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Regularity of eigenfunctions of Schrödinger operators with L^p-potential

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1 Introduction

This thesis deals with the smoothening properties of a second order elliptic operator L defined by

$$Lu = \Delta u + b_i D_i u + cu \quad \text{on} \quad \Omega, \tag{1.1}$$

where $\mathbf{b} = (b_1, \dots, b_n)$, c are prescribed L^p-functions and Ω is a bounded open set in $\mathbb{R}^n$ with $n \geqslant 3$. The results will be applied to eigenfunctions of the well-known Schrödinger operator

$$H = -\Delta + V \quad \text{on} \quad \Omega, \tag{1.2}$$

i.e. (weak) solutions to $H\psi = E\psi$, that describe a quantum particle in a domain Ω under the influence of a potential V. Our main goal is to single out the singular and the regular part of the eigenfunction ψ by means of the splitting

$$\psi = e^F \varphi, \tag{1.3}$$

where F is given by $\Delta F = V$ and thus contains the main singularities, while φ is smoother than the original eigenfunction ψ. This splitting was first introduced in [HHØ01] where it was applied for the potential $V(x) = |x|^{-1}$ and in a many-body setup. It was shown that e^F is Lipschitz continuous and φ has a locally α-Hölder continuous first derivative for any $\alpha \in (0, 1)$. In the present work we shall provide similar results for a broader class of potentials for a single particle. We shall discuss first the potential $V(x) = |x|^{-s}$ for $s < \frac{3}{2}$ and then generalize these results to an arbitrary L^p_{loc}-potential V with $p > \frac{2}{3}n$.

From the mathematical point of view, the study of existence, uniqueness and regularity properties of solutions to $Lu = f$ are crucial issues in the field of partial differential equations. We will discuss powerful techniques such as the Moser iteration and the approach via potential estimates, which are commonly used to prove regularity results. Notice that the whole analysis will be within the non-classical theory of elliptic equations where solutions are not classically differentiable and coefficients are only required to belong to some L^p-space. Notice also that L^p-integrability of the potential is the most reasonable assumption from a physical point of view since it monitors decay and singularity of V without prescribing its exact shape and smoothness.

Our main results are Theorem 7.2 and Theorem 7.4, where $V = |x|^{-s}$, $s < \frac{3}{2}$ and $V \in L^p_{\text{loc}}$, $p > \frac{2}{3}n$ respectively. The steps towards their proof will be worked out in detail in the preceding chapters. We shall start with stating and discussing standard definitions and recalling some well-known functional analysis theorems in Chapter 2. In Chapter 3 we shall establish crucial estimates on potentials and harmonic functions. The results will then be applied in Chapter 4 to prove interior Hölder regularity of bounded weak solutions. Chapter 4 will give the main contribution towards the regularity analysis of eigenfunctions. In practise, we shall distinguish two different types of elliptic operators. In Section 4.2 we will use Schauder's approach to derive properties of the operator L given by (1.1) with coefficients $\mathbf{b}$, c belonging to L^p for $p > n$ and properties of the operator $\bar{L}$ defined by

$$\bar{L}u = \Delta u + c,$$

with $c \in L^p$ for $p > \frac{n}{2}$. In both cases we will be able to derive an explicit Hölder coefficient $\alpha(n,p)$ depending only on the exponent p and the dimension n. In Section 4.3 we shall apply the method of Moser iteration to investigate the operator L as in (1.1) under weaker assumptions. Namely, we will only require $c \in L^p$ and $\mathbf{b} \in L^{2p}$ for $p > \frac{n}{2}$. Although we are still able to prove α-Hölder continuity in this case, the dependence of α turns out to be much more complicated than before. Chapter 5 proves the local boundedness of solutions under fairly weak assumptions, thus providing the main ingredient used in Chapter 4, where the boundedness of the solution was assumed throughout. This will conclude the first part of the thesis, that concerns the regularity properties of a general elliptic operator L.

In Chapter 6 we shall turn our attention towards existence and uniqueness questions. In view of our goal to derive regularity properties of eigenfunctions, we shall restrict ourselves to the special case of Schrödinger operators $H = -\Delta + V$ on whole $\mathbb{R}^n$. The assumptions will be slightly more general than before by allowing the potential V to be in L^p_{loc} for $p \geqslant \frac{n}{2}$ with some further restrictions on V. This, together with the regularity theory from the previous chapters will enable us to state and prove our results in Chapter 7.

Our presentation and the results can be extended in various ways. First of all one can exploit different methods. In fact, whereas in the present work we chose a purely analytic approach, many of the main estimates, like the weak Harnack inequality, may be as well derived via probabilistic methods by means of Markov processes and stopping times, see e.g. [Sim82]. Another possible extension is broadening the function classes used. As it turns out, assuming L^p-integrability of the coefficients is not the most natural class of functions to work with in the context of Elliptic/Hamilton operators. Broadly speaking, the crucial quantity is the integrability of the potential, rather than the function itself, see e.g. [Sim82], [LU68]. It might also be possible to weaken some assumptions on the L^p-properties of the coefficients. For instance, it was shown in [LU68] that the assumptions on $\mathbf{b}$ in Section 4.3 can be relaxed to $\mathbf{b}$ belonging to L^n. Concerning our main results in Chapter 7 it is possible to lift them suitably from the one-body to the many-body setup as in [HHØ01]. Another possible extension is the development of finer regularity results similar to [FHHS05], where it was shown that the derivative of the function φ in (1.3) is not only α-Hölder continuous for any $\alpha \in (0,1)$ but even Lipschitz continuous.

2 Preliminaries

In this chapter we state some preliminary definitions and lemmata as well as some well known functional analysis theorems. For reference see e.g. [LL01], [GT01] and [Eva10]. Throughout all integrals are taken w.r.t. the Lebesgue measure $\mathrm{d}\lambda$ and $n \geqslant 3$. For the measure of a set Ω we will write $|\Omega| := \lambda(\Omega)$. All functions are to be understood in the usual sense of equivalence classes. In particular sup and inf are to be understood as ess sup and ess inf and if we consider continuity it will mean that there exists a continuous representative in its equivalence class. Further we will use the summation convention, i.e. repeated indices i will be summed up, and denote all constants by the same letter C.

2.1 Sobolev spaces

Definition 2.1. (L^p-space)
Let $1 < p < \infty$ and $\Omega \subset \mathbb{R}^n$ be open. For a function $f : \Omega \to \mathbb{C}$ we define the norm

$$\|f\|_{p;\Omega} \equiv \|f\|_{L^p(\Omega)} := \left(\int_\Omega |f|^p\right)^{\frac{1}{p}}.$$

If $p = \infty$ we define

$$\|f\|_{\infty;\Omega} \equiv \sup_{x\in\Omega} |f(x)| := \inf\{\alpha > 0\colon \lambda\left(\{x \in \Omega\colon |f(x)| > \alpha\}\right) = 0\}$$

and $\inf_\Omega$ analogously.

We say

$$f \in L^p(\Omega) \quad :\Longleftrightarrow \quad \|f\|_{p;\Omega} < \infty.$$

If the function $\mathbf{f} = (f_1, \ldots, f_n)\colon \mathbb{R}^n \to \mathbb{C}^n$ is vector valued we say

$$\mathbf{f} \in L^p(\Omega) :\Longleftrightarrow \|f_i\|_{p;\Omega} < \infty$$

for every component f_i, $i = 1, \ldots, n$.

In the case $p = 2$ we further define a scalarproduct $\langle \cdot, \cdot \rangle$

$$\begin{aligned} \langle \cdot, \cdot \rangle \colon \quad L^2(\Omega) \times L^2(\Omega) &\longrightarrow \mathbb{C}, \\ (f, g) \qquad &\longmapsto \langle f, g \rangle = \int_\Omega \bar{f} g. \end{aligned}$$

Remark 2.2. The defined norms are indeed norms and $(\Omega, L^p(\Omega))$ is a Banach space. Moreover the scalar product is well defined and induces the L^2-norm.

Definition 2.3. ($L^p_{\mathrm{loc}}(\Omega)$-space)
Let $p \in [1, \infty]$ and $\Omega \subset \mathbb{R}^n$ be open. We say $f \colon \Omega \to \mathbb{C}$ is in $L^p_{\mathrm{loc}}(\Omega)$ iff

$$\|f\|_{p;K} < \infty \quad \text{for any } K \subset \Omega \text{ compact.}$$

Further for $\Omega \subset \mathbb{R}^n$ open and unbounded we say

$$f \in L^p(\Omega) + L^q(\Omega) \;:\Leftrightarrow\; \exists f_1 \in L^p(\Omega),\ f_2 \in L^q(\Omega) \colon f = f_1 + f_2.$$

Lemma 2.4. *(Interpolation inequality)*
Let $|\Omega| < \infty$ and $u \in L^r(\Omega)$. Then for any $\varepsilon > 0$ we have

$$\|u\|_q \leqslant \; \varepsilon \|u\|_r + \varepsilon^{-\mu} \|u\|_p,$$

where $p \leqslant q \leqslant r$ and $\mu = \left(\frac{1}{p} - \frac{1}{q}\right) / \left(\frac{1}{q} - \frac{1}{r}\right)$.

Lemma 2.5. *(Characterization of the L^2-norm)*
Let $f \in L^2(\Omega)$ then

$$\begin{aligned} \|f\|_2 &= \sup\{|\langle f, g \rangle| \colon g \in L^2(\Omega),\ \|g\|_2 = 1\} \\ &= \sup\{|\langle f, g \rangle| \colon g \in C_0^\infty(\mathbb{R}^n),\ \|g\|_2 = 1\}. \end{aligned}$$

Proof. We start with the first equality. Setting $g := f/\|f\|_2$ gives

$$\|f\|_2 \leqslant \sup\{|\langle f, g \rangle| \colon g \in L^2(\Omega),\ \|g\|_2 = 1\}$$

and by the Cauchy Schwarz inequality

$$\text{RHS} \leqslant \sup\{\|f\|_2 \|g\|_2 \colon g \in L^2(\Omega),\ \|g\|_2 = 1\} = \|f\|_2. \quad \#$$

The second equality is due to the denseness of $C_0^\infty(\Omega)$ in $L^2(\Omega)$ w.r.t. $\|\cdot\|_2$. ☐

Definition 2.6. (Strictly positive functions)
Let $\Omega \subset \mathbb{R}^n$ be open and $f \in L^1_{\mathrm{loc}}(\Omega)$. We will say f is strictly positive on Ω, denoted by $f > 0 :\Leftrightarrow$ for any compact $K \subset \Omega$, there exists $\varepsilon > 0$ s.t. $|\{x \in K \colon f(x) < \varepsilon\}| = 0$.

Definition 2.7. (Pseudo-L^p-norm)
Let $\Omega \subset \mathbb{R}^n$ be bounded, $p \in \mathbb{R}$ and $u > 0$ in Ω. Then we define the Pseudo-L^p-norm of u in Ω by:

$$\interleave u \interleave_{p;\Omega} := \left(\int_\Omega u(x)^p \, \mathrm{dx} \right)^{\frac{1}{p}}.$$

If we consider a specific ball $\Omega := B_r$ of radius $r > 0$ we will abbreviate it to:

$$\interleave u \interleave_{p;r} := \left(\int_{B_r} u(x)^p \, \mathrm{dx} \right)^{\frac{1}{p}}.$$

Proposition 2.8. (Properties of the pseudo-L^p-norm)
Let $\Omega \subset \mathbb{R}^n$ be open, bounded and $u > 0$ then

i) $\interleave \cdot \interleave_{p;\Omega}$ coincides with $\| \cdot \|_{p;\Omega}$ for $p \geqslant 1$ and otherwise is not even a norm.

ii) If $\Omega' \subset \Omega$ then $\interleave \cdot \interleave_{p;\Omega'} \leqslant \interleave \cdot \interleave_{p;\Omega}$ and thus for $0 < r' < r$ we have $\interleave \cdot \interleave_{p;r'} \leqslant \interleave \cdot \interleave_{p;r}$.

iii) If $0 < p' < p$ then $\interleave \cdot \interleave_{p';\Omega} \leqslant |\Omega|^{\frac{1}{p'} - \frac{1}{p}} \interleave \cdot \interleave_{p;\Omega}$.

iv) Assume $\sup_{1 \leqslant p < \infty} \|u\|_{p;\Omega} < \infty$ then $u \in L^\infty(\Omega)$ and $\lim_{p \to \infty} \|u\|_{p;\Omega} = \sup_\Omega u$.

v) $\lim_{p \to -\infty} \interleave u \interleave_{p;\Omega} = \inf_\Omega u$.

Proof. i)-ii) are trivial. Note that the triangle inequality fails to hold for $p < 1$. It remains to show iii) to v).

iii)

Since $\frac{p}{p'} > 1$ we can apply Hölder's inequality to

$$\interleave u \interleave_{p'}^{p'} = \int u^{p'} \cdot 1 \leqslant \|u^{p'}\|_{\frac{p}{p'}} \|1\|_{\frac{p}{p-p'}} = \interleave u \interleave_p^{p'} |\Omega|^{\frac{p-p'}{p}}.$$

Taking the p'-th root gives:

$$\interleave u \interleave_{p'} \leqslant |\Omega|^{\frac{1}{p'} - \frac{1}{p}} \interleave u \interleave_p.$$

iv)

In the following we will omit the subscript Ω and set $C := \sup_{1 \leqslant p < \infty} \|u\|_{p;\Omega}$. Assume $u \notin L^\infty$, i.e.

$$\|u\|_\infty = \inf\{\alpha > 0 \colon \lambda(\{x \colon |u(x)| > \alpha\}) = 0\} = \infty$$

and thus for all $\alpha > 0$ we have $\lambda(\{x \in \Omega \colon |u(x)| > \alpha\}) > 0$. We now estimate for all $\alpha > 0$, $p \in [1, \infty)$:

$$C \geqslant \|u\|_p \geqslant \left(\int_{x \colon |u(x)| > \alpha} |u|^p \right)^{\frac{1}{p}} \geqslant \alpha \left(\lambda(\{x \colon |u(x)| > \alpha\}) \right)^{\frac{1}{p}}$$

and thus for fixed α

$$\lim_{p\to\infty} \alpha\left(\lambda\left(\{x\colon |u(x)| > \alpha\}\right)\right)^{\frac{1}{p}} = \alpha \leqslant C < \infty$$

which contradicts the arbitrariness of α and thus we conclude $\|u\|_\infty < \infty$. #

For the second part we may assume $\|u\|_p > 0$ and choose $0 < \varepsilon < \|u\|_p$. Then as above we conclude

$$\|u\|_p \geqslant (\|u\|_\infty - \varepsilon)\lambda\left(\{x\colon |u(x)| > \|u\|_\infty - \varepsilon\}\right)^{\frac{1}{p}}$$

and by definition of $\|u\|_\infty$ it is

$$\lambda\left(\{x\colon |u(x)| > \|u\|_\infty - \varepsilon\}\right)^{\frac{1}{p}} > 0$$

and thus

$$\liminf_{p\to\infty} \|u\|_p \geqslant \|u\|_\infty - \varepsilon \qquad \forall\, \varepsilon > 0.$$

By letting $\varepsilon \to 0$ we obtain $\liminf_{p\to\infty} \|u\|_p \geqslant \|u\|_\infty$. On the other hand by iii) we can estimate $\|u\|_p$ from above by

$$\|u\|_p \leqslant |\Omega|^{\frac{1}{p}} \|u\|_\infty \to \|u\|_\infty \ \text{ for } p \to \infty$$

and arrive at the conclusion

$$\limsup_{p\to\infty} \|u\|_p \leqslant \|u\|_\infty \leqslant \liminf_{p\to\infty} \|u\|_p. \qquad \#$$

v)
Assume $q > 0$, then we want to show $\lim_{q\to\infty} |\!|\!|u|\!|\!|_{-q} = \inf u$. By definition of the norm we have $|\!|\!|u|\!|\!|_{-q}^{-1} = \|u^{-1}\|_q$ and by assumption for all $q > 1$

$$\|u^{-1}\|_q = \left(\int_\Omega \left(\frac{1}{u}\right)^q\right)^{\frac{1}{q}} \leqslant \frac{1}{\varepsilon}|\Omega| < \infty$$

and thus by the previous claim

$$\lim_{q\to\infty} |\!|\!|u|\!|\!|_{-q}^{-1} = \lim_{q\to\infty} \|u^{-1}\|_q = \sup u^{-1}.$$

Noting that $\inf u = (\sup u^{-1})^{-1}$ gives

$$\inf u = \left(\lim_{q\to\infty}\left(\int u^{-q}\right)^{\frac{1}{q}}\right)^{-1} = \lim_{q\to\infty}\left(\left(\int u^{-q}\right)^{\frac{1}{q}}\right)^{-1} = \lim_{q\to\infty} |\!|\!|u|\!|\!|_{-q},$$

where we used that the $x \mapsto x^{-1}$ is continuous away from 0. □

Definition 2.9. (Weak differentiability)
Let $\Omega \subset \mathbb{R}^n$ be open and $f \in L^1_{\text{loc}}(\Omega)$. Then f has weak derivative $D_i f$, $i = 1, \ldots, n$ iff there exists $g \in L^1_{\text{loc}}(\Omega)$ s.t.

$$\int_\Omega D_i \varphi f = -\int_\Omega \varphi g \quad \forall\, \varphi \in C_0^\infty(\Omega).$$

In this case we denote g by $D_i f$.

