text{for } n \geq 3. \end{cases} \tag{3.15}$$

Applying Fubini's theorem we obtain

$$\begin{aligned}\|f_\delta^*\|_2^2 &= \int_{S^{n-1}} |f_\delta^*(\xi)|^2 d\xi \\ &\lesssim (K_n(\delta))^2 \int_{S^{n-1}} \int_{\mathbb{R}^n} |\widehat{\psi_\xi}(\lambda)| |\widehat{f}(\lambda)|^2 (1+|\lambda|) d\lambda d\xi \\ &= (K_n(\delta))^2 \int_{\mathbb{R}^n} |\widehat{f}(\lambda)|^2 (1+|\lambda|) \int_{S^{n-1}} |\widehat{\psi_\xi}(\lambda)| d\xi d\lambda.\end{aligned} \tag{3.16}$$

Thus, if

$$\int_{S^{n-1}} \widehat{\psi}_\xi(\lambda) d\xi \lesssim \frac{1}{1+|\lambda|} \quad \text{for } \lambda \in \mathbb{R}^n \tag{3.17}$$

holds for all λ the assertion follows due to Plancherel's theorem, i.e.

$$\begin{aligned} \|f_\delta^*\|_2^2 &\lesssim (K_n(\delta))^2 \int_{\mathbb{R}^n} |\widehat{f}(\lambda)|^2 (1+|\lambda|) \int_{S^{n-1}} |\widehat{\psi_\xi}(\lambda)| d\xi d\lambda \\ &\lesssim (K_n(\delta))^2 \int_{\mathbb{R}^n} |\widehat{f}(\lambda)|^2 d\lambda \\ &= (K_n(\delta))^2 \|f\|_2^2 \end{aligned} \tag{3.18}$$

Clearly, (3.16) holds if $|\lambda \leq 1|$.
Fix $\lambda \in \mathbb{R}^n$ with $|\lambda| > 1$, and choose coordinates such that $\lambda = |\lambda| e_1$.
Consequently

$$\begin{aligned} \int_{S^{n-1}} \widehat{\psi}_\xi(\lambda) d\xi &\approx \int_{S^{n-1}} e^{-(\lambda \xi_1)^2} \prod_{i=2}^{n} e^{-(\delta|\lambda|)^2 \xi_i} d\xi \\ &\lesssim \int_0^{\pi/2} e^{-|\lambda|^2 \sin^2 \theta} d\theta \\ &\approx \int_0^{\pi/2} e^{-c|\lambda|^2 \theta^2} d\theta \\ &\leq \int_0^{\infty} e^{-c|\lambda|^2 \theta^2} d\theta \\ &= \frac{1}{|\lambda|} \int_0^{\infty} e^{-cx^2} dx \\ &\approx \frac{1}{1+|\lambda|}. \end{aligned} \tag{3.19}$$

□

3.2 A Restricted weak-type Inequality

Theorem 3.3. *Given $\epsilon > 0$ and $\epsilon' > 0$. Then there is a constant $c > 0$ such that*

$$|A| \geq c \cdot \delta^{\frac{2}{3}+\epsilon} \sigma^{\frac{7}{3}+\epsilon'} |\{(\chi_A)_\delta^* > \sigma\}| \tag{3.20}$$

holds for all $A \subset B(0,1)$, $0 \leq \sigma \leq 1$ and $0 < \delta \ll 1$.
If $|\{(\chi_A)_\delta^* > \sigma\}| > d$ for some $d > 0$, then (3.20) is equivalent to

$$|A| > c' \cdot \delta^{\frac{2}{3}+\epsilon} \sigma^{\frac{7}{3}+\epsilon'} \tag{3.21}$$

Proof. By setting $c' := c \cdot d$, inequality (3.20) clearly implies inequality (3.21). On the other hand, given inequality (3.21) consequently

$$|A| > c' \cdot \delta^{\frac{2}{3}+\epsilon}\sigma^{\frac{7}{3}+\epsilon'} \geq c''|S^2| \cdot \delta^{\frac{2}{3}+\epsilon}\sigma^{\frac{7}{3}+\epsilon'} \geq c'' \cdot \delta^{\frac{2}{3}+\epsilon}\sigma^{\frac{7}{3}+\epsilon'}|\{(\chi_A)^*_\delta > \sigma\}|.$$

Here, d is only a scaling parameter, thus we may assume $d = \frac{1}{2}$.
Let

$$\mathcal{D} := \{\xi \in S^2 : (\chi_A)^*_\delta > \sigma\} \tag{3.22}$$

and let $A \subset B(0,1)$ such that $\mathcal{D} > \frac{1}{2}$.
Define a *bush* as a collection of δ-tubes which have a common point (see figure 10).
The idea is to construct a sequence of bushes $\{B_i\}_{\{i=0,\dots s\}}$ such that on the one hand these bushes cover an essential part of A, namely

$$|\{(\chi_{\overline{A}})^*_\delta > \frac{\sigma}{2}\}| > \frac{1}{4}$$

where

$$\overline{A} := \bigcup_{i=0}^{s} B_i \cap A,$$

and on the other hand

$$\frac{|A|}{\sigma} \gtrsim \sum_{i=0}^{s} |B_i|.$$

Now let $\mathcal{E} \subset \mathcal{D}$ be a $\frac{20\delta}{\sigma}$-separated subset, i.e. a set of discrete directions $\xi \in \mathcal{D}$ which satisfies for any two distinct points $\xi, \xi' \in S^2$

$$|\xi - \xi'| \geq \frac{20\delta}{\sigma}.$$

Clearly the minimum number of $\frac{20\delta}{\sigma}$-discs to cover $\mathcal{D}$ is $\approx \left(\frac{\sigma}{20\delta}\right)^2$.
Thus with (2.11), $\mathcal{E}$ can be chosen such that

$$\#\mathcal{E} \geq c\Big(\frac{\sigma}{\delta}\Big)^2 \tag{3.23}$$

According to the definition of $\mathcal{D}$ it follows that for any $\xi \in \mathcal{E}$ there is a (ξ, δ)-tube τ_ξ such that