Lemma 2.10. *(Chain rule)*
Let $f\colon \mathbb{R} \to \mathbb{R}$ be differentiable with $f' \in L^\infty(\mathbb{R})$. Then if u is weakly differentiable in Ω we have $f \circ u$ is weakly differentiable and

$$D(f \circ u) = f'(u) Du.$$

We now want to generalize the concept of weak derivatives to a bigger class of functions and to an arbitrary derivative D^α, $\alpha \in \mathbb{N}^k$.

Definition 2.11. (The space of test functions $\mathcal{D}(\Omega)$)
Let $\Omega \subset \mathbb{R}^n$ be open. We define the space of test functions, denoted by $\mathcal{D}(\Omega)$, as $\mathcal{D}(\Omega) = C_0^\infty(\Omega)$ together with the following notion of convergence:

$$\varphi_n \to \varphi \text{ in } \mathcal{D}(\Omega) :\Leftrightarrow \begin{cases} \exists K \subset \Omega \text{ compact s.t. } \operatorname{supp}(\varphi_n - \varphi) \subset K \text{ and} \\ \forall\, \alpha \in \mathbb{N}^k \text{ multi-indices: } D^\alpha \varphi_n \to D^\alpha \varphi \text{ uniformly in } K. \end{cases}$$

Remark 2.12. In this case the sequential convergence suffices to characterize the topology (see e.g. [Rud91]). However we will only work with sequential convergence in the following.

Definition 2.13. (The space of distributions $\mathcal{D}'(\Omega)$)
Let $\Omega \subset \mathbb{R}^n$ be open and define the space of distributions $\mathcal{D}'(\Omega) := \mathcal{D}(\Omega)^*$ as the dual space of $\mathcal{D}(\Omega)$. Any element $T \in \mathcal{D}'(\Omega)$ will be called distribution on Ω. We further define sequential convergence on $\mathcal{D}'(\Omega)$ by

$$T_n \to T \text{ in } \mathcal{D}'(\Omega) \;:\Leftrightarrow T_n(\varphi) \to T(\varphi) \quad \forall\, \varphi \in \mathcal{D}(\Omega)$$

and we define for any multi-index α the derivative $D^\alpha T \in \mathcal{D}'(\Omega)$ of a distribution T by its action on $\varphi \in \mathcal{D}(\Omega)$:

$$(D^\alpha T)(\varphi) := (-1)^{|\alpha|} T(D^\alpha \varphi).$$

Additionally we define the kernel of a distribution $T \in \mathcal{D}'(\Omega)$ by

$$\mathcal{N}_T := \{\varphi \in \mathcal{D}(\Omega)\colon T(\varphi) = 0\}.$$

Lemma 2.14. *(Linear dependence of distributions)*
Let $S_1, \ldots, S_N \in \mathcal{D}'(\Omega)$ be distributions and suppose that $T \in \mathcal{D}'(\Omega)$ is a distribution

s.t. $T(\varphi) = 0$ *for all* $\varphi \in \bigcap_{i=1}^N \mathcal{N}_{S_i}$. *Then there exist* $c_1, \dots, c_N \in \mathbb{C}$ *s.t.*

$$T = \sum_{i=1}^N c_i S_i.$$

Definition 2.15. (Sobolev spaces H^1 and H_0^1)
Let $\Omega \subset \mathbb{R}^n$ be open and $f : \Omega \to \mathbb{C}$. Then

$$f \in H^1(\Omega) :\Leftrightarrow f \in L^2(\Omega) \text{ and } Df \in L^2(\Omega),$$

where Df is the distributional gradient of f.

Further $H^1(\Omega)$ can be equipped with a norm $\|\cdot\|_{H^1}$ defined by

$$\|\cdot\|_{H^1} : H^1(\Omega) \to \mathbb{R}^+; \quad f \mapsto (\|f\|_{2;\Omega} + \|Df\|_{2;\Omega})^{1/2}.$$

Moreover we define the space $H_0^1(\Omega)$ as the closure of $C_0^\infty(\Omega)$ w.r.t. $\|\cdot\|_{H^1}$.

Theorem 2.16. *(Properties of H^1 and H_0^1)*
Let $\Omega \subset \mathbb{R}^n$ be open. Then the following holds:

1. $\|\cdot\|_{H^1}$ *is induced by a scalar product*

$$\begin{aligned} \langle\cdot,\cdot\rangle_{H^1} : \quad & H^1 \times H^1 \longrightarrow \mathbb{C}, \\ & (f,g) \quad \longmapsto \langle f,g\rangle = \int \overline{f}g + \overline{Df}\cdot Dg. \end{aligned}$$

2. $H^1(\Omega)$ *and* $H_0^1(\Omega)$ *are Hilbert spaces w.r.t.* $\|\cdot\|_{H^1}$.
3. $C^\infty(\Omega) \cap H^1(\Omega)$ *is dense in* $H^1(\Omega)$ *w.r.t.* $\|\cdot\|_{H^1}$.
4. *If* $\Omega = \mathbb{R}^n$ *even* $C_0^\infty(\Omega) \cap H^1(\Omega)$ *is dense in* $H^1(\Omega)$ *w.r.t.* $\|\cdot\|_{H^1}$.

Theorem 2.17.
Let $\Omega \subset \mathbb{R}^n$ be open and $\{f_n\}_{n\in\mathbb{N}} \subset H^1(\Omega)$ with $\sup_j \|f_j\|_{H^1(\Omega)} < \infty$. Then there exists a subsequence f_{n_k} and $f \in H^1(\Omega)$ s.t.

$$f_{n_k} \rightharpoonup f \text{ weakly in } H^1(\Omega) \text{ as } k \to \infty.$$

Theorem 2.18. *(Rellich-Kondrachov)*
Let $B \subset \mathbb{R}^n$, $n \geqslant 3$ be open, bounded with a regular boundary and suppose $f_n \rightharpoonup f$ in $H^1(\mathbb{R}^n)$. Then for any q with $1 \leqslant q < \frac{2n}{n-2}$ we have

$$\int_B |f - f_n|^q \to 0 \text{ for } n \to \infty.$$

Or in short: $H^1(B) \Subset L^q(B)$, i.e. $H^1(B)$ is compactly imbedded in $L^q(B)$.

Lemma 2.19. *(Sobolev inequality for H^1)*
Let $\Omega \subset \mathbb{R}^n$, $n \geqslant 3$ be open, bounded and $u \in H^1(\Omega)$. Then there exists a constant $C = C(n, |\Omega|)$ such that

$$\|u\|_{\frac{2n}{n-2}} \leqslant C(n, |\Omega|)\|u\|_{H^1}.$$

Further for any domain $\Omega \subset \mathbb{R}^n$ and $u \in H_0^1(\Omega)$ there exists a universal constant $C_{Sob} = C_{Sob}(n)$ depending only on n such that

$$\|u\|_{\frac{2n}{n-2}} \leqslant C_{Sob}\|Du\|_2.$$

Corollary 2.20. *(Poincare inequality in $H_0^1(\Omega)$)*
Suppose that $u \in H_0^1(\Omega)$. Then there exists a constant $C = C(n, |\Omega|)$ such that

$$\|u\|_2 \leqslant C\|Du\|_2.$$

Theorem 2.21. *(Hardy-Littlewood-Sobolev inequality)*
Let $p,\, p' > 1$, $0 < \lambda < 1$ and $\frac{1}{p} + \frac{\lambda}{n} + \frac{1}{p'} = 2$. Then for any two functions $f \in L^p(\mathbb{R}^n)$ and $h \in L^{p'}(\mathbb{R}^n)$ we have

$$\left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x)|x-y|^{-\lambda} h(y) \text{ dx dy} \right| \leqslant C\|f\|_p\|h\|_{p'},$$

where $C = C(n, \lambda, p)$.

2.2 Harmonic functions and potentials

Definition 2.22. (Positive and negative part)
For any function u we define

$$u_+ := \max\{u, 0\} \qquad \text{the positive part of } u,$$
$$u_- := -\min\{u, 0\} \qquad \text{the negative part of } u.$$

With this definition we have $u = u_+ - u_-$ and $|u| = u_+ + u_-$.

Definition 2.23. (Balls in $\mathbb{R}^n$)
For any given point $x_0 \in \mathbb{R}^n$ and $r > 0$ we denote the set of points with distance smaller than r by $B_r(x_0) \subset \mathbb{R}^n$. I.e.

$$B_r(x_0) := \{x \in \mathbb{R}^n \colon |x - x_0| < r\}.$$

Definition 2.24. (The Laplacian and harmonic functions)
Let Ω be a domain in $\mathbb{R}^n$ and $u\colon \Omega \to \mathbb{R}$. We define the (distributional) Laplacian of u by

$$\Delta u = \sum_{i=1}^n D_{ii}u = \text{div} Du.$$

And we call u harmonic in Ω if it satisfies

$$\Delta u = 0 \quad \text{in} \quad \Omega.$$

Lemma 2.25. *(Properties of harmonic functions)*
Let u be harmonic in a domain $\Omega \subset \mathbb{R}^n$. Then the following holds:

1. *Mean-value theorem:*
   For any ball $B = B_R(y) \Subset \Omega$ we have

$$u(y) = \frac{1}{\omega_n R^n} \int_B u \text{ dx},$$

   where ω_n denotes the volume of the unit sphere.

2. *Maximum principle:*
   Assume additionally that Ω is bounded and $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ then

$$\inf_{\partial\Omega} u \leqslant u(x) \leqslant \sup_{\partial\Omega} u \qquad \forall\, x \in \Omega.$$

3. *Differentiability of u:*

$$u \text{ is real analytic in } \Omega.$$

Definition 2.26. (Fundamental solution)
For $n \geqslant 3$ define:

$$\Gamma(x-y) = \Gamma(|x-y|) := \frac{1}{n(2-n)\omega_n} |x-y|^{2-n},$$

where ω_n denotes the volume of the unit sphere in $\mathbb{R}^n$.

Lemma 2.27. *(Properties of Γ)*
Gamma is symmetric in x, y and the following estimates hold:

$$|D_i\Gamma(x-y)| \leqslant \frac{1}{n\omega_n}|x-y|^{1-n}, \tag{2.1}$$

$$|DD_i\Gamma(x-y)| \leqslant \frac{n+1}{\omega_n}|x-y|^{-n}. \tag{2.2}$$

Further in the sense of distributions

$$-\Delta_x\Gamma(x-y) = \delta_y,$$

where δ_y is the Dirac delta measure at y.

Proof. The symmetry holds by definition. By explicit computation we have

$$\begin{aligned} D_i\Gamma(x-y) &= \frac{1}{n\omega_n}(x_i-y_i)|x-y|^{-n} \\ D_{ij}\Gamma(x-y) &= \frac{1}{n\omega_n}\left(|x-y|^2\delta_{ij} - n(x_i-y_i)(x_j-y_j)\right)|x-y|^{-n-2}. \end{aligned} \tag{2.3}$$

Estimating $|(x_i - y_i)| \leqslant |x-y|$ leads to (2.1). Similarly one estimates each term on the right-hand side of (2.3) by $(n+1)|x-y|^{-n}$ to arrive at (2.2).

For the second part see Th. 6.20 in [LL01]. □

Definition 2.28. (Newton potential)
Let $\Omega \subset \mathbb{R}^n$ be open and $f \in L^1_{\text{loc}}(\Omega)$ s.t. the function $y \mapsto \Gamma(x-y)f(y)$ is integrable over Ω for a.e. $x \in \Omega$. Then we define for a.e. $x \in \mathbb{R}^n$ the function

$$w(x) = w_f(x) := \int_\Omega \Gamma(x-y)f(y)\mathrm{d}y,$$

called the Newton potential of f.

Lemma 2.29. *(Properties of the Newton potential)*
Let $\Omega \subset \mathbb{R}^n$ be open and w_f the Newton potential of a function f on Ω. Then $w_f \in L^1_{loc}(\Omega)$ and

$$-\Delta w_f = f$$

in distributional sense in Ω. Moreover any solution u of Poisson's equation

$$-\Delta u = f \quad \text{in } \mathcal{D}'(\Omega)$$

can be written as $u = h + w_f$, where h is harmonic in Ω.

Lemma 2.30. *Let $\Omega \subset \mathbb{R}^n$ be open, $V \in L^p(\Omega)$, $p > \frac{n}{2}$ and $w = w_V$ the corresponding Newton potential. Then we have*

$$D_i w \in L^{p'}(\mathbb{R}^n),\ i = 1, \ldots, n$$

where $p' = \frac{np}{n-p}$.

Proof. By (2.1) we can estimate

$$|D_i w(x)| \leqslant C(n) \int_\Omega |x-y|^{1-n}|V(y)|\,\mathrm{d}y = C(n)\int_{\mathbb{R}^n} |x-y|^{1-n}\tilde{V}(y)\,\mathrm{d}y,$$

where $\tilde{V}(y) = |V(y)|\mathbb{1}_\Omega(y)$ and thus $\|\tilde{V}\|_{p;\mathbb{R}^n} = \|V\|_{p;\Omega}$. We now test this equation against an arbitrary function $f \in L^q(\mathbb{R}^n)$ with q given by $\frac{1}{q} = 2 - \frac{1}{p} - \frac{n-1}{n}$ and apply the HLS-inequality (Theorem 2.21) to obtain

$$\int_{\mathbb{R}^n} |f(x)||D_i w(x)| \,\mathrm{dx} \leqslant C(n) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(x)||x-y|^{1-n} \tilde{V}(y) \,\mathrm{dy}\,\mathrm{dx}$$
$$\leqslant C(n,p) \|f\|_q \|\tilde{V}\|_p < \infty.$$

Therefore $D_i w(x) \in L^q(\mathbb{R}^n)^* \cong L^{p'}(\mathbb{R}^n)$ with

$$\frac{1}{p'} = 1 - \frac{1}{q} = \frac{1}{p} - \frac{1}{n} = \frac{n-p}{np}.$$

□

Lemma 2.31. *Let $\Omega \subset \mathbb{R}^n$ be open, $V \in L^p(\Omega)$, $1 < p < \frac{n}{2}$ and $w = w_V$ the corresponding Newton potential. Then*

$$w \in L^{p'}(\mathbb{R}^n),$$

where $p' = \frac{np}{n-2p}$.

Proof. The proof is analogous to the previous proof. Only that we now apply the HLS inequality with $\lambda = \frac{n-2}{n}$ and thus have a different q and p' given by

$$\frac{1}{p'} = 1 - \frac{1}{q} = 1 - 2 + \frac{1}{p} + \frac{n-2}{n} = \frac{n-2p}{np}.$$

□

2.3 Weak solutions

Let $\Omega \subset \mathbb{R}^n$, $n \geqslant 3$ be open, bounded with C^1 boundary. In the following we consider weak partial differential equations of the form

$$\Delta u + b_i D_i u + cu = f + D_i g_i \quad \text{on} \;\; \Omega.$$

The above formulation is to be understood in the following sense:

Definition 2.32. (Weak solutions)
We consider the operator L with $Lu = \Delta u + b_i D_i u + cu$ and assume that the functions $D_i u + b_i u + cu$, $i = 1, \dots, n$ are locally integrable. We further define the corresponding bilinear form $\mathfrak{L}$ by

$$\mathfrak{L}(u,v) = \int_\Omega D_i u D_i v - (b_i D_i u + cu)v. \tag{2.4}$$

We say u is a weak solution of $Lu = 0 :\Leftrightarrow \mathfrak{L}(u,v) = 0 \quad \forall\, v \in C^1_0(\Omega)$.

For given functions f, $g_i \in L^1_{loc}(\Omega)$, $i = 1, \dots, n$ we define the dual element F on $C^1_0(\Omega)$ by

$$F(v) = \int_\Omega fv - g_i D_i v \, \mathrm{d}x.$$

We say u is a weak solution of $Lu = f + D_i g_i :\Leftrightarrow \mathfrak{L}(u, v) = F(v) \quad \forall\, v \in C^1_0(\Omega)$.

Remark 2.33. The formulation (2.4) requires only weak differentiabilty of the solution u. If we have e.g. $u \in H^1(\Omega)$, assuming $\mathbf{b} \in L^2_{\text{loc}}(\Omega)$ and $c \in L^{\frac{2n}{n+2}}$ guarantees the local integrability of $D_i u + b_i u + cu$.

Definition 2.34. (Weak sub-/supersolutions)
A weakly differentiable function u is called weak subsolution of $Lu = f$ if

$$\mathfrak{L}(u, v) - F(v) = \int_\Omega D_i u D_i v - (b_i D_i u + cu - f)v \leqslant 0 \tag{2.5}$$

for all $v \in C^1_0(\Omega)$ with $v \geqslant 0$.

A weakly differentiable function u is called weak supersolution of $Lu = f$ if

$$\mathfrak{L}(u, v) - F(v) = \int_\Omega D_i u D_i v - (b_i D_i u + cu - f)v \geqslant 0 \tag{2.6}$$

for all $v \in C^1_0(\Omega)$ with $v \geqslant 0$.

Remark 2.35. Let $\Omega \subset \mathbb{R}^n$ be open, bounded and $V \in L^p(\Omega)$ with $p = \frac{n}{2}$ then for fixed $u \in H^1(\Omega)$ the mapping

$$\mathfrak{L}(u, \cdot)\colon H^1_0(\Omega) \to \mathbb{R}, \quad v \mapsto \mathfrak{L}(u, v) = \int_\Omega D_i u D_i v - Vuv$$

is a bounded linear functional on $H^1_0(\Omega)$. And thus the validity of the above relations for $v \in C^1_0(\Omega)$ with $v \geqslant 0$ imply their validity for $v \in H^1_0(\Omega)$ with $v \geqslant 0$.

Proof. By Hölder's inequality we have

$$|\mathfrak{L}(u, v)| \leqslant \|Du\|_2 \|Dv\|_2 + \|V\|_p \|uv\|_q,$$

where $q = \frac{n}{n-2}$ is the conjugate Hölder exponent of p. Applying the generalized Hölder inequality to the second term gives

$$|\mathfrak{L}(u, v)| \leqslant \|u\|_{H^1} \|v\|_{H^1} + \|V\|_p \|u\|_{q_1} \|v\|_{q_1}$$

with $q_1 = 2q = \frac{2n}{n-2}$. Applying Sobolev's inequality for H^1 to the second term then shows

$$|\mathfrak{L}(u, v)| \leqslant C(n, |\Omega|, \|V\|_p) \|u\|_{H^1} \|v\|_{H^1}.$$

Therefore the boundedness for $H_0^1(\Omega)$ is readily established by the bounded linear extension principle. □

Remark 2.36. By the above remark, since also $C_0^\infty(\Omega) \subset H_0^1(\Omega)$ dense w.r.t. $\|\cdot\|_{H^1}$, the concepts of distributional solutions and weak solutions naturally coincide. E.g. the solution and the subsequent theory is independent of the test function space used, as long as it is dense in $H_0^1(\Omega)$.

2.4 Hölder spaces

Definition 2.37. (Hölder coefficient)
Let f be a function on a domain Ω and $0 < \alpha \leqslant 1$. Then

$$[f]_{\alpha;\Omega} := \sup_{\substack{x,\, y \in \Omega \\ x \neq y}} \frac{|f(x) - f(y)|}{|x-y|^\alpha}$$

is called the α-Hölder coefficient of f in Ω.

We call f uniformly α-Hölder-continuous in Ω :⇔ $[f]_{\alpha;\Omega}$ is finite.

We call f locally α-Hölder-continuous :⇔ $[f]_{\alpha;K}$ is finite for all $K \Subset \Omega$ compact.

In the case $\alpha = 1$ we will also say that f is Lipschitz continuous.

Definition 2.38. (Hölder space)
Let $\Omega \subset \mathbb{R}^n$ be open and k a non-negative integer. Define the Hölder space as

$$C^{k,\alpha}(\Omega) := \{f \in C^k(\Omega) \colon \text{k-th order partial derivatives are locally } \alpha\text{-Hölder continuous}\}.$$

For simplicity we will write $C^{0,\alpha}(\Omega) =: C^\alpha(\Omega)$ and $C^{k,0}(\Omega) =: C^k(\Omega)$.

Definition 2.39. (Hölder norms)
Let us define the following seminorms on $C^k(\Omega)$ and $C^{k,\alpha}(\Omega)$ respectively

$$[u]_{k,0;\Omega} = |D^k u|_{0;\Omega} := \sup_{|\beta|=k} \sup_{\Omega} |D^\beta u|,$$
$$[u]_{k,\alpha;\Omega} = [D^k u]_{\alpha;\Omega} := \sup_{|\beta|=k} [D^\beta u]_{\alpha;\Omega}.$$

Now we can define the related norms

$$\|u\|_{C^k(\Omega)} = |u|_{k;\Omega} = \sum_{j=0}^k [u]_{j,0;\Omega},$$
$$\|u\|_{C^{k,\alpha}(\Omega)} = |u|_{k,\alpha;\Omega} = |u|_{k;\Omega} + [u]_{k,\alpha;\Omega}.$$

Definition 2.40. (Non-dimensional Hölder norms on bounded sets)
If $\Omega \subset \mathbb{R}^n$ is open and bounded with $d = \operatorname{diam}\, \Omega$ we define

$$\|u\|'_{C^k(\Omega)} = |u|'_{k;\Omega} = \sum_{j=0}^{k} d^j [u]_{j,0;\Omega},$$
$$\|u\|'_{C^{k,\alpha}(\Omega)} = |u|'_{k,\alpha;\Omega} = |u|'_{k;\Omega} + d^{k+\alpha}[u]_{k,\alpha;\Omega}.$$

3 Basic estimates

In this chapter we will provide useful estimates for the subsequent development of interior Hölder regularity.

3.1 Potential estimates

We will start with estimates on the Newton potential. The following estimates follow the idea of Lemma 4.4 in [GT01].

3.1.1 Newton potential estimates

Lemma 3.1. *Let $f \in L^p(B)$, $p > n$, where $B = B_R(x_0)$ and $w = w_f$ the corresponding Newton potential. Then for $\alpha = 1 - \frac{n}{p}$ we have*

$$[Dw]_{\alpha;B} \leqslant C(n, \alpha)\|f\|_{p;B}$$

and

$$|Dw|_{0;B} \leqslant C(n, \alpha)R^{\alpha}\|f\|_{p;B},$$

where C is a constant depending only on n and α.

Proof. Let us first extend f to $\mathbb{R}^n$ by setting $f \equiv 0$ outside of B. We will start with proving the second claim. Therefore pick any $x \in B$ and estimate

$$\begin{aligned} |D_i w(x)| &\leqslant \int_B |D_i \Gamma(x-y)||f(y)| \text{ dy} \\ &\leqslant \frac{1}{n\omega_n} \int_B |x-y|^{1-n} |f(y)| \mathrm{d}y \\ &\leqslant \frac{1}{n\omega_n} \||x-\cdot|^{1-n}\|_{q;B}\|f\|_{p;B} \end{aligned} \tag{3.1}$$

by inequality (2.1) and Hölder's inequality, where $q = \frac{n}{n-1+\alpha}$ is the conjugate Hölder exponent of p.

To estimate the q-norm, we note that $\||x-\cdot|^{1-n}\|_{q;B=B_R(x_0)} \leqslant \||\cdot|^{1-n}\|_{q;B_{2R}(0)}$. An explicit calculation shows:

$$\left(\int\limits_{B_{2R}(0)} |y|^{q(1-n)}\mathrm{d}y\right)^{\frac{1}{q}} \leqslant C(n,\alpha)\left(R^{q(1-n)+n}\right)^{\frac{1}{q}}$$
$$= C(n,\alpha)R^{\alpha}$$

and finishes the proof of the second claim. #

To prove the first claim let x, $\overline{x} \in B$. Then we have

$$D_i w(x) = \int\limits_B D_i\Gamma(x-y)f(y)\mathrm{d}y,$$
$$D_i w(\overline{x}) = \int\limits_B D_i\Gamma(\overline{x}-y)f(y)\mathrm{d}y.$$

We now set $\delta = |x-\overline{x}|$ and $\xi = \frac{1}{2}(x+\overline{x})$. Expanding the difference gives

$$D_i w(\overline{x}) - D_i w(x) = \overbrace{\int\limits_{B_\delta(\xi)} D_i\Gamma(\overline{x}-y)f(y)\mathrm{d}y}^{I_1} + \overbrace{\int\limits_{B_\delta(\xi)} -D_i\Gamma(x-y)f(y)\mathrm{d}y}^{I_2}$$
$$+ \underbrace{\int\limits_{B\setminus B_\delta(\xi)} \left(D_i\Gamma(x-y) - D_i\Gamma(\overline{x}-y)\right)f(y)\mathrm{d}y}_{J}.$$

I_1 and I_2 can be estimated as in (3.1) by

$$|I| \leqslant \frac{1}{n\omega_n}\||x-\cdot|^{1-n}\|_{q;B_\delta(\xi)}\|f\|_{p;B},$$

and further as in the proof of the second claim we estimate

$$\||x-\cdot|^{1-n}\|_{q;B_\delta(\xi)} \leqslant \||\cdot|^{1-n}\|_{q;B_{\frac{3\delta}{2}}(0)} \leqslant C(n,\alpha)\delta^{\alpha}$$

to conclude

$$|I| \leqslant C(n,\alpha)\delta^{\alpha}\|f\|_{p;B}. \tag{3.2}$$

It remains to estimate J. The mean-value theorem on a line gives

$$|J| \leqslant |x-\overline{x}| \int\limits_{B\setminus B_\delta(\xi)} |DD_i\Gamma(\hat{x}-y)||f(y)|\mathrm{d}y$$

for some $\hat{x}$ between x and $\overline{x}$.

By (2.2) and Hölder we obtain

$$\begin{aligned}|J| &\leqslant \delta\frac{n+1}{\omega_n}\int\limits_{B\setminus B_\delta(\xi)}|\hat{x}-y|^{-n}|f(y)|\mathrm{d}y\\ &\leqslant \delta C(n)\||\hat{x}-\cdot|^{-n}\|_{q;B\setminus B_\delta(\xi)}\|f\|_{p;B}.\end{aligned}\tag{3.3}$$

Noting that $|\xi-y|\leqslant 2|\hat{x}-y|$ an explicit calculation shows

$$\begin{aligned}\left(\int\limits_{B\setminus B_\delta(\xi)}|\hat{x}-y|^{-qn}\mathrm{d}y\right)^{\frac{1}{q}} &\leqslant \left(2^{qn}\int\limits_{|y-\xi|\geqslant\delta}|\xi-y|^{-qn}\mathrm{d}y\right)^{\frac{1}{q}}\\ &\leqslant C(n,\alpha)\left(\delta^{-qn+n}\right)^{\frac{1}{q}}\\ &= C(n,\alpha)\delta^{-1+\alpha}.\end{aligned}\tag{3.4}$$

By combining (3.3) and (3.4) we obtain

$$|J|\leqslant C(n,\alpha)\delta^{\alpha}\|f\|_{p;B}$$

and hence together with (3.2) and recalling the definition of δ we conclude

$$\frac{|D_iw(\overline{x})-D_iw(x)|}{|\overline{x}-x|^{\alpha}}\leqslant C(n,\alpha)\|f\|_{p;B}.$$

Taking the supremum over all x, $\bar{x}\in B$ finishes the proof. □

Corollary 3.2. *Let $f\in L^p(B)$, $\frac{n}{2}<p\leqslant n$, where $B=B_R(x_0)$ and $w=w_f$ the corresponding Newton potential. Then for $\alpha=2-\frac{n}{p}$ we have*

$$[w]_{\alpha;B}\leqslant C(n,\alpha)\|f\|_{p;B}$$

and

$$|w|_{0;B}\leqslant C(n,\alpha)R^{\alpha}\|f\|_{p;B},$$

where C is a constant depending only on n and α.

Proof. The proof goes analogously to the previous lemma by replacing D_iw with w and adjusting the estimates accordingly. □

3.1.2 Riesz potential estimates

We now turn our attention to estimates on the Riesz potential which can be viewed as a generalization of the Newton potential. The goal of this section is to prove Theorem 3.8, which will be an important step in the development of the 'General interior Hölder regularity'. We will follow the presentation of Chapter 7 in [GT01] to establish the main steps towards the proof of the theorem.