$$\frac{1}{|\tau_\xi|}\int_{\tau_\xi} \chi_A > \sigma, \tag{3.24}$$

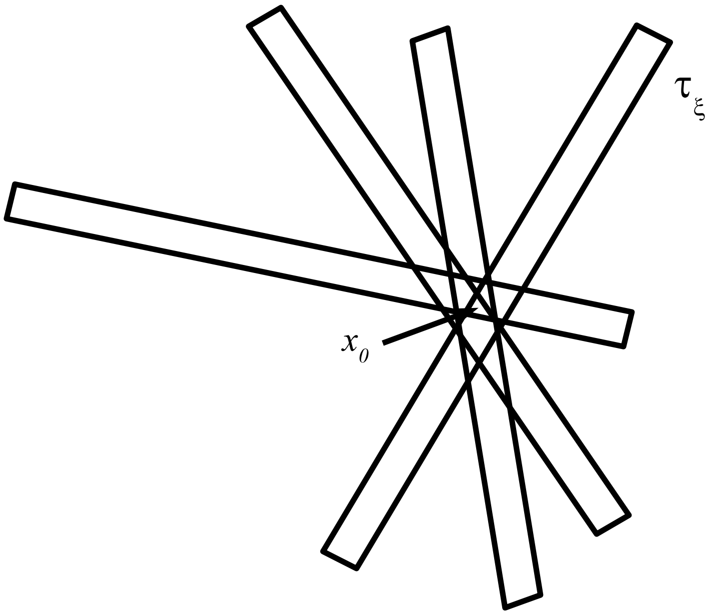

Figure 10: A bush consisting of four δ-tubes with common point x_0

which can also be written as

$$|A \cap \tau_\xi| = \int_{\tau_\xi} \chi_A > \pi\sigma\delta^2. \tag{3.25}$$

For $\sigma \leq \delta$ this already implies (3.21) since then

$$|A| > \pi\sigma\delta^2 \geq \pi\sigma^{\frac{7}{3}}\delta^{\frac{2}{3}} \geq \pi\delta^{\frac{2}{3}+\epsilon}\sigma^{\frac{7}{3}+\epsilon'}.$$

Thus we assume $\delta < \sigma$.
Inequalities (3.23) and (3.25) imply

$$\sum_{\xi\in\mathcal{E}} \int_{\tau_\xi} \chi_A \gtrsim \sigma^3, \tag{3.26}$$

consequently

$$\sigma^3 \lesssim \int_A \sum_{\xi\in\mathcal{E}} \chi_{\tau_\xi} \leq |A| \cdot \|\chi_A \sum_{\xi\in\mathcal{E}} \chi_{\tau_\xi}\|_\infty. \tag{3.27}$$

Note that $\|\chi_A \sum_{\xi\in\mathcal{E}} \chi_{\tau_\xi}\|_\infty$ is the highest multiplicity of a point $x \in A$, where multiplicty of x means the number of δ-tubes τ_ξ for which $\{\tau_\xi \cap x\} = \{x\}$.

Let $x_0 \in A$ be such a point of maximal multiplicity. Consider a set $\mathcal{F}_0 \subset \mathcal{E}$ of all directions $\xi \in \mathcal{E}$ for which τ_ξ intersects x_0.
The number of such directions is equal to the multiplicity of x_0, hence

$$\#\mathcal{F}_0 \gtrsim \frac{\sigma^3}{|A|}. \tag{3.28}$$

The corresponding set of δ-tubes forms the bush

$$B_0 := \bigcup_{\xi \in \mathcal{F}_0} \tau_\xi \tag{3.29}$$

Consider δ-tubes $\tau_{\xi_i}(i = 1, ..., \#\mathcal{F}_0)$ and unit lines $I_{\xi_i} \subset \tau_{\xi_i}$ with $x_0 \in I_{\xi_i}$. For two points $x_i \in I_{\xi_i}, x_j \in I_{\xi_j}$ with distance to x_0 not less than $\frac{\sigma}{3}$

$$|x_i - x_j| \leq \frac{20\delta}{3}, \tag{3.30}$$

since $\mathcal{F}_0$ is $\frac{20\delta}{\sigma}$-separated.
Therefore

$$(\tau_{\xi_i} \cap \tau_{\xi_j}) \setminus B(x_0, \frac{\sigma}{3}) = \emptyset \tag{3.31}$$

holds:
Assume there is a point $y \in \tau_{\xi_i} \cap \tau_{\xi_j}$ with $|y - x_0| \geq \frac{\sigma}{3}$. Then for all $x_1 \in I_{\xi_i}, x_2 \in I_{\xi_j}$ with $|x_1 - x_0| \geq \frac{\sigma}{3}$, $|x_2 - x_0| \geq \frac{\sigma}{3}$

$$4\delta \geq |y - x_1| + |y - x_2| \geq |x_1 - x_2| \geq \frac{20\delta}{3}$$

which is a contradiction. Thus the sets $\left(\tau_\xi \backslash B(x_0, \frac{\sigma}{3})\right)_{\xi \in \mathcal{F}_0}$ indeed are mutually disjoint.
Since

$$|A \cap (\tau_\xi \cap B(x_0, \frac{\sigma}{3}))| \leq \frac{2}{3}\pi\sigma\delta^2 \tag{3.32}$$

and

$$\pi\sigma\delta^2 < |A \cap \tau_\xi| = |A \cap (\tau_\xi \backslash B(x_0, \frac{\sigma}{3}))| + |A \cap (\tau_\xi \cap B(x_0, \frac{\sigma}{3}))| \tag{3.33}$$

we find

$$|A \cap (\tau_\xi \setminus B(x_0, \frac{\sigma}{3}))| \geq \pi\frac{\sigma}{3}\delta^2. \tag{3.34}$$

Summing up these mutually disjoint sets for all $\xi \in \mathcal{F}_0$ yields

$$\frac{3}{\sigma}|A \cap B_0| \gtrsim \#\mathcal{F}_0 \delta^2 \approx |B_0|. \tag{3.35}$$

Let

$$A_1 := A \setminus B_0 \tag{3.36}$$

and

$$\mathcal{D}_1 := \left\{ \xi \in S^2 \middle| (\chi_{A_1})^*_\delta > \frac{\sigma}{10} \right\} \tag{3.37}$$

If $|\mathcal{D}_1| < \frac{1}{10}$ we can go directly to (3.50) with $s = 1$.
Otherwise we iterate the construction from $\mathcal{D}_t$ to $\mathcal{D}_{t+1}$ as long as $|\mathcal{D}_t| < \frac{1}{10}$:
Consider a $\frac{20\delta}{\sigma}$-separated set $\mathcal{E}_t \subset \mathcal{D}_t$ with

$$\mathcal{E}_t \geq c_1 \left(\frac{\sigma}{\delta}\right)^2 \tag{3.38}$$

where for each $\xi \in \mathcal{E}_t$ exists a (ξ, δ)-tube s.t.

$$|A_t \cap \tau_\xi| \geq \frac{1}{10} \sigma \delta^2. \tag{3.39}$$

Again, there is a subset $\mathcal{F}_t \subset \mathcal{E}_t$ and a point $x_t \in A_t$ such that

$$x_t \in \tau_\xi \quad \text{for} \quad \xi \in \mathcal{F}_t \tag{3.40}$$

and

$$\#\mathcal{F}_t \gtrsim \frac{\sigma^3}{|A_t|} > \frac{\sigma^3}{|A|}. \tag{3.41}$$

Define

$$B_t := \bigcup_{\xi \in \mathcal{F}_t} \tau_\xi. \tag{3.42}$$

The same arguments used for (3.35) yield

$$|B_t| \lesssim \frac{1}{\sigma} |A_t \cap B_t|. \tag{3.43}$$

For the next step let

$$A_{t+1} := A_t \setminus B_t = A \setminus \bigcup_{i=0}^{t} B_t \tag{3.44}$$

and

$$\mathcal{D}_{t+1} := \left\{ \xi \in S^2 \middle| (\chi_{A_{t+1}})^*_\delta > \frac{\sigma}{10} \right\}. \tag{3.45}$$

Suppose $|\mathcal{D}_{t_1}| \geq \frac{1}{10}$, i.e. the construction continues up to t_1 steps.
The sets $\{A_i \cap B_i\}_{0 \leq i \leq t_1}$ are mutually disjoint, since

$$B_i \cap A_{i+1} = B_i \cap (A_i \setminus B_i) = \emptyset$$

and $A_j \subset A_i$ for $i \leq j$.
Hence

$$|A| \geq \sum_{i=0}^{t_1} |A_i \cap B_i|. \tag{3.46}$$

By (3.28) and (3.41) we also have

$$|B_i| \gtrsim \frac{\sigma^3}{|A|}\delta^2 \tag{3.47}$$

hence together

$$\frac{|A|}{\sigma} \overset{(3.46)}{\geq} \frac{1}{\sigma}\sum_{i=0}^{t_1} |A_i \cap B_i| \overset{(3.43)}{\gtrsim} \sum_{i=0}^{t_1} |B_i| \overset{(3.47)}{\gtrsim} t_1 \frac{\sigma^3}{|A|}\delta^2 \tag{3.48}$$

and thus

$$t_1 \leq C\frac{|A|^2}{\sigma^4\delta^2} \tag{3.49}$$

i.e. the construction stops after s steps with $s \leq C\frac{|A|^2}{\sigma^4\delta^2}$.
Let

$$\overline{\mathcal{D}} := \mathcal{D} \setminus \mathcal{D}_s \tag{3.50}$$

and

$$\overline{A} := A \setminus A_s = \bigcup_{i=1}^{s} (A \cap B_i). \tag{3.51}$$

Observe

$$|\overline{\mathcal{D}}| \geq |\mathcal{D}| - |\mathcal{D}_s| > \frac{1}{4}. \tag{3.52}$$

By sublinearity of the maximal operator (according to (2.43)), for each $\xi \in \overline{\mathcal{D}}$

$$\sigma \leq (\chi_A)^*_\delta(\xi) = (\chi_{\overline{A}} + \chi_{A_s})^*_\delta(\xi) \leq (\chi_{\overline{A}})^*_\delta(\xi) + (\chi_{A_s})^*_\delta(\xi). \tag{3.53}$$

Since $\xi \notin \mathcal{D}_s$

$$(\chi_{A_s})^*_\delta(\xi) \le \frac{\sigma}{2}$$

consequently

$$(\chi_{\overline{A}})^*_\delta(\xi) > \frac{\sigma}{2}, \tag{3.54}$$

in particular there is some $x \in B(0,1)$ such that the δ-tube $\tau_\xi(x)$ satisfies

$$|\overline{A} \cap \tau_\xi(x)| > \frac{\sigma}{2}\delta^2. \tag{3.55}$$

By

$$|\overline{A} \cap \tau_\xi(x)| = \Big|\bigcup_{i=0}^{s}(A \cap B_i) \cap \tau_\xi(x)\Big| \le \sum_{i=0}^{s}|B_i \cap \tau_\xi(x)| \tag{3.56}$$

we finally obtain

$$\sum_{i=0}^{s}|\tau_\xi \cap B_i| > \frac{\sigma}{2}\delta^2. \tag{3.57}$$

Fix $0 \le t \le s$. Define the *parallelepiped* $T_\xi(x, x_t)$ as the smallest parallelepiped which contains the δ-neighborhood of a parallelogram generated by the unit lines $I_\xi(x)$ and $I_\xi(x_t)$ centered at x, x_t respectively , i.e.

$$T_\xi(x, x_t) :=$$

$$\{a \in \mathbb{R}^3 | a = x + \gamma_1 x_t + \lambda\xi + \gamma_2\delta\frac{\xi \times (x_t - x)}{|\xi \times (x_t - x)|};\ -\delta \le \gamma_1, \gamma_2 \le 1+\delta,\ -\frac{1}{2}-\delta \le \lambda \le \frac{1}{2}+\delta\}$$

For these parallelepipeds

$$\frac{1}{|\tau_\xi(x)|}|\tau_\xi(x) \cap B_t| \lesssim \frac{1}{|T_\xi(x, x_t)|}|T_\xi(x, x_t) \cap B_t| \tag{3.58}$$

holds:
Choose coordinates such that $x = 0$ and assume $|\xi \times x_t| > \delta$ (otherwise (3.58) is obvious). Consider a plane H_ξ generated by the directions ξ and $\xi \times x_t$ translated such that the area of $h_\xi := \tau_\xi \cap B_t \cap H_\xi$ is maximal, and therefore $|\tau_\xi \cap B_t| \lesssim \delta|h_\xi|$. The top x_t and the base area h_ξ form a (generalized) cone which lies in $B_t \cap T_\xi$ and has measure $\frac{1}{3}|\xi \times x_t| \cdot |h_\xi|$, whereas $|T_\xi| \approx \frac{|\xi \times x_t|}{\delta}|\tau_\xi|$. Altogether

$$\frac{1}{|\tau_\xi|}|\tau_\xi \cap B_t| \lesssim \frac{1}{|\tau_\xi|}\delta|h_\xi| \approx \frac{|\xi \times x_t|}{|T_\xi|}|h_\xi| \le \frac{1}{|T_\xi|}|T_\xi \cap B_t|.$$

Define a set of parallelepipeds

$$\mathbf{T}_\xi := \{T_\xi(x,y)\big|x,y \in \mathbb{R}^3 \wedge 0 \le |(x-y) + ((x-y)\cdot\xi)\xi| \le 1\} \tag{3.59}$$

and thus the maximal function

$$\mathcal{M}_\delta f(\xi) := \sup_{\mathbf{T}_\xi} \frac{1}{|T_\xi|}\int_{T_\xi} f(x)dx. \tag{3.60}$$

Clearly $\mathcal{M}_\delta f(\xi) \gtrsim f_\delta^*(\xi)$, since $\mathbf{T}_\xi$ also contains the set of tubes τ_ξ. Hence by (3.54), (3.51) and the sublinearity of the Kakeya maximal operator

$$\frac{\sigma}{2} \overset{(3.54)}{<} (\chi_{\overline{A}})_\delta^* \overset{(3.51)}{\le} \sum_{t=0}^{s}(\chi_{B_t})_\delta^* \le \sum_{t=0}^{s}\sum_{\delta<2^{-k}<1}(\chi_{B_t^{2^{-k}}})_\delta^* \lesssim \sum_{t=0}^{s}\sum_{\delta<2^{-k}<1}\mathcal{M}_\delta(\chi_{B_t^{2^{-k}}}) \tag{3.61}$$

for each $\xi \in \overline{\mathcal{D}}$, where

$$B_t^r := (B_t - x_t)\cap[B(0,2r)\setminus B(0,r)]. \tag{3.62}$$

In the following, $\sum_r$ means the sum over all dyadic values of $\delta < r < 1$, so the number of summands is $\approx \log\frac{1}{\delta}$.
Squaring the left and right hand side of (3.61) and applying the inequality of Cauchy-Schwartz, i.e. $\Big(\sum_{i=1}^n a_i\Big)^2 \le n\cdot\sum_{i=1}^n a_i^2$, yields

$$\sigma^2 \lesssim s\log\frac{1}{\delta}\sum_{t=0}^{s}\sum_r \mathcal{M}_\delta(\chi_{B_t^r})^2. \tag{3.63}$$

Lemma 3.4. *Given $\epsilon > 0$ and let $0 < r < 1$. Then*

$$\|\mathcal{M}_\delta f\|_2^2 \lesssim r^{-1}\delta^{-\epsilon}\|f\|_2^2 \tag{3.64}$$

holds for any $0 < \delta \ll 1$ if supp $f \subset B(0,2r)\setminus B(0,r)$.