Definition 3.3. (Riesz potential)
Let $\Omega \subset \mathbb{R}^n$, $n \geqslant 3$ be open, bounded and define for any $\mu \in (0,1]$ the operator V_μ by:

$$V_\mu \colon L^1(\Omega) \to L^1(\Omega),\ f \mapsto (V_\mu f)(x) := \int_\Omega |x-y|^{n(\mu-1)} f(y)\ \mathrm{dy}.$$

Lemma 3.4. *(Properties of V_μ)*
The operator V_μ is a well defined and the the following L^1-bound on $V_\mu f$ holds:

$$\|V_\mu f\|_{1;\Omega} \leqslant \mu^{-1}\omega_n^{1-\mu}|\Omega|^\mu \|f\|_{1;\Omega}.$$

Proof. For the full proof we refer to Lemma 7.12 in [GT01]. We will satisfy ourselves with a plausibility check by scaling. Although $p = 1$ is not an allowed value in the HLS-inequality (Theorem 2.21) we will apply it here with $\lambda = n(1-\mu)$ to check the scaling behavior:

$$\|V_\mu f\|_{1;\Omega} \equiv \int_\Omega \int_\Omega |f(x)||x-y|^{-\lambda}\ 1\ \mathrm{dx}\ \mathrm{dy} \leqslant C(n,\mu)\|f\|_{1;\Omega}\|1\|_{r;\Omega}$$

with $r^{-1} = 2 - 1 - \frac{\lambda}{n} = \mu^{-1}$. □

Lemma 3.5. *Let $\Omega \subset \mathbb{R}^n$, $n \geqslant 3$ be open, bounded and let $n^{-1} < \mu \leqslant 1$. Further assume that f is an integrable function on Ω s.t. there exists a constant K s.t.*

$$\int_{\Omega\cap B_R} |f| \leqslant K R^{n-1} \tag{3.5}$$

for any ball B_R. Then the following pointwise estimate on $V_\mu f$ holds:

$$|V_\mu f(x)| \leqslant K\frac{n-1}{n\mu-1} d^{n\mu-1} \quad a.e.\ in\ \Omega,$$

where $d = diam(\Omega)$.

Proof. We start with extending f to be zero outside of Ω and for any radius $\rho \geqslant 0$ we define

$$v(\rho) := \int_{B_\rho(x)} |f(y)|\ \mathrm{dy}.$$

We now set $\rho := |x-y|$ and obtain

$$|V_\mu f(x)| \leqslant \int_\Omega \rho^{n(\mu-1)}|f(y)|\ \mathrm{dy}.$$

Transforming the dy integral into a radial integral and an integration by parts gives

$$\begin{aligned}\int_\Omega \rho^{n(\mu-1)}|f(y)|\ \mathrm{dy} &= \int_0^d \rho^{n(\mu-1)} v'(\rho)\,\mathrm{d}\rho \\ &= d^{n(\mu-1)}v(d) + n(1-\mu)\int_0^d \rho^{n(\mu-1-1)}v(\rho)\,\mathrm{d}\rho.\end{aligned}$$

By (3.5) we have $v(\rho) \leqslant K\rho^{n-1}$ and can therefore bound the first summand by $Kd^{n\mu-1}$. For the second summand we bound $v(\rho) \leqslant K\rho^{n-1}$ for each ρ and perform the $\mathrm{d}\rho$ integration. We thus have

$$|V_\mu f(x)| \leqslant Kd^{n\mu-1} + K\frac{n-n\mu}{n\mu-1}d^{n\mu-1} = K\frac{n-1}{n\mu-1}d^{n\mu-1}.$$

□

The following lemma will exploit both the pointwise and the L^1-estimate on V_μ to derive a bound on $\int_\Omega \exp(|V_{1/n}f(x)|)\,\mathrm{dx}$.

Lemma 3.6. *Let $\Omega \subset \mathbb{R}^n$, $n \geqslant 3$ be open and bounded. Further assume that f is an integrable function on Ω that satisfies the condition* (3.5) *and set $g := V_{1/n}f$. Then there exist constants c_1, c_2 depending only on n s.t.*

$$\int_\Omega \exp\left(\frac{|g|}{c_1 K}\right)\,\mathrm{dx} \leqslant c_2 d^n$$

with $d = diam(\Omega)$.

Proof. The idea of the proof is to bound the quantity $(q!)^{-1}\int_\Omega |g|^q$ for any $q \geqslant 1$ by some constant $C(q)$ that is summable over q. First for any $q \geqslant 1$ let us write

$$|x-y|^{n-1} = |x-y|^{\left(\frac{1}{nq}-1\right)n\frac{1}{q}}|x-y|^{\left(\frac{1}{nq}+\frac{1}{n}-1\right)n\left(1-\frac{1}{q}\right)}.$$

By Hölder's inequality we thus obtain

$$|g(x)| \leqslant \left(V_{1/(nq)}|f|(x)\right)^{\frac{1}{q}}\left(V_{1/n+1/(nq)}|f|(x)\right)^{1-\frac{1}{q}}.$$

By Lemma 3.5 with $\mu = 1/n + 1/(nq)$ we can bound the second factor pointwise a.e. by

$$(V_{1/n+1/(nq)}|f|)(x) \leqslant Kq(n-1)d^{\frac{1}{q}}$$

and by Lemma 3.4 with $\mu = 1/(nq)$ we can bound the L^1-norm of the first factor by

$$\begin{aligned}\int_\Omega V_{1/(nq)}|f|(x)\,\mathrm{dx} &\leqslant qn\omega_n^{1-1/qn}|\Omega|^{1/qn}\|f\|_{1;\Omega}\\ &\leqslant Kqn\omega_n d^{n-1+\frac{1}{q}},\end{aligned}$$

where we used that $|\Omega| \leqslant \omega_n d^n$ and the property (3.5) in the second line. By combining the two above estimates we thus have

$$\begin{aligned}\int_\Omega |g|^q &\leqslant K^q\omega_n q^q n(n-1)^{q-1}d^n\\ &\leqslant \frac{n\omega_n}{n-1}\left(K(n-1)q\right)^q d^n\end{aligned}$$

and therefore

$$\int_\Omega \sum_{m=0}^{N} \frac{|g|^m}{m!(c_1K)^m}\,\mathrm{dx} \leqslant \frac{n\omega_n}{n-1}d^n \sum_{m=0}^{N}\left(\frac{n-1}{c_1}\right)^m \frac{m^m}{m!}$$
$$\leqslant c_2 d^n$$

if $e(n-1) < c_1$. And finally by letting $N \to \infty$ we complete the proof. □

The next lemma will be the second main ingredient towards the proof of Th. 3.8.

Lemma 3.7. *Let $\Omega \subset \mathbb{R}^n$ be open, bounded and convex. Further assume that $u \in H^1(\Omega)$. Then*

$$|u(x) - u_\Omega| \leqslant \frac{d^n}{n|\Omega|}\int_\Omega |x-y|^{1-n}|Du(y)|\,\mathrm{dy}$$

with $u_\Omega := |\Omega|^{-1}\int_\Omega u\,\mathrm{dx}$ and $d = diam(\Omega)$.

Proof. By the denseness of $C^\infty \cap H^1$ in H^1 it suffices to show the estimate for $u \in C^1(\Omega)$ only. By the fundamental theorem of calculus, on a line between x and y, we may write for x, $y \in \Omega$:

$$u(x) - u(y) = -\int_0^{|x-y|} D_r u(x + r\xi)\,\mathrm{dr}$$

with $\xi = \frac{y-x}{|y-x|}$. After an integration w.r.t. dy we obtain

$$|\Omega|(u(x) - u_\Omega) = -\int_\Omega \mathrm{dy}\int_0^{|x-y|} D_r u(x + r\xi)\,\mathrm{dr}.$$

We now define

$$W(x) := \begin{cases} |D_r u(x)|, & x \in \Omega \\ 0, & x \notin \Omega \end{cases}$$

and thus have

$$|u(x) - u_\Omega| \leqslant \frac{1}{|\Omega|}\int_{|x-y|<d} \mathrm{dy}\int_0^\infty W(x + r\xi)\,\mathrm{dr},$$

where we used that $B_d(x) \supset \Omega$. An explicit calculation in polar coordinates shows

$$\begin{aligned} |u(x) - u_\Omega| &\leqslant \frac{1}{|\Omega|}\int_0^\infty \int_{|\xi|=1}\int_0^d W(x + r\xi)\rho^{n-1}\,\mathrm{d}\rho\,\mathrm{d}\xi\,\mathrm{dr} \\ &= \frac{d^n}{n|\Omega|}\int_0^\infty \int_{|\xi|=1} W(x + r\xi)\,\mathrm{d}\xi\,\mathrm{dr} \\ &= \frac{d^n}{n|\Omega|}\int_\Omega |x-y|^{n-1}|D_r u(y)|\,\mathrm{dy}. \end{aligned}$$

□

With the previous two lemmata we now easily derive the main theorem of this section.

Theorem 3.8. *Let $\Omega \subset \mathbb{R}^n$ be open, bounded and convex. Further assume $u \in H^1(\Omega)$ and that there exists $K > 0$ s.t.*

$$\int_{\Omega \cap B_R} |Du| \, \mathrm{dx} \leqslant K R^{N-1} \quad \text{for all balls } B_R \subset \mathbb{R}^n.$$

Then there exist positive constants σ_0 and C depending only on n s.t.

$$\int_\Omega \exp\left(\frac{\sigma}{K}|u - u_\Omega|\right) \, \mathrm{dx} \leqslant C d^n,$$

where $d = diam(\Omega)$, $\sigma = \sigma_0 |\Omega| d^{-n}$ and $u_\Omega := |\Omega|^{-1} \int_\Omega u \, \mathrm{dx}$.

Proof. By Lemma 3.7 we have

$$\frac{n|\Omega|}{d^n}|u(x) - u_\Omega| \leqslant \int_\Omega |x-y|^{1-n} |Du(y)| \, \mathrm{dy} = V_{1/n}|Du|$$

and thus by the monotonicity of exp and by Lemma 3.6 applied with $g := V_{1/n}|Du|$ we conclude

$$\begin{aligned}\int_\Omega \exp\left(\frac{n|\Omega|}{d^n}|u(x) - u_\Omega| \frac{1}{c_1 K}\right) &\leqslant \int_\Omega \exp\left(\frac{1}{c_1 K} V_{1/n}|Du|\right) \\ &\leqslant c_2 d^n\end{aligned}$$

and therefore the theorem with $\sigma_0 := n c_1^{-1}$. □

3.2 Estimates on harmonic functions

The proof of the following theorem corresponds to Th. 2.10 in [GT01] and Th. 2.2.7 in [Eva10].

Theorem 3.9. *(Bounds on derivatives of harmonic functions)*
Let u be harmonic in Ω and let $\Omega' \Subset \Omega$. Then for any multi-index β we have

$$\sup_{\Omega'} |D^\beta u| \leqslant \left(\frac{n|\beta|}{d}\right)^{|\beta|} \sup_\Omega |u|,$$

where $d = dist(\Omega', \partial\Omega)$.

Proof. Let $B = B_R(y) \Subset \Omega$. Since u is analytic, $D^\alpha u$ is well defined for all multi-indices α. Differentiating $\Delta u = 0$ shows that also $D^\alpha u$ is harmonic. Hence by the mean value property and the divergence theorem we obtain

$$D_i u(y) = \frac{1}{\omega_n R^n} \int_B D_i u \, \mathrm{dx} = \frac{1}{\omega_n R^n} \int_{\partial B} u \nu_i \, \mathrm{ds}$$

and after performing the surface integral we have

$$|D_i u(y)| \leqslant \frac{n}{R} \sup_{\partial B} |u|.$$

To prove the theorem we now iterate this process. Therefore let α be any multi-index with $|\alpha| = k$ and $y \in \Omega'$ with R s.t. $B_R(y) \subset \Omega$. We now consider the ball $B_{R/k}(y)$ and estimate $D^\alpha u(y)$ as above

$$|D^\alpha u(y)| \leqslant \frac{nk}{R} \sup_{\partial B_{R/k}(y)} |D^{\alpha_1} u|,$$

where $|\alpha_1| = k - 1$. Now let $y_1 \in \partial B_{R/k}(y)$ and consider another ball $B_{R/k}(y_1)$ about y_1 to obtain

$$|D^\alpha u(y)| \leqslant \left(\frac{nk}{R}\right)^2 \sup_{\partial B_{R/k}(y_1)} |D^{\alpha_2} u|,$$

where $|\alpha_2| = k - 2$. After k steps we arrive at

$$|D^\alpha u(y)| \leqslant \left(\frac{nk}{R}\right)^k \sup_{\partial B_{R/k}(y_{k-1})} |u|$$

and by the choice of R and k it is $\partial B_{R/k}(y_{k-1}) \subset \Omega$ and thus the claim follows. □

4 Interior Hölder regularity

In this chapter we will exploit the potential estimates to derive regularity properties of weak solutions. We will start with Poisson's equation as the specimen of elliptic equations and then extend the obtained estimates to more general equations. The proofs and theorems are based on Chapter 6 and 8 in [GT01] providing more details than in the book.

4.1 Interior regularity of Possion's equation

Theorem 4.1. *(Interior $C^{1,\alpha}$-regularity of Poisson's equation in balls)*
Let $\Omega \subset \mathbb{R}^n$ be open, bounded, $f \in L^p(\Omega)$, $p > n$ and let u be a distributional solution of Poisson's equation

$$\Delta u = f.$$

Then for any two concentric balls $B_1 := B_R(x_0)$ and $B_2 := B_{2R}(x_0) \subset \Omega$ we have

$$|u|'_{1,\alpha;B_1} \leqslant C(n,\alpha)\left(|u|_{0;B_2} + R^{1+\alpha}\|f\|_{p;B_2}\right),$$

where $\alpha = 1 - \frac{n}{p}$.

Proof. Let $x \in B_2$ and by Lemma 2.29 write u as

$$u(x) = w(x) + h(x),$$

where w is the Newtonian potential of f and h is harmonic. Recalling the definition of the norm and after a triangle inequality we obtain:

$$|u|'_{1,\alpha;B_1} \leqslant |u|_{0;B_1} + R|Dw|_{0;B_1} + R^{1+\alpha}[Dw]_{\alpha;B_1} + R|Dh|'_{0,\alpha;B_1}. \tag{4.1}$$

By Lemma 3.1 we have $[Dw]_{\alpha;B_1} \leqslant C\|f\|_{p;B_1}$ and $|Dw|_{0;B_1} \leqslant CR^{\alpha}\|f\|_{p;B_1}$. Further by definition $|Dh|'_{0,\alpha;B_1} = |Dh|_{0;B_1} + R^{\alpha}[Dh]_{\alpha;B_1}$ and by Theorem 3.9 the first term can be estimated by

$$|Dh|_{0;B_1} \leqslant C(n)R^{-1}|h|_{0;B_2}.$$

Let now $x, y \in B_1$, $x \neq y$ be arbitrary. The harmonicity and the mean-value theorem gives

$$(2R)^{\alpha-1}\frac{|Dh(x)-Dh(y)|}{|x-y|^\alpha} \leqslant \frac{|Dh(x)-Dh(y)|}{|x-y|} \leqslant |D^2h|_{0;B_1}.$$

Since the RHS is independent of x and y, taking the supremum leads to

$$R^{\alpha-1}[Dh]_{\alpha;B_1} \leqslant 2|D^2h|_{0;B_1} \leqslant C(n)R^{-2}|h|_{0;B_2},$$

where in the last step we again used Theorem 3.9. With the above estimates we can therefore write (4.1) as

$$|u|'_{1,\alpha;B_1} \leqslant C\left(|u|_{0;B_1} + R^{1+\alpha}\|f\|_{p;B_1} + |h|_{0;B_2}\right).$$

Finally by rewriting $h = u - w$ we estimate

$$|h|_{0;B_2} \leqslant |u|_{0;B_2} + |w|_{0;B_2} \leqslant |u|_{0;B_2} + C(n,\alpha)R^{1+\alpha}\|f\|_{p;B_2},$$

where we estimated $|w|_{0;B_2}$ by Corollary 3.2. □

Theorem 4.2. *Let $\Omega \subset \mathbb{R}^n$ be open, bounded, $f \in L^p(\Omega)$, $\frac{n}{2} < p \leqslant n$ and let u be a distributional solution of Poisson's equation*

$$\Delta u = f.$$

Then for any two concentric balls $B_1 := B_R(x_0)$ and $B_2 := B_{2R}(x_0) \Subset \Omega$ we have

$$|u|'_{\alpha;B_1} \leqslant C(n,\alpha)\left(|u|_{0;B_2} + R^\alpha\|f\|_{p;B_2}\right),$$

where $\alpha = 2 - \frac{n}{p}$.

Proof. The proof goes analogously to the previous theorem where some estimates even simplify and can therefore be skipped. However we state the proof explicitly:

Again let $x \in B_2$ and write u as

$$u(x) = w(x) + h(x),$$

where w is the Newtonian potential of f and h is harmonic. Recalling the definition of the norm and after a triangle inequality we obtain:

$$|u|'_{\alpha;B_1} \leqslant |u|_{0;B_1} + R^\alpha[w]_{\alpha;B_1} + R^\alpha[h]_{\alpha;B_1}. \tag{4.2}$$

By Corrollary 3.2 we have $[w]_{\alpha;B_1} \leqslant C\|f\|_{p;B_1}$. Let now $x, y \in B_1$, $x \neq y$ be arbitrary. The harmonicity and the mean-value theorem gives

$$(2R)^{\alpha-1}\frac{|h(x)-h(y)|}{|x-y|^{\alpha}} \leqslant \frac{|h(x)-h(y)|}{|x-y|} \leqslant |Dh|_{0;B_1}.$$

Since the RHS is independent of x and y, taking the supremum leads to

$$R^{\alpha-1}[h]_{\alpha;B_1} \leqslant 2|Dh|_{0;B_1} \leqslant C(n)R^{-1}|h|_{0;B_2},$$

where in the last step we again used Theorem 3.9. With the above estimates we can therefore write (4.2) as

$$|u|'_{\alpha;B_1} \leqslant C\left(|u|_{0;B_1} + R^{\alpha}\|f\|_{p;B_1} + |h|_{0;B_2}\right).$$

Finally by rewriting $h = u - w$ we estimate

$$|h|_{0;B_2} \leqslant |u|_{0;B_2} + |w|_{0;B_2} \leqslant |u|_{0;B_2} + C(n,\alpha)R^{\alpha}\|f\|_{p;B_2},$$

where the estimate of $|w|_{0;B_2}$ is given by Corrollary 3.2. This concludes the proof. □

4.2 Interior regularity of weak solutions

In this section we will extend the estimates for Poisson's equation to more general elliptic equations. The proof will make use of the following concept of interpolation of Hölder norms.

Definition 4.3. (Interior non-dimensional Hölder norms)
Let $x, y \in \Omega \subset \mathbb{R}^n$ and set $d_x := \operatorname{dist}(x, \partial\Omega)$ and $d_{x,y} := \min\{d_x, d_y\}$. For $u \in C^k(\Omega)$ and $u \in C^{k,\alpha}(\Omega)$ respectively we define the following interior seminorms:

$$[u]^*_{k,0;\Omega} = [u]^*_{k;\Omega} := \sup_{\substack{x\in\Omega\\|\beta|=k}} d_x^k|D^{\beta}u(x)|,$$
$$[u]^*_{k,\alpha;\Omega} := \sup_{\substack{x,\,y\in D\\|\beta|=k}} d_{x,y}^{k+\alpha}\frac{|D^{\beta}u(x)-D^{\beta}u(y)|}{|x-y|^{\alpha}} \qquad \alpha \in (0,1].$$

And the related norms

$$\|u\|^*_{C^k(\Omega)} = |u|^*_{k;\Omega} = \sum_{j=0}^{k}[u]^*_{j;\Omega},$$
$$\|u\|^*_{C^{k,\alpha}(\Omega)} = |u|^*_{k,\alpha;\Omega} = |u|^*_{k;\Omega} + [u]^*_{k,\alpha;\Omega}.$$

Remark 4.4. If Ω is bounded with $d = \operatorname{diam}(\Omega)$ these interior norms are related to the usual norm by

$$|u|^*_{k,\alpha;\Omega} \leqslant \max\{1, d^{k+\alpha}\}|u|_{k,\alpha;\Omega}$$

and if $\Omega' \Subset \Omega$ with $\delta = \operatorname{dist}(\Omega', \partial\Omega)$ by

$$\min\{1, \delta^{k+\alpha}\}|u|_{k,\alpha;\Omega} \leqslant |u|^*_{k,\alpha;\Omega}.$$

Lemma 4.5. *(Interior Hölder interpolation inequality)*
Let $\Omega \subset \mathbb{R}^n$ be open and bounded. Suppose $j + \beta < k + \alpha$, where j, $k \in \{0, 1\}$ and $0 \leqslant \beta, \alpha \leqslant 1$ and assume $u \in C^{k,\alpha}(\Omega)$.

Then for any $\varepsilon > 0$ there exists $C = C(\varepsilon, j, k)$ s.t.

$$\begin{aligned} [u]^*_{j,\beta;\Omega} &\leqslant C|u|_{0;\Omega} + \varepsilon[u]^*_{k,\alpha;\Omega}, \\ |u|^*_{j,\beta;\Omega} &\leqslant C|u|_{0;\Omega} + \varepsilon[u]^*_{k,\alpha;\Omega}. \end{aligned} \tag{4.3}$$

Proof. For notational convenience we omit the subscript Ω. We will only establish the estimates for the seminorms. The estimates for the norms follow immediately. We distinguish several cases:

i) $j = 0$; $\beta = 0$
This case is trivial with $\varepsilon = 0$.

ii) $j = k = 1$; $\beta = 0$, $\alpha > 0$
Let $x \in \Omega$ be arbitrary and let $0 < \mu \leqslant \frac{1}{2}$ be a positive constant. Further set $d := \mu d_x$ and $B := B_d(x)$. Denote by x', x'' the endpoints of the segment of length $2d$ parallel to the x_i axis with its center at x. By the mean-value theorem we have for some $\bar{x}$ on this segment:

$$|D_i u(\bar{x})| = \frac{|u(x') - u(x'')|}{2d} \leqslant \frac{1}{d}\sup_{y\in B}|u(y)| \leqslant \frac{1}{d}|u|_0$$

and

$$\begin{aligned} |D_i u(x)| &\leqslant |D_i u(\bar{x})| + |D_i u(x) - D_i u(\bar{x})| \\ &\leqslant \frac{1}{d}|u_0| + d^\alpha \sup_{y\in B} d_{x,y}^{-1-\alpha} \sup_{y\in B} d_{x,y}^{1+\alpha}\frac{|D_i u(x) - D_i u(y)|}{|x-y|^\alpha}. \end{aligned}$$

Now since $d_{x,y} > d_x - d = (1-\mu)d_x \geqslant d_x/2$ for all $y \in B$ it follows that

$$[u]^*_1 = \sup_{\substack{x\in\Omega \\ i=1,\dots,n}} d_x|D_i u(x)| \leqslant \mu^{-1}|u|_0 + 2^{1+\alpha}\mu^\alpha[u]^*_{1,\alpha} \tag{4.4}$$

and thus the claim with $4\mu^\alpha \leqslant \varepsilon$ and $C = \mu^{-1}$.

iii) $j = 0$, $k = 1$; $\beta > 0$, $\alpha = 0$
Let x, $y \in \Omega$ be arbitrary with $d_x \leqslant d_y$, i.e. $d_{x,y} = d_x$ and μ, d, B defined as

before. For $y \in B$ and $0 < \beta \leqslant 1$ we obtain by the mean-value theorem

$$d_x^\beta \frac{|u(x) - u(y)|}{|x-y|^\beta} \leqslant \mu^{1-\beta} d_x |Du|_{0;B} \leqslant 2\mu^{1-\beta}[u]_1^*,$$

where we used $d_y > d_x/2$. However for $y \notin B$ we have

$$d_x^\beta \frac{|u(x) - u(y)|}{|x-y|^\beta} \leqslant 2\mu^{-\beta}|u|_0.$$

Combining the inequalities we obtain for $0 < \beta \leqslant 1$

$$[u]_{0,\beta}^* = \sup_{x,\, y \in \Omega} d_{x,y}^\beta \frac{|u(x) - u(y)|}{|x-y|^\beta} \leqslant 2\mu^{-\beta}|u|_0 + 2\mu^{1-\beta}[u]_1^* \tag{4.5}$$

and thus the claim with $2\mu^{1-\beta} \leqslant \varepsilon$. Note that $\beta = 1$ is not a valid choice for the given j and k. But together with *i)* the inequality (4.5) gives the estimate of $[u]_{0,1}^*$ in terms of $[u]_{1,\alpha}^*$ for $\alpha > 0$.

iv) $j \leqslant k;\ \beta > 0,\ \alpha > 0$
If $j = 0,\ k = 1$ we first interpolate $[u]_{0,\beta}^* \leqslant C|u|_0 + \varepsilon[u]_1^*$ by (4.5) and then $[u]_1^* \leqslant C|u|_0 + \varepsilon[u]_{1,\alpha}^*$ by (4.4). It remains to show the case $j = k$ and hence $\alpha > \beta$. In the case $j = k = 0$ with the same notation as above we have for $y \in B$

$$d_x^\beta \frac{|u(x) - u(y)|}{|x-y|^\beta} \leqslant \mu^{\alpha-\beta} d_x^\alpha \frac{|u(x) - u(y)|}{|x-y|^\alpha}$$

and for $y \notin B$

$$d_x^\beta \frac{|u(x) - u(y)|}{|x-y|^\beta} \leqslant 2\mu^{-\beta}|u|_0.$$

By combining these inequalities and taking the supremum over $x,\, y \in \Omega$ with $\varepsilon = \mu^{\alpha-\beta}$ and $C = 2\mu^{-\beta}$ we obtain

$$[u]_{0,\beta}^* \leqslant C|u|_0 + \varepsilon[u]_{0,\alpha}^*.$$

In the case $j = k = 1$ we have for $y \in B$

$$d_x^{\beta+1} \frac{|D_i u(x) - D_i u(y)|}{|x-y|^\beta} \leqslant \mu^{\alpha-\beta} d_x^{\alpha+1} \frac{|D_i u(x) - D_i u(y)|}{|x-y|^\alpha}$$

and for $y \notin B$

$$\begin{aligned} d_x^{\beta+1} \frac{|D_i u(x) - D_i u(y)|}{|x-y|^\beta} &\leqslant \mu^{-\beta} \left(d_x |D_i u(x)| + d_y |D_i u(y)| \right) \\ &\leqslant 2\mu^{-\beta}[u]_1^*. \end{aligned}$$

Together with (4.4) we can estimate the RHS to arrive at

$$d_x^{\beta+1} \frac{|D_i u(x) - D_i u(y)|}{|x-y|^\beta} \leqslant 2\mu^{-\beta-1}[u]_0 + 8\mu^{\alpha-\beta}[u]_{1,\alpha}^*.$$

Combining the inequalities for $y \in B$ and $y \notin B$ and taking the supremum over $x, y \in \Omega$ with $\varepsilon = 9\mu^{\alpha-\beta}$ completes the proof.

□

We are now ready to approach the two main theorems of this section. The first theorem assumes higher L^p-regularity of the coefficients and inhomogeneities ($p > n$) and shows Hölder regularity up to the first derivative of the solution. The second theorem assumes less L^p-regularity of the coefficients and inhomogeneities ($\frac{n}{2} < p \leqslant n$) and only shows Hölder regularity of the solution for equations with no divergence term ($\mathbf{b} \equiv 0$).

Theorem 4.6. *($C^{1,\alpha}$-regularity of weak solutions)*
Let $\Omega \subset \mathbb{R}^n$ be open, bounded and u be a bounded weak solution of

$$Lu \equiv \Delta u + b_i(x)D_i u + c(x)u = f, \tag{4.6}$$

where $\mathbf{b}$, c, $f \in L^p(\Omega)$, $p > n$.

Then $u \in C^{1,\alpha}(\Omega')$ with $0 < \alpha \leqslant 1 - \frac{n}{p}$ for any subdomain $\Omega' \Subset \Omega$ and further we have

$$|u|_{1,\alpha;\Omega'} \leqslant C(|u|_{0;\Omega} + \|f\|_{p;\Omega}), \tag{4.7}$$

where $C = C(n, p, \|\mathbf{b}\|_{p;\Omega}, \|c\|_{p;\Omega}, \delta, \phi)$ with $\delta = \operatorname{dist}(\Omega', \partial\Omega)$ and $\phi = diam(\Omega)$.

Proof. By Remark 4.4 it suffices to show the above estimate for the norm $|u|^*_{1,\alpha;\Omega}$ and by the interpolation inequality it suffices to only estimate $[u]^*_{1,\alpha;\Omega}$. Thus it remains to show that

$$[u]^*_{1,\alpha;\Omega} \leqslant C(|u|_{0;\Omega} + \|f\|_{p;\Omega}).$$

Therefore let x_0, $y_0 \in \Omega$ with $x_0 \neq y_0$ and suppose $d_{x_0} = d_{x_0,y_0} = \min\{d_{x_0}, d_{y_0}\}$. Let further $0 < \mu \leqslant \frac{1}{2}$ be a positive constant and define $d = \mu d_{x_0}$. In the following we will consider a ball $B = B_d(x_0)$ of radius d about x_0. Before we start we rewrite (4.6) in the form

$$\Delta u = -b_i D_i u - cu + f \equiv F$$

and consider it as an equation in B. If $y_0 \in B_{d/2}(x_0)$ we have by Theorem 4.1 with $R \equiv \frac{d}{2}$

$$\left(\frac{d}{2}\right)^{1+\alpha} \frac{|Du(x_0) - Du(y_0)|}{|x_0 - y_0|^\alpha} \leqslant C\left(|u|_{0;B} + (d/2)^{1+\alpha}\|F\|_{p;B}\right),$$

where Du denotes any first derivative. By the definition of d this gives

$$d_{x_0}^{1+\alpha} \frac{|Du(x_0) - Du(y_0)|}{|x_0 - y_0|^\alpha} \leqslant \frac{C}{\mu^{1+\alpha}}|u|_{0;B} + d_{x_0}^{1+\alpha}\|F\|_{p;B}.$$

On the other hand if $y_0 \notin B_{d/2}(x_0)$ we have

$$d_{x_0}^{1+\alpha}\frac{|Du(x_0)-Du(y_0)|}{|x_0-y_0|^\alpha} \leqslant \left(\frac{2}{\mu}\right)^\alpha (d_{x_0}|Du(x_0)| + d_{y_0}|Du(y_0)|) \leqslant 4\mu^{-\alpha}[u]^*_{1;\Omega}.$$

By combining the two inequalities we obtain

$$d_{x_0}^{1+\alpha}\frac{|Du(x_0)-Du(y_0)|}{|x_0-y_0|^\alpha} \leqslant \frac{C}{\mu^{1+\alpha}}|u|_{0;B} + d_{x_0}^{1+\alpha}\|F\|_{p;B} + 4\mu^{-\alpha}[u]^*_{1;\Omega}. \tag{4.8}$$