Given the lemma, integration on the r.h.s. of (3.63) yields

$$s\log\frac{1}{\delta}\sum_{t=0}^{s}\sum_r\int_{\overline{\mathcal{D}}}\mathcal{M}_\delta(\chi_{B_t^r})^2 \le s\log\frac{1}{\delta}\sum_{t=0}^{s}\sum_r\|\mathcal{M}_\delta(\chi_{B_t^r})\|_2^2 \lesssim s\delta^{-\epsilon}\sum_{t=0}^{s}\sum_r\frac{1}{r}|B_t^r| \tag{3.65}$$

and by integration on the l.h.s of (3.63) we find

$$\int_{S^2} \sigma^2 \approx \int_{\overline{\mathcal{D}}} \sigma^2 \approx \sigma^2.$$

since $|\overline{\mathcal{D}}| \approx |S^2|$, together

$$\sigma^2 \lesssim s\delta^{-\epsilon} \sum_{t=0}^{s} \sum_{r} \frac{1}{r} |B_t^r| \tag{3.66}$$

For each translation of a (ξ, δ)-tube τ_ξ

$$|\tau_\xi \cap B(0, 2r) \setminus B(0, r)| \lesssim r\delta^2$$

hence

$$|B_t^r| = \Big| \bigcup_{\xi \in \mathcal{F}_t} [\tau'_\xi \cap B(0, 2r) \setminus B(0, r)] \Big| \lesssim r \cdot \#\mathcal{F}_t \delta^2 \approx r|B_t| \tag{3.67}$$

where $\tau'_\xi := \tau_\xi - x_t$.
Applying (3.48), i.e.

$$\frac{|A|}{\sigma} \gtrsim \sum_{t=0}^{s} |B_t|$$

along with (3.67) and (3.49) finally yields

$$\sigma^2 \lesssim s\delta^{-\epsilon'} \sum_{t=0}^{s} \log \frac{1}{\delta} |B_t| \lesssim s\delta^{-\epsilon''} \frac{|A|}{\sigma} \overset{(3.49)}{\lesssim} \delta^{-2-\epsilon''} \frac{|A|^3}{\sigma^5} \tag{3.68}$$

which is inequality (3.21). □

Proof. (Lemma 3.4)
We will use the 2-dimensional bound in (3.2).
For $e \in S^2$ let $L(e)$ be the corresponding 2-dimensional subspace of $\mathbb{R}^3$, i.e. a plane through 0 with normal vector e, and let $S_{L(e)} \cong S^1$ be the unit sphere in $L(e)$. Consider a parallelepiped $T_\xi \in \mathbf{T}_\xi$ with $\xi \in S_{L(e)}$ and define $L_t(e) := L(e) + te$ with the corresponding spheres $S_{L_t(e)}$. Fix $k \geq 0$ such that for $\epsilon_k := 2^{-k}$

$$T_\xi \cap (L_{-\epsilon_k}(e) \cup L_{\epsilon_k}(e)) = \emptyset$$

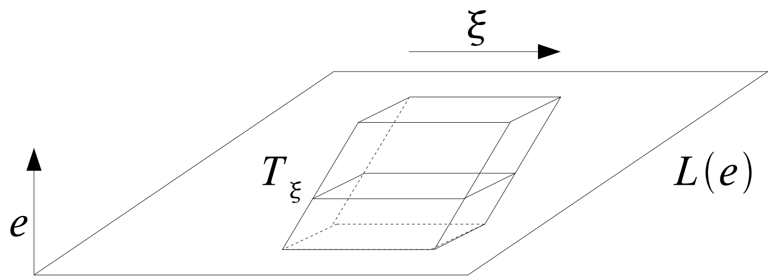

Figure 11: The Parllelepiped in the plane L

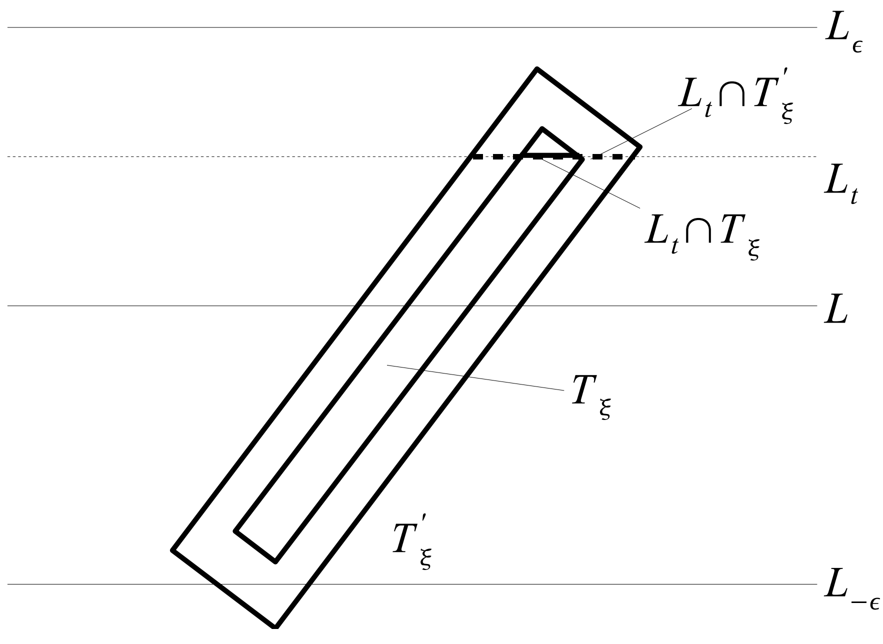

Figure 12: Here, ξ is perpendicular to the plane of projection

and

$$T_\xi \cap (L_{-\epsilon_{k-1}}(e) \cup L_{\epsilon_{k-1}}(e)) \neq \emptyset.$$

Let T'_ξ be a parallelepiped $T_\xi(x,y) := T_\xi$ elongated by 2δ on both sides along $x-y$ (see figure 12) such that $T_\xi \cap L_t(e) \neq \emptyset$ implies $|T'_\xi \cap L_t| \geq 2\delta$, while $|T'_\xi| \approx |T_\xi|$. We use the convention $\frac{1}{|\emptyset|}\int_\emptyset f := 0$. For each t we have

$$\frac{1}{|T'_\xi \cap L_t|}\int_{T_\xi \cap L_t(e)} |f(x) d\mu_2(x)| \leq \frac{1}{|T'_\xi \cap L_t(e)|}\int_{T'_\xi \cap L_t(e)} |f(x) d\mu_2(x)| \lesssim (f|_{L_t(e)})^*_\delta(\xi),$$

since the δ-rectangles τ_ξ in the 2-dimensional Kakeya maximal function are subsets of the rectangles $T'_\xi \cap L_t(e)$. Thus

$$\begin{aligned}\frac{1}{|T_\xi|}\int_{T_\xi} |f| &\approx \frac{1}{|T'_\xi|}\int_{T_\xi} |f| \\ &\approx \frac{1}{\epsilon_k}\int_{-\epsilon_k}^{\epsilon_k} \frac{1}{|T'_\xi \cap L_t|}\Big(\int_{T_\xi \cap L_t(e)} |f(x)| d\mu_2(x)\Big) dt \\ &\lesssim \frac{1}{\epsilon_k}\int_{-\epsilon_k}^{\epsilon_k} (f|_{L_t(e)})^*_\delta(\xi) dt,\end{aligned} \tag{3.69}$$

where $\mu_2(x)$ denotes the 2-dimensional Lebesgue measure on $L_t(e)$. Considering (3.69) for all $T_\xi \in \mathbf{T}_\xi$ we find

$$\mathcal{M}_\delta f(\xi) \lesssim \sup_{\epsilon_k} \frac{1}{\epsilon_k}\int_{-\epsilon_k}^{\epsilon_k} (f|_{L_t(e)})^*_\delta(\xi) dt \leq \sum_{\delta < \epsilon_k < 1} \frac{1}{\epsilon_k}\int_{-\epsilon_k}^{\epsilon_k} (f|_{L_t(e)})^*_\delta(\xi) dt \tag{3.70}$$

where again $\epsilon_k := 2^{-k}$. By squaring (3.70) and exploiting Cauchy-Schwarz twice, i.e.
$\Big(\sum_{i=1}^n a_i\Big)^2 \leq n \cdot \sum_{i=1}^n a_i^2$ and $\Big(\frac{1}{\epsilon}\int_{-\epsilon}^{\epsilon} f\Big)^2 \leq \frac{2}{\epsilon}\int_{-\epsilon}^{\epsilon} f^2$, we get

$$|\mathcal{M}_\delta f(\xi)|^2 \lesssim \Big(\log \frac{1}{\delta}\Big) \sum_{\epsilon_k} \frac{1}{\epsilon_k}\int_{-\epsilon_k}^{\epsilon_k} [(f|_{L_t})^*_\delta(\xi)]^2 dt. \tag{3.71}$$

Thus integrating over $S_{L(e)}$ and $S_{L_t(e)}$ respectively and applying the 2-dimensional bound yields

$$\begin{aligned}\int_{S_{L(e)}} |\mathcal{M}_\delta f|^2 &\lesssim \Big(\log \frac{1}{\delta}\Big) \sum_{\epsilon_k} \frac{1}{\epsilon_k}\int_{-\epsilon_k}^{\epsilon_k} \Big(\int_{S_{L_t(e)}} [(f|_{L_t(e)})^*_\delta(\xi)]^2 d\xi\Big) dt \\ &\overset{(3.2)}{\leq} \Big(\log \frac{1}{\delta}\Big)^2 \sum_{\epsilon_k} \frac{1}{\epsilon_k}\int_{-\epsilon_k}^{\epsilon_k} \Big(\int_{L_t(e)} |f(x)|^2 d\mu_2(x)\Big) dt.\end{aligned} \tag{3.72}$$

We are going to integrate (3.72) over the set of all 2-dimensional subspaces $L(e)$, i.e. over all $e \in S^2$.
Define the sets

$$A_{L_t(e)} := L_t(e) \cap (B(0,2r) \setminus B(0,r))$$

and their union

$$D_{e,\epsilon} := \bigcup_{-\epsilon<t<\epsilon} A_{L_t(e)},$$

Let $S^2_\rho := \{\rho\xi | \xi \in S^2\}$, $r \leq \rho \leq 2r$ be a scaled sphere and define

$$C_{\rho,\epsilon}(e) := S^2_\rho \cap D_{e,\epsilon}.$$

Since f is supported on $B(0,2r) \setminus B(0,r)$

$$\int_{-\epsilon}^{\epsilon} \Big(\int_{L_t(e)} |f(x)|^2 d\mu_2(x) \Big) dt = \int_{D_{e,\epsilon}} |f(x)|^2 dx, \tag{3.73}$$

furthermore

$$\int_{D_{e,\epsilon}} |f(x)|^2 dx = \int_r^{2r} \Big(\int_{C_{\rho,\epsilon}(e)} |f(x)|^2 d\sigma^2_\rho(x) \Big) d\rho \tag{3.74}$$

where $\sigma^2_\rho(x)$ denotes the 2-dimensional surface measure on S^2_ρ.

Lemma 3.5. *Let $f \in L^1(S^2_\rho$. Let $R(e) \subset S^2_\rho$, $|R(e)| \neq 0$ rotation invariant with respect to the direction $e \in S^{n-1}$, i.e. the position of $R(e)$ is well-defined by e. Then*

$$\int_{S^2} g(\rho\xi) d\xi = \frac{1}{|R(e)|} \int_{e\in S^2} \Big(\int_{R(e)} g(x) d\sigma^2_\rho(x) \Big) de. \tag{3.75}$$

In particular

$$\int_{S^2} g(\rho\xi) d\xi \approx \frac{1}{\rho\epsilon} \int_{e\in S^2} \Big(\int_{C_{\rho,\epsilon}(e)} g(x) d\sigma^2_\rho(x) \Big) de. \tag{3.76}$$

Furthermore

$$\int_{S^2} g(\xi) d\xi = \frac{1}{|S_{L(e)}|} \int_{e\in S^2} \Big(\int_{S_{L(e)}} g(x) d\sigma_1(x) \Big) de, \tag{3.77}$$

where $\sigma_1(x)$ denotes the 1-dimensional arc measure on $S_{L(e)}$ and accordingly $|S_{L(e)}| := \int_{S_{L(e)}} d\sigma^1(x)$.