We now want to estimate the RHS in terms of $|u|_{0;\Omega}$, $\|f\|_{p;\Omega}$ and $\varepsilon[u]^*_{1,\alpha;\Omega}$ with a sufficiently small ε so that we can bring the terms containing $[u]^*_{1,\alpha;\Omega}$ to the LHS. By the interpolation inequality (4.3) applied with $\varepsilon = \mu^{2\alpha}/4$ the third summand can be bounded by

$$4\mu^{-\alpha}[u]^*_{1;\Omega} \leqslant C(\mu)|u|_{0;\Omega} + \mu^\alpha[u]^*_{1,\alpha;\Omega}.$$

For the second term we start with a triangle inequality

$$\|F\|_{p;B} \leqslant \|c\|_{p;\Omega}|u|_{0;\Omega} + \|f\|_{p;\Omega} + \|\mathbf{b}\|_{p;\Omega}|Du|_{0;B} \tag{4.9}$$

and as in the proof of the interpolation inequality we estimate

$$\begin{aligned}|Du|_{0;B} &= \sup_{x\in B}|Du(x)| \\ &\leqslant \sup_{x\in B} d_x^{-1} \sup_{x\in B} d_x|Du(x)| \\ &\leqslant 2d_{x_0}^{-1}[u]^*_{1;\Omega},\end{aligned}$$

where in the last line we used $d_x \geqslant d_{x_0} - d = (1-\mu)d_{x_0} \geqslant 1/2 d_{x_0}$. After collecting the constants we obtain

$$d_{x_0}^{1+\alpha}\|F\|_{p;B} \leqslant C(d_{x_0}^{1+\alpha}|u|_{0;\Omega} + d_{x_0}^{1+\alpha}\|f\|_{p;B} + d_{x_0}^{\alpha}[u]^*_{1;\Omega})$$

with $C = C(n, p, \|\mathbf{b}\|_{p;\Omega}, \|c\|_{p;\Omega})$ and since $d_{x_0} < \phi \equiv \operatorname{diam}(\Omega)$ the expression can be simplified to

$$d_{x_0}^{1+\alpha}\|F\|_{p;B} \leqslant C(\mu)(|u|_{0,\Omega} + \|f\|_{p;B}) + C\mu^\alpha[u]^*_{1,\alpha;\Omega},$$

where we again made use of the interpolation inequality (4.3) with a sufficiently small ε. Combining these results we can now estimate the LHS of (4.8) by

$$d_{x_0}^{1+\alpha}\frac{|Du(x_0)-Du(y_0)|}{|x_0-y_0|^\alpha} \leqslant C(\mu)(|u|_{0;\Omega} + \|f\|_{p;B}) + C\mu^\alpha[u]^*_{1,\alpha;\Omega}.$$

Since the RHS of this inequality is independent of x_0, y_0 taking the supremum over all x_0, $y_0 \in \Omega$ gives

$$[u]^*_{1,\alpha;\Omega} \leqslant C(\mu)(|u|_{0;\Omega} + \|f\|_{p;B}) + C\mu^\alpha [u]^*_{1,\alpha;\Omega}$$

and finally by choosing and fixing $\mu = \mu_0$ such that $C\mu_0^\alpha \leqslant 1/2$ we arrive at the desired estimate. □

Theorem 4.7. *(C^α-regularity of weak solutions)*
Let $\Omega \subset \mathbb{R}^n$ be open, bounded and u be a bounded weak solution of

$$Lu \equiv \Delta u + c(x)u = f, \tag{4.10}$$

where c, $f \in L^p(\Omega)$, $\frac{n}{2} < p \leqslant n$.

Then $u \in C^\alpha(\Omega')$ with $0 < \alpha \leqslant 2 - \frac{n}{p}$ for any subdomain $\Omega' \Subset \Omega$ and further we have

$$|u|_{\alpha;\Omega'} \leqslant C(|u|_{0;\Omega} + \|f\|_{p;\Omega}),$$

where $C = C(n, p, \|c\|_{p;\Omega}, \delta, \phi)$ with $\delta = \operatorname{dist}(\Omega', \partial\Omega)$ and $\phi = diam(\Omega)$.

Proof. Analogously to the previous theorem it suffices to only estimate $[u]^*_{\alpha;\Omega}$. With the same notation as above we consider

$$\Delta u = -cu + f \equiv F$$

as an equation in B. By distinguishing the cases of $y_0 \in B_{d/2}(x_0)$ and $y_0 \notin B_{d/2}(x_0)$ and using Corollary 3.2 instead of Theorem 4.1, we obtain the analogous estimate to (4.8)

$$d^\alpha_{x_0} \frac{|Du(x_0) - Du(y_0)|}{|x_0 - y_0|^\alpha} \leqslant \frac{C}{\mu^\alpha}|u|_{0;B} + d^\alpha_{x_0}\|F\|_{p;B}.$$

However the estimate of the RHS is now straight forward

$$\|F\|_{p;B} \leqslant |u|_{0;B}\|c\|_{p;B} + \|f\|_{p;B} \leqslant C(|u|_{0;\Omega} + \|f\|_{p;\Omega})$$

and after fixing $\mu = \mu_0$ und using $d_{x_0} \leqslant \phi \equiv \operatorname{diam}(\Omega)$ we conclude

$$[u]^*_{\alpha;\Omega} \leqslant C(|u|_{0;\Omega} + \|f\|_{p;\Omega}).$$

□

Remark 4.8. If $\frac{n}{2} < p \leqslant n$ the case of

$$Lu \equiv \Delta u + b_i D_i u + cu = f$$

with $0 \neq \mathbf{b}$, c, $f \in L^p(\Omega)$ cannot be proven with the same method since we cannot interpolate $D_i u$ analogously to Theorem 4.6.

4.3 General interior regularity of weak solutions

In this section we will introduce the method of iteration of p-norms to develop an estimate similar to Theorem 4.7 for elliptic equations *with* divergence term.

Theorem 4.9. *(Weak Harnack inequality for supersolutions)*
Let $\Omega \subset \mathbb{R}^n$, $n \geqslant 3$ be open, bounded and $u \in H^1(\Omega)$ be a weak supersolution of

$$Lu \equiv \Delta u + b_i D_i u + cu = g \quad \text{in } \Omega,$$

where $\mathbf{b} \in L^{2p}(\Omega)$, c, $g \in L^p(\Omega)$ with $p > n/2$. Assume further that $u \geqslant 0$ in a ball $B_{4R}(y) \subset \Omega$ then

$$R^{-n} \|u\|_{1;B_{2R}(y)} \leqslant C \left(\inf_{B_R(y)} u + R^{2-\frac{n}{p}} \|g\|_{p;\Omega} \right),$$

where $C = C(n, p, M_R)$ with $M_R := R^{2-\frac{n}{p}} \left(\|\mathbf{b}\|_{2p}^2 + \|c\|_p \right)$.

Proof. Let us define

$$k = k(R) = R^{2-\frac{n}{p}} \|g\|_{p;\Omega}$$

and abbreviate $B_R := B_R(y)$. We first prove the claim for $R = 1$ and $k > 0$. The final result will be concluded by a coordinate transformation. The case when $g \equiv 0$ will follow from letting $k \to 0$. If not indicated otherwise, all norms taken are to be understood over entire Ω. Let us further define

$$B : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}, \quad B(x, v, p) := b_i(x) p_i + c(x) v - g(x).$$

With this notation, u being a weak supersolution reads

$$\int_\Omega D_i v D_i u - v B(x, u, Du) \, \mathrm{dx} \geqslant 0 \qquad \text{for all } \ 0 \leqslant v \in C_0^1(\Omega). \tag{4.11}$$

Roughly the proof will be established in the following way:

We first choose a test function v depending on u. With the help of the PDE we morally obtain an upper bound on $\|D\left(u^{-\beta}\right)\|_2$, $\beta > 0$, which enables us to exploit Sobolev's inequality to obtain estimates on higher L^p-norms in terms of lower L^p-norms of $u^{-\beta}$. This result will then be iterated for nested balls and the conclusion will be made with Proposition 2.8 v.

Let us start with the test function v. Therefore let $0 \leqslant \eta \in C_0^1(B_4)$, $\beta \in \mathbb{R}^+$ and $\bar{u} = u + k$ and define

$$v = \eta^2 \bar{u}^{-\beta}.$$

Hereby v defines a proper test function in B_4. By definition it clearly vanishes on the boundary of B_4 (and can be extended by 0 to Ω). Taking formally the weak derivative of v gives

$$Dv = 2\eta D\eta \bar{u}^{-\beta} + \beta\eta^2\bar{u}^{\beta-1}Du.$$

Since $\bar{u} \geqslant k > 0$ is bounded away from zero the first summand is even bounded. Similarly for the second summand the prefactor of Du is bounded and $Du \in L^2(B_4)$ by assumption. Thus $Dv \in L^2(B_4)$ and by Poincare $v \in H_0^1(B_4)$. This also justifies the validity of taking the weak derivative above.

Plugging v into (4.11) gives

$$-\beta\int_\Omega \eta^2\bar{u}^{-\beta-1}D_iuD_iu + 2\int_\Omega \eta\bar{u}^{-\beta}D_i\eta D_iu - \int_\Omega \eta^2\bar{u}^{-\beta}B(x,u,Du) \geqslant 0. \quad (4.12)$$

For any $0 < \varepsilon \leqslant \frac{1}{2}$ by Cauchy's inequality and by writing $\bar{u}^{-\beta} = \sqrt{\bar{u}^{-\beta+1}\bar{u}^{-\beta-1}}$ the second summand can be bounded (pointwise a.e.) by

$$\left|\eta^2\bar{u}^{-\beta}D\eta Du\right| \leqslant \frac{\varepsilon}{2}\bar{u}^{-\beta-1}\eta^2|Du|^2 + \frac{1}{2\varepsilon}|D\eta|^2\bar{u}^{-\beta+1}.$$

The third summand can be bounded as follows

$$\begin{aligned}\left|\eta^2\bar{u}^{-\beta}(b_iD_iu + cu - g)\right| &\leqslant \eta^2\bar{u}^{-\beta+1}(|c| + k^{-1}|g|) + \eta^2\bar{u}^{-\beta}|b||Du| \\ &\leqslant \eta^2\bar{u}^{-\beta+1}(|c| + k^{-1}|g|) + \eta^2(\frac{\varepsilon}{2}\bar{u}^{-\beta-1}|Du|^2 + \frac{1}{2\varepsilon}\bar{u}^{-\beta+1}|b|^2) \\ &\leqslant \frac{\varepsilon}{2}\bar{u}^{-\beta-1}\eta^2|Du|^2 + \frac{1}{2\varepsilon}\bar{b}\eta^2\bar{u}^{-\beta+1},\end{aligned}$$

where $\bar{b} := |b|^2 + |c| + k^{-1}|g|$. This particular form of $\bar{b}$ guarantees that $\|\bar{b}\|_p$ is independent of g and will eventually single out the dependence of $\|g\|_p$ from the constant C in the claim. Using the above estimates in (4.12) gives

$$\begin{aligned}\int_\Omega \eta^2\bar{u}^{-\beta-1}|Du|^2 &\leqslant \frac{1}{\beta}\int_\Omega |\eta^2\bar{u}^{-\beta}B(x,u,Du)| + \frac{2}{\beta}\int_\Omega |\eta^2\bar{u}^{-\beta}D\eta Du| \\ &\leqslant \frac{3\varepsilon}{2\beta}\int_\Omega \eta^2\bar{u}^{-\beta-1}|Du|^2 + \frac{1}{\varepsilon\beta}\int_\Omega(\bar{b}\eta^2 + |D\eta|^2)\bar{u}^{\beta+1}\end{aligned}$$

and then by choosing $\varepsilon = \min\{\frac{1}{2}, \frac{\beta}{3}\}$ we obtain

$$\int_\Omega \eta^2\bar{u}^{-\beta-1}|Du|^2 \leqslant C(\beta)\int_\Omega(\bar{b}\eta^2 + |D\eta|^2)\bar{u}^{\beta+1}, \quad (4.13)$$

where $C(\beta) = 2\max\{2/\beta, 3/\beta^2\}$. For notational convenience we introduce the function w

$$w := \begin{cases}\bar{u}^{\frac{-\beta+1}{2}} & \text{for } \beta \neq 1,\\ \log\bar{u} & \text{for } \beta = 1,\end{cases} \quad \text{i.e. } Dw = \begin{cases}\frac{-\beta+1}{2}\bar{u}^{\frac{-\beta-1}{2}}Du & \text{for } \beta \neq 1,\\ \bar{u}^{-1}Du & \text{for } \beta = 1,\end{cases}$$

and set $\gamma := -\beta + 1$.

With the new notation (4.13) reads

$$\int_\Omega |\eta Dw|^2 \leqslant \begin{cases} \frac{C(\beta)}{4}\gamma^2 \int_\Omega \left(\bar{b}\eta^2 + |D\eta|^2\right) w^2, & \beta \neq 1 \\ 6\int_\Omega \left(\bar{b}\eta^2 + |D\eta|^2\right), & \beta = 1. \end{cases} \tag{4.14}$$

We now apply Sobolev's inequality to ηw to obtain

$$\|\eta w\|^2_{\frac{2n}{n-2}} \leqslant C_{\text{Sob}}(n)\int_\Omega |D(\eta w)|^2 \leqslant 2C_{\text{Sob}}(n)\int_\Omega |\eta Dw|^2 + |wD\eta|^2. \tag{4.15}$$

Now by using first Hölder's and then the interpolation inequality we obtain for any $\varepsilon > 0$

$$\begin{aligned} \int_\Omega \bar{b}(\eta w)^2 &\leqslant \|\bar{b}\|_p \|\eta w\|^2_{\frac{p}{2p-2}} \\ &\leqslant \|\bar{b}\|_p \left(\varepsilon\|\eta w\|_{\frac{2n}{n-2}} + \varepsilon^{-\sigma}\|\eta w\|_2\right)^2, \quad \sigma = \frac{n}{2p-n}. \end{aligned}$$

With the help of the above estimates we can write (4.15) as

$$\|\eta w\|^2_{\frac{2n}{n-2}} \leqslant C(n,\beta)\gamma^2\|\bar{b}\|_p(\varepsilon\|\eta w\|_{\frac{2n}{n-2}} + \varepsilon^{-\sigma}\|\eta w\|_2)^2 + C(n,\beta)(1+\gamma)^2\int_\Omega |wD\eta|^2.$$

Choosing $\varepsilon = 1/(2C(n,\beta)\gamma^2\|\bar{b}\|_p)$ and thus $\gamma^2\varepsilon^{-\sigma} = \gamma^{2\sigma+2}(2C(n,\beta)\|\bar{b}\|_p)^\sigma$ leads to

$$\|\eta w\|_{\frac{2n}{n-2}} \leqslant C(n,\beta,\|\bar{b}\|_p)(1+|\gamma|)^{\sigma+1}\|(\eta + |D\eta|)w\|_2.$$

Let us now specify the cutoff $\eta \in C^0_1(B_4) \subset C^1_0(\Omega)$ precisely. Let $1 \leqslant R_1 < R_2 \leqslant 3$ s.t. $\eta \equiv 1$ in B_{R_1} and $\eta \equiv 0$ in $\Omega \backslash B_{R_2}$ with $|D\eta| \leqslant 2/(R_2 - R_1)$. By setting $\chi := \frac{n}{n-2}$ the previous estimate now reads

$$\begin{aligned} \|w\|_{2\chi;R_1} &\leqslant C(1+|\gamma|)^{1+\sigma}\|(\eta + D\eta)w\|_2 \\ &\leqslant C(1+|\gamma|)^{1+\sigma}\left(1 + 2/(R_2 - R_1)\right)\|w\|_{2;R_2}. \end{aligned}$$