Proof. Note that g can be written as a linear combination of Dirac-δ-distributions, i.e. $g(x) = \int g(y)\delta(x-y)dy$. By linearity of integrals it is therefore enough to show that (3.75) holds for Dirac-δ-functions.
Let $g(x) := \delta_a(x)$ be a spherical δ-function supported on $a \in S^2_\rho$, i.e.

$$\int_{S^2} \delta_a(\rho\xi)d\xi = \int_{S^2_\rho} \delta_a(x)\rho^{-2}d\sigma^2_\rho(x) = \rho^{-2}. \tag{3.78}$$

On the right hand side of (3.75) we also obtain

$$\begin{aligned}\int_{e\in S^2}\Big(\int_{R(e)} \delta_a(x)d\sigma^2_\rho(x)\Big)de &= \int_{e\in S^2} \chi_{R(e)}(a)de \\ &= \rho^{-2}\int_{e\in S^2} \chi_{R(\rho^{-1}a)}(\rho e)de \\ &= \rho^{-2}|R(e)|\end{aligned} \tag{3.79}$$

which shows (3.75).
Now, (3.76) follows immediately since $|C_{\rho,\epsilon}(e)| \approx \rho\epsilon$.
For (3.77) consider $g \in C_0^\infty(S^2)$. We have

$$\int_{S^2} g(\xi)d\xi = \frac{1}{|C_{1,\epsilon}(e)|}\int_{e\in S^2}\Big(\int_{C_{1,\epsilon}(e)} g(x)d\sigma^2_\rho(x)\Big)de \tag{3.80}$$

for all $\epsilon > 0$.
Observe that

$$\lim_{\epsilon\to 0}\frac{1}{2\epsilon}\int_{C_{1,\epsilon}(e)} g(x)d\sigma^2(x)de = \lim_{\epsilon\to 0}\frac{1}{2\epsilon}\int_{-\epsilon}^{\epsilon}\int_{S_{L(e)}} g(\frac{\xi}{\sqrt{1-t^2}}+te)d\xi dt = \int_{S_{L(e)}} g(x)d\sigma^1(x)de$$

and

$$\lim_{\epsilon\to 0}\frac{|C_{1,\epsilon}(e)|}{\epsilon} = 2|S_{L(e)}|.$$

Consequently

$$\begin{aligned}\int_{S^2} g(\xi)d\xi &= \lim_{\epsilon\to 0}\frac{1}{|C_{1,\epsilon}(e)|}\int_{e\in S^2}\Big(\int_{C_{1,\epsilon}(e)} g(x)d\sigma^2_\rho(x)\Big)de \\ &= \frac{1}{|S_{L(e)}|}\int_{e\in S^2}\Big(\int_{S_{L(e)}} g(x)d\sigma_1(x)\Big)de.\end{aligned}$$

□

Using (3.77) and integrating the left hand side of (3.72) we find

$$\int_{e_L \in S^2} \Big(\int_{S_L} |\mathcal{M}_\delta f(\xi)|^2 d\xi \Big) de \overset{(3.77)}{=} |S^1| \int_{S^2} |\mathcal{M}_\delta f(\xi)|^2 d\xi \approx \|\mathcal{M}_\delta f\|_2^2 \quad (3.81)$$

and together with the right hand side

$$\begin{aligned}
\|\mathcal{M}_\delta f\|_2^2 &\overset{(3.73)}{\lesssim} \Big(\log\frac{1}{\delta}\Big)^2 \sum_{\epsilon_k} \frac{1}{\epsilon_k} \int_{e \in S^2} \Big(\int_{D_{e,\epsilon_k}} |f(x)|^2 dx \Big) de \\
&\overset{(3.74)}{=} \Big(\log\frac{1}{\delta}\Big)^2 \sum_{\epsilon_k} \int_r^{2r} \frac{1}{\epsilon_k} \int_{S^2} \Big(\int_{C_{\rho,\epsilon_k}(e)} |f(x)|^2 d\sigma_2(x) \Big) de\, d\rho \\
&\overset{(3.76)}{=} \Big(\log\frac{1}{\delta}\Big)^2 \sum_{\epsilon_k} \int_r^{2r} \frac{1}{\epsilon_k} |C_{\rho,\epsilon_k}| \int_{S^2} |f(\rho\xi)|^2 d\xi\, d\rho \\
&\approx \Big(\log\frac{1}{\delta}\Big)^3 \int_r^{2r} \rho \int_{S^2} |f(\rho\xi)|^2 d\xi\, d\rho \qquad (3.82) \\
&= \Big(\log\frac{1}{\delta}\Big)^3 \int_r^{2r} \int_{S^2_\rho} |f(x)|^2 \frac{d\sigma_2(x)}{\rho} d\rho \\
&= \Big(\log\frac{1}{\delta}\Big)^3 \int_{B(0,2r)\backslash B(0,r)} \frac{|f(x)|^2}{|x|} dx \\
&\leq \Big(\log\frac{1}{\delta}\Big)^3 r^{-1} \|f\|_2^2.
\end{aligned}$$

□

Finally, we are almost done with the proof of theorem 3.1, since inequality (3.1) follows now by *Marcinkiewicz interpolation* (theorem 2.5).

Proof. (Theorem 3.1)
Note that (3.20) states that for the Kakeya maximal function a restricted weak type $(\frac{7}{3}+\epsilon, \frac{7}{3}+\epsilon)$ bound holds, whereas the 3-dimensional bound (3.3) states a restricted weak type (2,2) bound, which is actually even strong type (2,2).
Thus the Kakeya maximal operator satisfies the conditions of the Marcinkeiwicz interpolation theorem with $p_1 = q_1 = \frac{7}{3} + \epsilon$, $q_1 = q_2 = 2$, $C_1 \approx \delta^{-\frac{1}{2}}$ and $C_2 \approx \delta^{-\frac{2}{7}-\epsilon}$. Obviously $q_\theta \geq p_\theta$, hence the Kakeya maximal operator is

strong-type(p_θ, q_θ) with constant $C_\theta = C_1^\theta C_2^{1-\theta}$, where

$$\theta = \frac{p_1 p_2}{p_\theta(p_1 - p_2)} - \frac{p_2}{p_1 - p_2} = \frac{14}{p_\theta} - 6$$

and consequently

$$C_\theta = \left(\delta^{-\frac{1}{2}}\right)^\theta \left(\delta^{-\frac{2}{7}-\epsilon}\right)^{1-\theta} = \delta^{-\frac{3}{14}\theta - \frac{2}{7} - \epsilon'} = \delta^{-\frac{3}{p_\theta}+1-\epsilon'}.$$

□

Essentially the same arguments we used here for an estimate in 3 dimensions can also be used for higher dimensions for which we obtain

$$\|f_\delta^*\|_p \leq C_\epsilon \delta^{-\frac{n}{p}+1-\epsilon} \|f\|_p \quad \text{for } 2 \leq p \leq \frac{p}{(}n) \tag{3.83}$$

where $p(n)$ is implicitly given by $p(n) = \frac{p(n-1)(n+2)-n}{2p(n-1)-1}$, which is $p(n) = \frac{n+1}{2} + \epsilon_n$ for some ϵ_n. In particular

$$\begin{aligned} p(2) &= 2 \\ p(3) &= \frac{7}{3} \\ p(4) &= \frac{30}{11} \\ p(5) &= \frac{155}{49}. \end{aligned} \tag{3.84}$$

These bounds will improve the already known $\frac{n+1}{2}$-bound due to Drury (1983)[18], where for the upper Minkowski dimension the proof goes like this[4]: Assume a Kakeya set E has upper Minkowski dimension not greater than α, i.e.

$$|E_\delta| \leq C\delta^{1-\alpha}. \tag{3.85}$$

Analogously to the proof of inequality (3.21) we show that then

$$|E_\delta| \geq c\delta^{\alpha-1} \tag{3.86}$$

holds, which implies $\alpha \geq \frac{n+1}{2}$.
Considering the maximal δ-separated subset of S^{n-1} there must be points x_0

[4]Actually Drury proved x-ray and k-plane transform estimates from which this bound follows for both the Minkowski and the Hausdorff dimension

of high multiplicity $\approx \delta^{-(n-\alpha)}$ and the tubes intersecting x_0 are essentially disjoint away from a small neighborhood of x_0. Thus

$$|E_\delta| \geq c\delta^{-(n-\alpha)} \cdot \delta^{n-1} = c\delta^{\alpha-1}.$$

The same bound for Kakeya maximal functions was shown first by Christ-Duoandikoetxea-Rubio de Francia in 1987 [11].

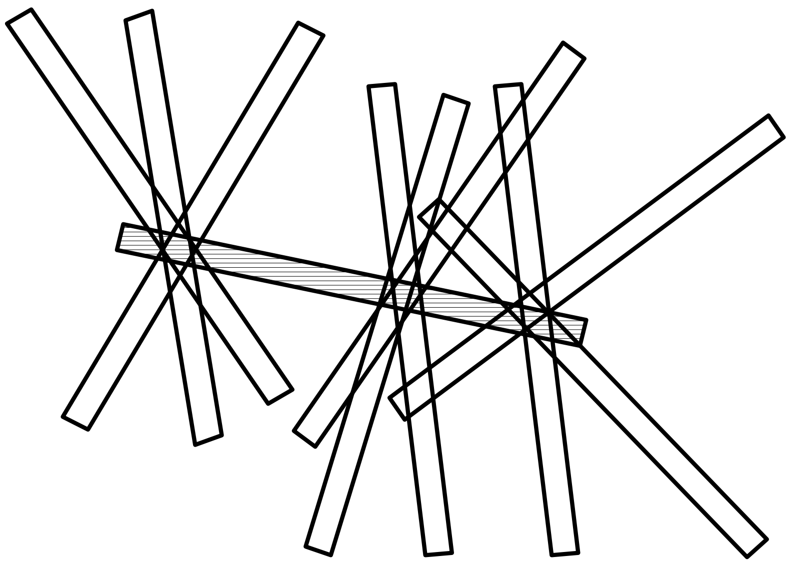

Figure 13: A "hairbrush"

4 An Overview of Present Results

The obtained bounds for the Hausdorff dimension due to Bourgain mentioned in the last section were improved by Wolff (1995) [50] to $\frac{n+2}{2}$. He used a "hairbrush" construction, which is more efficient than Bourgain's "bush" construction. While in the bush construction we considered only one point of high multiplicity in each step, the hairbrush construction exploits the fact that in Besicovitch sets there are many tubes which have many points of high multiplicity. Such a tube together with the tubes it intersects at its points of high multiplicity will form a hairbrush (see figure 13). Using the "bush" construction and a planar estimate due to Cordoba [12] enables to show that the bristols of the hairbrush are essentially disjoint. Hence the Minkowski dimension can be estimated with an estimate of the volume of that hairbrush. Again, restricted weak type estimates provide the proof for the corresponding constant for the Kakeya maximal estimate.
For $n > 4$ there is a better Hausdorff bound $(2-\sqrt{2})(n-4)+3$ due to Katz and Tao (2000) [27]. In that paper they also prove that the Kakeya maximal function satisfies the bound

$$\|f_\delta^*\|_{n+\frac{3}{4}} \leq C_\epsilon \delta^{-\frac{7n}{4n+3}+1-\epsilon} \|f\|_{\frac{4n+3}{7}}, \tag{4.87}$$

which improves Wolff's bound for $n \geq 9$.
For the Minkowski dimension, again the lower bound $(2-\sqrt{2})(n-4)+3$ is for $23 \geq n \geq 5$ the best result so far. Katz, Laba and Tao [25][30] proved a bound $\frac{n+2}{2}+\epsilon_n$ where ϵ_n is a fixed number (about 10^{-10} for $n=3,4$) which depends on n only. This statement means a slight improvement in 3 and 4 dimensions. Again, Katz and Tao [27] found a Minkowski bound $\frac{n-1}{\alpha}+1$, where α is the largest solution of $\alpha^3-4\alpha+2=0$ i.e. $\alpha=1.67513\ldots$, which is the biggest lower bound for $n \geq 23$. This estimate improved the previous bounds for $\alpha=\frac{7}{3}$ in [26] and $\alpha=\frac{25}{13}$ in [9].
The following table (see also [28]) illustrates the best presently known lower bounds p for the Minkowski dimension and the Hausdorff dimension for Kakeya sets. In the last two lines p means the largest value for which it is proved that the Kakeya maximal inequality (2.60), i.e.

$$\|f_\delta^*\|_{L^q(S^{n-1})} \leq C_\epsilon \delta^{-\frac{n}{p}+1-\epsilon} \|f\|_{L^p}$$

holds for any $0<\delta \ll 1$ with a suitable q which explicitly depends on p.

Result	*Dimension*	*Value for p*	*Reference*
Minkowski	$n=3$	$\frac{5}{2}+\epsilon$	Katz-Laba-Tao 1999 [25]
Minkowski	$n=4$	$3+\epsilon$	Laba-Tao 2000 [30]
Minkowski	$23 \geq n \geq 5$	$(2-\sqrt{2})(n-4)+3$	Katz-Tao 2000 [27]
Minkowski	$n>23$	$\frac{n-1}{\alpha}+1$	Katz-Tao 2000 [27]
Hausdorff	$n=3,4$	$\frac{n+2}{2}$	Wolff 1995 [50]
Hausdorff	$n=5$	$(2-\sqrt{2})(n-4)+3$	Katz-Tao 2000 [27]
Maximal	$8 \geq n \geq 3$	$\frac{n+2}{2}$	Wolff 1995 [50]
Maximal	$n \geq 9$	$\frac{4n+3}{7}$	Katz-Tao 2000 [27]

Since the Kakeya problem is related to various fields like in additive number theory, analysis, PDE and combinatorics one may expect that further progress will be made soon. As we have seen, combinatorics became a standard tool for harmonic analysis because of that problem. Applications to additive number theory are presented in I. Laba [31] and a short overview about more related topics can be found in [46].

References

[1] H.J. Alphen, *Uitbreiding van een stelling von Besicovitch*, Mathematica Zutphen B, **10** (1942), 144–157.

[2] A.S. Besicovitch, *Sur deux questions d'intégrabilité des fonctions*, J. Soc. Phys.-Math. (Perm) **2** (1919), 105–123.

[3] A.S. Besicovitch, *On Kakeya's problem and a similar one*, Math. Zeitschrift **27** (1928), 312–320.

[4] A.S. Besicovitch, *The Kakeya problem*, American Math. Monthly **70** (1963), 697–706.