And since $2/(R_2 - R_1) > 1$ we obtain

$$\|w\|_{2\chi;R_1} \leqslant \frac{C(1+|\gamma|)^{1+\sigma}}{R_2 - R_1}\|w\|_{2;R_2}, \quad C = C(n,\beta,\|\bar{b}\|_p)$$

which will be the key estimate to start the iteration of p-norms. Recalling the definition of $w^2 = \bar{u}^\gamma$ for $\gamma \neq 0$ we obtain

$$\left(\int_{B_{R_1}} \bar{u}^{\gamma\chi}\right)^{\frac{1}{2\chi}} \leqslant \frac{C(1+|\gamma|)^{1+\sigma}}{R_2 - R_1}\left(\int_{B_{R_2}} \bar{u}^\gamma\right)^{\frac{1}{2}}.$$

And by raising this equation to the power $2\gamma^{-1}$ we get

$$\|\bar{u}\|_{\chi\gamma;R_1} \leqslant \left(\frac{C(1+|\gamma|)^{1+\sigma}}{R_2-R_1}\right)^{\frac{2}{|\gamma|}} \|\bar{u}\|_{\gamma;R_2} \qquad \text{if } \gamma > 0, \tag{4.16}$$

$$\|\bar{u}\|_{\gamma;R_2} \leqslant \left(\frac{C(1+|\gamma|)^{1+\sigma}}{R_2-R_1}\right)^{\frac{2}{|\gamma|}} \|\bar{u}\|_{\chi\gamma;R_1} \qquad \text{if } \gamma < 0. \tag{4.17}$$

Our next goal is to use both estimates to prove the following three inequalities that essentially yield the desired result. Therefore let $0 < p_0 < 1 < \chi = \frac{n}{n-2}$ then

(a) $\|\bar{u}\|_{1;2} \leqslant C\|\bar{u}\|_{p_0;3}$,

(b) $\|\bar{u}\|_{p_0;3} \leqslant C\|\bar{u}\|_{-p_0;3}$,

(c) $\|\bar{u}\|_{-p_0;3} \leqslant C\|\bar{u}\|_{-\infty;1} = C\inf_{B_1} \bar{u}$.

The proof of (b) will follow from Theorem 3.8. For (a), since $\chi > 1$ there exists $M \in \mathbb{N}$ s.t. $\chi^M p_0 > 1$ and thus by Proposition 2.8. iii.

$$\|\bar{u}\|_{1;2} \leqslant |B_2|^{1-\chi^{-M}p_0^{-1}} \|\bar{u}\|_{\chi^M p_0;2}.$$

We now set $\gamma_m := \chi^m p_0$ and $r_m := 2 + 2^{-m}$, $m = 1, \ldots, M$ in (4.16) to obtain

$$\begin{aligned}\|\bar{u}\|_{\chi^M p_0;2} \leqslant \|\bar{u}\|_{\chi^M p_0;r_M} &\leqslant \left(\frac{C(1+\chi^M p_0)^{1+\sigma}}{2^{-M}}\right)^{2\chi^{-M}p_0^{-1}} \|\bar{u}\|_{\chi^{M-1}p_0;r_{M-1}} \\ &\leqslant \ldots \leqslant C(n, \|\bar{b}\|_p, p_0, p)\|\bar{u}\|_{p_0;3},\end{aligned}$$

where we also made use of Proposition 2.8 ii. Since we only used a finite number of steps the constant C is clearly finite. Combining the two estimates proves (a). #

For (c) we have to investigate the above constant more carefully and extend the range of R_1 slightly to $R_1 \in (1-\varepsilon, R_2)$ for a fixed $\varepsilon << 1$. We again set $\gamma_m := \chi^m p_0$ but now $r_0 = 3$, $r_m := 1 - \varepsilon + 2^{-m}$, $m = 1, 2, \ldots$ to obtain by (4.17)

$$\|\bar{u}\|_{-p_0;3} \equiv \|\bar{u}\|_{-\gamma_0;r_0} \leqslant \left(\frac{C(1+\gamma_0)^{1+\sigma}}{1+1/2+\varepsilon}\right)^{2\gamma_0^{-1}} \|\bar{u}\|_{-\gamma_1;r_1} \equiv C_0\|\bar{u}\|_{-\gamma_1;r_1}.$$

So that for any $N \in \mathbb{N}$ large enough s.t. $r_N < 1$ we have by iterating the above estimate and by Proposition 2.8 ii.

$$\|\bar{u}\|_{-p_0;3} \leqslant C_N\|\bar{u}\|_{-\gamma_N;r_N} \leqslant C_N\|\bar{u}\|_{-\gamma_N;1}. \tag{4.18}$$

We now would like to take $N \to \infty$ and therefore have to make sure that the constant C_N is uniformly bounded w.r.t. N. After the N-th iteration C_N amounts to

$$C_N = C_0 \prod_{m=1}^{N} \left(\frac{C(1+\gamma_m)^{1+\sigma}}{2^{-m+1}}\right)^{2\gamma_m^{-1}}.$$

Apart from the first M factors we have $\gamma_m > 1$ and thus

$$C_N \leqslant C_M \prod_{m=1}^{N} \left(2^{m+\sigma} C \gamma_m^{1+\sigma}\right)^{2\gamma_m^{-1}}$$
$$\leqslant C_M 2^{2p_0^{-1}(\sum m\chi^{-m} + \sigma \sum \chi^{-m})} C^{2p_0^{-1}\sum \chi^{-m}} p_0^{2p_0^{-1}(1+\sigma)\sum \chi^{-m}} \chi^{2p_0^{-1}(1+\sigma)\sum m\chi^{-m}}.$$

And since the right sum

$$\sum_{m=1}^{N} m\chi^{-m} < \sum_{m=1}^{\infty} m\chi^{-m} < \infty$$

is finite, the constant $C_N = C(n, \|\bar{b}\|_p, p_0, p)$ is independent of N. Now by taking $N \to \infty$ in (4.18) and by using Proposition 2.8 v. we obtain (c). #

To establish (b) we recall the estimate (4.14) for $\beta = 1$. Let therefore $B_{2R_0} \subset B_4$ and choose $\eta \in C_0^1(\Omega)$ s.t. $\eta \equiv 0$ in $\Omega \backslash B_{2R_0}$ and $\eta \equiv 1$ in B_{R_0} with $|D\eta| \leqslant 2{R_0}^{-1}$. From Hölder's inequality we obtain

$$\|Dw\|_{1;R_0} \leqslant \|1\|_{2;R_0} \|Dw\|_{2;R_0} = C(n) {R_0}^{\frac{n}{2}} \|Dw\|_{2;R_0},$$

where $C(n) = \sqrt{\omega_n}$. We now use (4.14) to estimate

$$\|Dw\|_{2;R_0}^2 \leqslant \|\eta Dw\|_{2;2R_0}^2 \leqslant C \int_\Omega \bar{b}\eta^2 + |D\eta|^2 \,\mathrm{dx}$$
$$\leqslant C\|\bar{b}\|_p \|\eta\|_{\frac{p}{2p-2}}^2 + \mathrm{vol}(B_{2R_0}) \frac{4}{{R_0}^2} \leqslant C\|\bar{b}\|_p \|1\|_{\frac{2p}{2p-2};2R_0} + C(n){R_0}^{n-2}$$
$$\leqslant C(\|\bar{b}\|_p, n){R_0}^{n-2}({R_0}^{2-\frac{n}{p}} + 1).$$

And since $\frac{n}{p}$, $R_0 < 2$ and thus ${R_0}^{2-\frac{n}{p}} \leqslant 4$ we obtain together with the previous estimate

$$\int_{B_{R_0}} |Dw| \leqslant C(\|\bar{b}\|_p, n){R_0}^{n-1}.$$

Hence by Theorem 3.8 applied with $\Omega := B_3$ convex it exists $p_0 > 0$ s.t. for $w_0 := |B_3|^{-1}\|w\|_{1;3}$ we have

$$\int_{B_3} e^{p_0|w-w_0|} \,\mathrm{dx} \leqslant C(\|\bar{b}\|_p, n)$$

and thus by using the simple inequality $e^x \leqslant e^{|x|}$ we obtain

$$\int_{B_3} e^{p_0 w} = e^{p_0 w_0} \int_{B_3} e^{p_0(w-w_0)} \leqslant Ce^{p_0 w_0},$$
$$\int_{B_3} e^{-p_0 w} = e^{-p_0 w_0} \int_{B_3} e^{-p_0(w-w_0)} \leqslant Ce^{-p_0 w_0}.$$

Multiplying the two inequalities then gives

$$\int_{B_3} e^{p_0 w} \int_{B_3} e^{-p_0 w} \leqslant C e^{p_0 w_0} e^{-p_0 w_0} = C$$

and by recalling the definition of $w = \log \bar{u}$ we conclude

$$\left(\int_{B_3} \bar{u}^{p_0} \right)^{\frac{1}{p_0}} \leqslant C(\|\bar{b}\|_p, n) \left(\int_{B_3} \bar{u}^{-p_0} \right)^{-\frac{1}{p_0}}$$

and thus prove (b). #

Now by combining (a), (b) and (c) we obtain

$$\int_{B_2} \bar{u} \text{ dx} \leqslant C \inf_{B_1} \bar{u} \tag{4.19}$$

and therefore the result for $R = 1$ and $k(1) \equiv \|g\|_p > 0$

$$\int_{B_2} u \text{ dx} + \text{vol}(B_2) k \leqslant C(\inf_{B_1} u + k),$$

after possibly increasing $C = C(\|\bar{b}\|_p, n, p)$ s.t. $C > \text{vol}(B_2)$. We now specify the constant by using Jensen's inequality

$$\|\bar{b}\|_p = \|\mathbf{b}^2 + c + k^{-1}(1) g\|_p \leqslant \|\mathbf{b}\|_{2p}^2 + \|c\|_p + \|g\|_p^{-1} \|g\|_p$$

and thus have $C = C(n, p, \|\mathbf{b}\|_{2p}^2 + \|c\|_p)$.

For $k(R) = R^{2-\frac{n}{p}} \|g\|_{p;\Omega} > 0$ and generic R, we reduce the proof to the previous result. Therefore for a given function $\tilde{u}$ fulfilling

$$\Delta_x \tilde{u} + (\tilde{b}_i) D_{x_i} \tilde{u} + \tilde{c} \tilde{u} = \tilde{g}, \quad x \in B_{4R}$$

in B_{4R} we define

$$u(y) := \tilde{u}(x), \quad y = R^{-1} x \in B_4.$$

The equation in the new variable y transforms by the chain rule to

$$R^{-2} (\Delta u)(y) + \tilde{b}_i(x) R^{-1} (D_i u)(y) + \tilde{c}(x) u(y) = \tilde{g}(y)$$

and hence by further defining

$$b_i(y) := R \tilde{b}_i(x), \quad c(y) := R^2 \tilde{c}(x), \quad g(y) := R^2 \tilde{g}(x)$$

we are able to apply the previous result to u, b, c, g and after an easy computation arrive at the desired result. The case $k(R) = 0$ can be obtained by taking the limit $k \to 0$ in the above proof. However this particular case will not be of interest in the following. □

Remark 4.10. The ratio between the balls of radii $4R$, $2R$ and R in the assumption of the previous theorem is due to the choice of the radii r_m in the iteration procedure in (a) and (c). Thus the same result, with the price of a larger constant, can be obtained for any two concentric nested balls with decreasing radii inside a given ball.

In order to prove the main theorem of this section we will need the following:

Lemma 4.11. *Fix $0 < \gamma, \tau < 1$ and let $\omega\colon (0, R_0] \to \mathbb{R}$ be increasing (i.e. $\omega(x) \leqslant \omega(y)$ for all $x < y$) s.t. for all $R \leqslant R_0$ we have*

$$\omega(\tau R) \leqslant \gamma\omega(R) + \sigma(R), \tag{4.20}$$

where $\sigma : (0, R_0] \to \mathbb{R}$ is also increasing. Then for any $\mu \in (0, 1)$, $R \leqslant R_0$ we have

$$\omega(R) \leqslant C\left((R/R_0)^\alpha \omega(R_0) + \sigma(R^\mu R_0^{1-\mu})\right),$$

where $C = C(\gamma, \tau)$, $\alpha = (1-\mu)\frac{\log\gamma}{\log\tau} > 0$.

Proof. We first fix $R_1 \leqslant R_0$. Then for any $R \leqslant R_1$ we have

$$\omega(\tau R) \leqslant \gamma\omega(R) + \sigma(R_1)$$

since σ is increasing. By iterating this inequality we obtain for any $m \in \mathbb{N}$

$$\begin{aligned}
\omega(\tau^m R_1) &\leqslant \omega(\tau^{m-1} R_1) + \sigma(R_1) \\
&\leqslant \gamma\left(\omega(\tau^{m-2} R_1) + \sigma(R_1)\right) + \sigma(R_1) \\
&\leqslant \gamma^m \omega(R_1) + \sigma(R_1) \sum_i \gamma^i \\
&\leqslant \gamma^m \omega(R_0) + (1-\gamma)^{-1}\sigma(R_1).
\end{aligned}$$

Now for any given $R \leqslant R_1$ we choose m s.t. $\tau^m < R \leqslant \tau^{m-1} R_1$ and thus obtain

$$\omega(R) \leqslant \omega(\tau^{m-1} R_1) \leqslant \frac{1}{\gamma}\gamma^m \omega(R_0) + \frac{\sigma(R_1)}{1-\gamma}.$$

Further since $\tau^m < \frac{R}{R_1}$ we have $m > \log\left(\frac{R}{R_1}\right)(\log\tau)^{-1}$ and therefore

$$\omega(R) \leqslant \frac{1}{\gamma}\left(\frac{R}{R_1}\right)^{\frac{\log\gamma}{\log\tau}} \omega(R_0) + \frac{\sigma(R_1)}{1-\gamma}.$$

Now specifying $R_1 = R_0^{1-\mu} R^\mu$ gives the desired result

$$\omega(R) \leqslant \frac{1}{\gamma}\left(\frac{R}{R_0}\right)^{(1-\mu)\frac{\log\gamma}{\log\tau}} \omega(R_0) + \frac{\sigma(R_0^{1-\mu} R^\mu)}{1-\gamma}.$$

□

Definition 4.12. (Oscillation)
Let $\Omega \subset \mathbb{R}^n$ and define the oscillation of a given function u in Ω by

$$\operatorname*{osc}_{\Omega} u := \sup_{\Omega} u - \inf_{\Omega} u.$$

Theorem 4.13. *(C^α-regularity of weak solutions)*
Let $\Omega \subset \mathbb{R}^n$ be open, bounded and $u \in H^1(\Omega)$ be a weak solution of

$$Lu \equiv \Delta u + b_i D_i u + cu = g \quad \text{in } \Omega,$$

where $\mathbf{b} \in L^{2p}(\Omega)$, c, $g \in L^p(\Omega)$ with $p > n/2$. Then u is locally Hölder continuous in Ω and for any ball $B_0 = B_{R_0(y)} \subset \Omega$, $R < R_0$ we have

$$\operatorname*{osc}_{B_R(y)} u \leqslant CR^\alpha (R_0^{-\alpha} \sup_{B_0} |u| + \|g\|_{p;\Omega}), \tag{4.21}$$

where $C = C(n, \|\mathbf{b}\|_{2p}, \|c\|_p, p, R_0)$ and $\alpha = \alpha(n, p, M_{R_0})$, $M_{R_0} := R_0^{2-\frac{n}{p}} \left(\|\mathbf{b}\|_{2p}^2 + \|c\|_p\right)$ are strictly positive constants. Further for any subdomain $\Omega' \Subset \Omega$ we have

$$|u|_{0;\alpha;\Omega'} \leqslant C(|u|_{0;\Omega} + \|g\|_{p;\Omega}),$$

with C and α as above and $R_0 := \operatorname{dist}(\Omega', \partial\Omega)$.