[5] A.S. Besicovitch, *On fundamental geometric properties of plane line sets*, Journal of the London Math. Society **39** (1964), 441–448.

[6] A.S. Besicovitch, R. Rado, *A plane set of measure zero containing circumferences of every radius*, J. London Math. Soc. **43** (1968), 717–719.

[7] J. Bourgain, *Averages in the plane over convex curves and maximal operators*, J. Analyse Math. **47** (1986), 69–85.

[8] J. Bourgain, *Besicovitch type maximal operators and applications to Fourier analysis*, Geometric and Functional Analysis **1** (1991), 147–187.

[9] J. Bourgain, *On the dimension of Kakeya sets and related maximal inequalities*, Geom. Funct. Anal. **9** (1999), 256–282.

[10] J. Bourgain, *L^p estimates for oscillatory integrals in several variables*, Geometric and Functional Analysis **1** (1991), 321–374

[11] M. Christ, J. Duoandikoetxea, and J.L. Rubio de Francia, *Maximal operators associated to the Radon transform and the Calderon-Zygmund method of rotations*, Duke Math. J. **53** (1986), 189–209.

[12] A. Cordoba, *The Kakeya maximal function and spherical summation multipliers*, American Journal of Mathematics **99** (1977), 1–22.

[13] F. Cunningham, *The Kakeya problem for simply connected and for star-shaped sets*, American Mathematical Monthly **78** (1971), 114–129.

[14] F. Cunningham, *Three Kakeya problems*, American Math. Monthly **81** (1974), 582–592.

[15] F. Cunningham, I.J. Schoenberg, *On the Kakeya constant*, Canad. J. Math. **17** (1965), 946–956

[16] R.O. Davies, *Some remarks on the Kakeya problem*, Proc. Cambridge Phil. Soc. **69** (1971), 417–421.

[17] R.O. Davies, *Another thin set of circles*, J. London Math. Soc., (2) **5** (1972), 191–192.

[18] S. Drury, *L^p estimates for the x-ray transform*, Ill. J. Math. **27** (1983), 125-129.

[19] K.J. Falconer, *Continuity properties of k-plane integrals and Besicovitch sets*, Math. Proc. Cambridge Philos. Soc. **87** (1980), 221-226.

[20] K.J. Falconer, *The geometry of fractal sets*, Cambridge University Press (1985).

[21] K.J. Falconer, *Fractal Geometry — Mathematical Foundations and Applications*, John Wiley & Sons Ltd. (1990).

[22] B. Fisher, *On a Problem of Besicovitch*, The American Mathematical Monthly **80**, 7 (1973), 785–787.

[23] M. Fujiwara, S. Kakeya, *On some problems of maxima and minima for the curve of constant breadth and the in-revolvable curve of the equilateral triangle*, Tôhoku Mathematical Journal **11** (1917), 92–110.

[24] S. Kakeya, *Some problems on minima and maxima regarding ovals*, Tôhoku Science Reports, **6** (1917), 71–88.

[25] N.H. Katz, I. Laba and T. Tao, *An improved bound on the Minkowski dimension of Besicovitch sets in $\mathbb{R}^3$*, Annals of Math. (2) **152**, 2 (2000), 383–446.

[26] N.H. Katz, T. Tao, *Bounds on arithmetic projections, and applications to the Kakeya conjecture*, Math. Res. Letters **6** (1999), 625–630.

[27] N.H. Katz, T. Tao, *New bounds for Kakeya problems*, J. Anal. Math. **87** (2002), 231-263.

[28] N.H. Katz, T. Tao, *Recent progress on the Kakeya conjecture*, Publicacions Matematiques, Proceedings of the 6th El Escorial International Conference on Harmonic Analysis and Partial Differential Equations, U. Barcelona, 2002, 161–180.

[29] J.R. Kinney, *A thin set of circles*, Amer. Math. Monthly **75** (1968), 1077–1081.

[30] I. Laba, T. Tao, *An improved bound for the Minkowski dimension of Besicovitch sets in medium dimension*, Geometric and Functional Analysis **11** (2001), 773–806.

[31] I. Laba, *From harmonic analysis to arithmetic combinatorics*, Bulletin (New Series) of the AMS **45**, 1, (2008) 77-115

[32] E. H. Lieb, M. Loss, *Analysis: Second Edition*, AMS, 2001

[33] J.M. Marstrand, *Packing circles in the plane*, Proc. London. Math. Soc. **55** (1987), 37-58.

[34] J.M. Marstrand *Packing Planes in* $\mathbb{R}^3$, Mathematika **26** (1979) 180-183.

[35] O.M. Nikodym, *Sur la mesure des ensembles plans dont tous les oints sont rectilinéairement accessibles*, Fund. Math., **10** (1927), 116-168.

[36] J. Pàl, *Ein Minimumproblem für Ovale*, Math. Annalen **83**, 311–319.

[37] O. Perron, *Über einen Satz von Besicovitch*, Math. Zeitschrift, **28** (1928), 383–386.

[38] H.A. Rademacher, *On a theorem from Besicovitch*, In Studies in Mathematical Analysis-Essays in Honor of G. Pòlya, Stanford University Press (1962), 294–296.

[39] W. Rudin, *Functional Analysis*, McGraw-Hill, Inc., New York, 1991.

[40] I.J. Schoenberg, *On certain minima related to the Besicovitch-Kakeya Problem*, Mathematica (Cluj), **4** (27) (1962), 145–148.

[41] I.J. Schoenberg, *On the Besicovitch-Perron solution of the Kakeya Problem*, In Studies in Mathematical Analysis-Essays in Honor of G. Pòlya, Stanford University Press (1962), 359–363

[42] E.M. Stein, *Maximal functions: Spherical means*, Proc. Natl. Acad. Sci. USA **73** (1976), 2174-2175.

[43] E.M. Stein, *Harmonic Analysis*, Princeton University Press, 1993.

[44] E.M. Stein, G. Weiss, *Introduction to Fourier Analysis on Euclidean spaces*, Princeton University Press, Princeton, N.J. 1990.

[45] T. Tao *Lecture notes 1 for 247A*, http://www.math.ucla.edu/%7Etao/247a.1.06f/notes1.dvi

[46] T. Tao, *From Rotating Needles to Stability of Waves: Emerging Connections between Combinatorics, Analysis, and PDE*, in *Notices of the AMS* **48**, 3 (2001).

[47] J.B. Wilker, *The non-Euclidean Kakeya Problem*, Preceedings of the 25th Summer Meeting of the Canadian Mathematical Congress (1971), 603–607.

[48] T. Wolff, *A Kakeya problem for circles*, American Journal of Mathematics **119** (1997), 985–1026.

[49] T. Wolff, *On some variants of the Kakeya problem*, Pacific Journal of Mathematics **190** (1999), 111–154.

[50] T. Wolff, *An improved bound for Kakeya type maximal functions*, Revista Mat. Iberoamericana **11**(1995), 651–674.

[51] T. Wolff, *Lectures in harmonic analysis*, I. Laba and C. Shubin, eds., American Mathematical Society, Providence, R.I. (2002).