Proof. The proof of the first part will be performed by choosing two different functions u in the weak Harnack inequality (Theorem 4.9) to obtain an equation of the form (4.20) from the previous lemma. We will omit the subscript Ω.

Let us set $M_0 := \sup_{B_0} |u|$ and by Remark 4.10 we may assume that $R < \frac{R_0}{4}$. Further we define the following four quantities

$$M_4 := \sup_{B_{4R}} u,\ m_4 := \inf_{B_{4R}} u,\ M_1 := \sup_{B_R} u,\ m_1 := \inf_{B_R} u.$$

Then the functions $M_4 - u$ and $u - m_4$ are both positive in B_{4R} and fulfill

$$\begin{aligned} L(M_4 - u) &= M_4 c - g, \\ L(u - m_4) &= -m_4 c + g. \end{aligned}$$

By viewing the right-hand sides as the inhomogeneities g in Theorem 4.9, we obtain

$$\begin{aligned} R^{-n}\|M_4 - u\|_{1;2R} &\leqslant C \left(\inf_{B_R} (M_4 - u) + R^{2-\frac{n}{p}} (\|g\|_p + M_4 \|c\|_p) \right) \\ &\leqslant C \left(M_4 - M_1 + \bar{k}(R) \right), \\ R^{-n}\|u - m_4\|_{1;2R} &\leqslant C \left(\inf_{B_R} (u - m_4) + R^{2-\frac{n}{p}} (\|g\|_p + m_4 \|c\|_p) \right) \\ &\leqslant C \left(m_1 - m_4 + \bar{k}(R) \right), \end{aligned}$$

where $\bar{k}(R) := R^{2-\frac{n}{p}}(\|g\|_p + M_0\|c\|_p)$. Now by adding the two inequalities we arrive at

$$R^{-n}\mathrm{vol}(B_{2R})(M_4 - m_4) \leqslant C\left(M_4 - m_4 + m_1 - M_1 + \bar{k}(R)\right). \tag{4.22}$$

Hence by setting $\omega(R) := \mathrm{osc}_{B_R}\, u$ we obtain

$$\omega(4R) \leqslant C\left(\omega(4R) - \omega(R) + \bar{k}(R)\right)$$

and we note that $\omega(R)$ and $\bar{k}(R)$ are increasing w.r.t R.

In the case $C \leqslant 1$ the above estimate holds in particular for $C = 1$ and we have

$$\operatorname*{osc}_{B_R} u = \omega(R) \leqslant \bar{k}(R) = R^{2-\frac{n}{p}}(\|g\|_p + \|c\|_p \sup_{B_0}|u|)$$

and thus the claim of the first part with $\alpha = 2 - \frac{n}{p}$. We note that this α coincides with the α-value from Theorem 4.7 for equations with no divergence term.

If $C > 1$ we rewrite (4.22) to

$$\omega(R) \leqslant (1 - C^{-1})\omega(4R) + \bar{k}(R)$$

and apply Lemma 4.20 with $\sigma(R) = \bar{k}(R)$, $\tau = \frac{1}{4}$, $\gamma = 1 - \frac{1}{C}$ to obtain

$$\omega(R) \leqslant C\left(\left(\frac{R}{R_0}\right)^{(1-\mu)\frac{\log\gamma}{\log\tau}}\omega(R_0) + \bar{k}(r^\mu R_0^{1-\mu})\right).$$

Now by choosing μ s.t. $(2 - \frac{n}{p})\mu = (1-\mu)\frac{\log\gamma}{\log\tau} =: \alpha$ we conclude

$$\begin{aligned}\operatorname*{osc}_{B_R} u &\leqslant R^\alpha R_0^{-\alpha} M_0 + R^\alpha R_0^{(1-\mu)(2-\frac{n}{p})}(\|g\|_p + M_0\|c\|_p)\\ &\leqslant CR^\alpha\left(R_0^{-\alpha}\sup_{B_0}|u| + \|g\|_p\right)\end{aligned}$$

with $C = C(n, \|\mathbf{b}\|_{2p}, \|c\|_p, p, R_0)$. #

The second part of the thereom will follow by setting $R_0 := \mathrm{dist}(\Omega', \partial\Omega)$ in the above estimate. For any x, $y \in \Omega'$ we will distinguish two cases:

If $\frac{1}{2}|x - y| \geqslant R_0$ we estimate

$$\frac{|u(x) - u(y)|}{|x - y|^\alpha} \leqslant 2R_0^{-\alpha}|u|_{0;\Omega}$$

and if $R := \frac{1}{2}|x-y| < R_0$ we set $z := \frac{1}{2}(x+y)$ and by the choice of R_0 it is $B_{R_0}(z) \subset \Omega$. Applying (4.21) then gives

$$\begin{aligned}\frac{|u(x)-u(y)|}{|x-y|^\alpha} &\leqslant 2R^{-\alpha} \operatorname*{osc}_{B_R(z)} u \\ &\leqslant C(R_0^{-\alpha} \sup_{B_{R_0}(z)} |u| + \|g\|_{p;\Omega}) \\ &\leqslant C(R_0^{-\alpha}|u|_{0;\Omega} + \|g\|_{p;\Omega})\end{aligned}$$

and thus for any $x,\, y \in \Omega'$

$$\frac{|u(x)-u(y)|}{|x-y|^\alpha} \leqslant C(|u|_{0;\Omega} + \|g\|_{p;\Omega})$$

with $C = C(n, \|\mathbf{b}\|_{2p}, \|c\|_p, p, \operatorname{dist}(\Omega', \partial\Omega))$ uniformly in x and y. Taking the supremum over all $x \neq y$ in Ω' yields the desired result. # □

5 Boundedness of solutions

In this chapter we will establish assumptions under which the weak solution of an elliptic equation is locally bounded. The first theorem, based on Th. 11.7 in [LL01], will show the local boundedness under very general assumptions without providing an explicit bound. The second theorem, based on Chapter 8 in [GT01], will give an explicit bound in terms of the L^2-norm under stricter assumptions.

Theorem 5.1. *(Local boundedness of weak solutions)*
Let B_1 be a ball in $\mathbb{R}^n$, $n \geqslant 3$ and $u \in H^1(B_1)$ be a weak solution of

$$-\Delta u + Vu = 0 \quad \text{in} \quad B_1,$$

where $V \in L^p(B_1)$, $p > \frac{n}{2}$. Then $u \in C^{\alpha}_{loc}(B_1)$ is locally Hölder continuous for $\alpha = 2 - \frac{n}{p}$ and in particular u is locally bounded in B_1.

Proof. We will show that the conclusion holds for any arbitrary ball $B \subset B_1$ with strictly smaller radius and therefore in particular for any compact set in B_1. The proof will be performed inductively. At first we show that u is locally bounded and then conclude that $Vu \in L^p(B)$ and apply Corollary 3.2. By Lemma 2.29 any solution u to Poisson's equation $\Delta u = f$ can be represented by

$$u(x) = w_f(x) + h(x),$$

where $w_f(x)$ is the Newton potential of f and h harmonic. Since any harmonic function is continuous and therefore locally bounded, we will assume w.l.o.g. $h \equiv 0$. By assumption $u \in H^1(B_1)$ and thus by Sobolev's inequality $u \in L^{\frac{2n}{n-2}}(B_1)$. Applying the general Hölder inequality in B_1 shows

$$\|Vu\|_{s_1;B_1} \leqslant \|V\|_{p;B_1} \|u\|_{\frac{2n}{n-2};B_1}$$

for an $s_1 > 1$ given by Hölder's inequality. Then

$$u(x) = w(x) = \int_{B_1} \Gamma(x-y)V(y)u(y)\, \mathrm{dy} \quad \text{for } x \in B_2 \subsetneq B_1.$$

To avoid $\Gamma(x-y)$ being singular on the boundary, we restrict ourselves to interior points of B_1, i.e. to $x \in B_2$ concentric with B_1 and with a strictly smaller radius.

Now we apply Lemma 2.31 to obtain

$$u \in L^t(B_2), \quad t \leqslant \frac{ns_1}{n-2s_1}.$$

Further by assumption $V \in L^p(B_1)$, $\frac{n}{2} < p \leqslant n$ and thus we can write

$$\frac{1}{p} = \frac{2}{n} - \varepsilon, \quad 0 < \varepsilon \leqslant \frac{1}{n}.$$

Applying the general Hölder inequality once more in B_2 yields

$$\|Vu\|_{s_2;B_2} \leqslant \|V\|_{p;B_2} \|u\|_{s_1;B_2}$$

with

$$\frac{1}{s_2} = \frac{1}{p} + \frac{1}{t} \geqslant \frac{2}{n} - \varepsilon + \frac{1}{s_1} - \frac{2}{n} = \frac{1-\varepsilon s_1}{s_1}$$

and thus in particular $s_2 < s_1/(1-\varepsilon)$. By iterating the above estimates we arrive after finitely many steps at

$$Vu \in L^{s_k}(B_k) \quad \text{with } s_k > \frac{n}{2},$$

where B_k is a series of nested balls inside B_1 but arbitrarily close. We are now in the situation to apply Corollary 3.2 and conclude that u is Hölder continuous in B_k. Thus u is bounded on any compact set inside B_k in particular for $\bar{B} \supset B = B_R(y_0)$ with a strictly smaller radius. Therefore

$$Vu \in L^p(B).$$

We now apply Corrollary 3.2 once more to obtain

$$[w]_{\alpha;B} \leqslant C(n,\alpha)|u|_{0;B}\|V\|_{p;B}$$

with $\alpha = 2 - \frac{n}{p}$ and thus the desired result. □

Lemma 5.2. *(Boundedness from above of weak subsolutions)*
Let $\Omega \subset \mathbb{R}^n$, $n \geqslant 3$ be open, bounded and $u \in H^1(\Omega)$ be a weak subsolution of

$$Lu = \Delta u + Vu = f,$$

where V, $f \in L^p(\Omega)$ for some $p > \frac{n}{2}$. Then if $u \leqslant 0$ on $\partial\Omega$ we have

$$\sup_{\Omega} u \leqslant C(\|u_+\|_2 + \|f\|_p),$$

where $C = C(n, \|V\|_p, |\Omega|, p)$.

Proof. By a special choice of the test function v in (2.5) we want to bound a higher L^{p_1} norm of u_+ by a lower L^{p_2} norm and then conclude the result by induction. To define this test function we set $k := \|f\|_p$ and consider for any $\beta \geqslant 1$ and $N > k$ a function $H \in C^1([k,\infty))$ that grows with power β up to N and is cut-off linearly after:

$$H(z) := \begin{cases} z^\beta - k^\beta & z \in [k,N], \\ \beta N^{\beta-1}z + (1-\beta)N^\beta - k^\beta & z \geqslant N. \end{cases} \tag{5.1}$$

We now set $w := u_+ + k \geqslant k$ and define $G(w) := \int_k^w |H'(s)|^2 \mathrm{ds}$ and take $v := G(w)$ as a test function in (2.5). The above choice of H guarantees that v is indeed a proper test function:

Since $u_+ = 0$ on $\partial\Omega$ we also have $v = \int_k^{w=k} ... = 0$ on $\partial\Omega$ and clearly $v \geqslant 0$. Further by the chain rule (Lemma 2.10) we have $Dv = G'(w)Dw$ and therefore

$$\|Dv\|_2 = \|G'(w)Dw\|_2 = \|(H'(w))^2 Dw\|_2 \leqslant \beta^2 N^{2\beta-2}\|Dw\|_2 \leqslant C(N,\beta)\|Du\|_2.$$

Now by Poincares inequality it follows that $v \in H_0^1(\Omega)$.

Plugging v into (2.5) leads to

$$\int_\Omega D_i v D_i u \,\mathrm{dx} = \int_\Omega G'(w) D_i w D_i w \,\mathrm{dx} \leqslant \int_\Omega G(w)(|V|u_+ + |f|) \,\mathrm{dx},$$

where we used that supp $G(w) \subset$ supp u_+ and $D_i w = D_i u$ for $v = G(w) > 0$. Since $G(s) \leqslant sG'(s)$ we have

$$\begin{aligned} \int_\Omega G'(w)|Dw|^2 \,\mathrm{dx} &\leqslant \int_\Omega G'(w)w(|V|w + |f|) \,\mathrm{dx} \\ &\leqslant \int_\Omega G'(w)w^2(|V| + k^{-1}|f|) \,\mathrm{dx}. \end{aligned}$$

By the definition of v this can be written as

$$\int_\Omega |DH(w)|^2 \,\mathrm{dx} \leqslant \int_\Omega \left(|V| + k^{-1}|f|\right)(H'(w)w)^2 \,\mathrm{dx}. \tag{5.2}$$

Now since $w|_{\partial\Omega} = 0$ and $H \in C^1[k,\infty)$ we have $H(w) \in H^1(\Omega)$ and can therefore apply Sobolev's inequality to $H(w)$

$$C(n)^{-1}\|H(w)\|_{\frac{2n}{n-2}} \leqslant \left(\int_\Omega |DH(w)|^2 \,\mathrm{dx}\right)^{\frac{1}{2}}.$$

Together with (5.2) and Hölder's inequality we obtain

$$\begin{aligned} \|H(w)\|_{\frac{2n}{n-2}} &\leqslant C(n)\left(\int_\Omega (|V| + k^{-1}|f|)(H'(w)w)^2 \,\mathrm{dx}\right)^{\frac{1}{2}} \\ &\leqslant C(n)\left(\|V\|_p + k^{-1}\|f\|_p\right)^{1/2}\|H'(w)w\|_{\frac{2p}{p-1}}. \end{aligned}$$

Remark : If $k = 0$, i.e. $f \equiv 0$ the above estimates continues to hold.

With the chosen k we thus have

$$\|H(w)\|_{\frac{2n}{n-2}} \leqslant C\|wH'(w)\|_{\frac{2p}{p-1}},$$

where $C = C(n, \|V\|_p)$. This estimate holds for all N in the definition of H. Thus letting $N \to \infty$ and recalling that $w \geqslant k$ gives

$$\|w^\beta\|_{\frac{2n}{n-2}} \leqslant \tilde{C}\beta\|w^\beta\|_{\frac{2p}{p-1}} + k^\beta|\Omega|^{\frac{n-2}{2n}} \leqslant C\beta\|w^\beta\|_{\frac{2p}{p-1}},$$

where $C = C(n, \|V\|_p, |\Omega|)$. This shows that $w \in L^{\beta\frac{2p}{p-1}}(\Omega)$ implies the stronger inclusion $w \in L^{\beta\frac{2n}{n-2}}(\Omega)$. Setting $p^* := \frac{2p}{p-1} > \frac{2n}{n-2}$ and $\rho := \frac{n(p-1)}{p(n-2)} > 1$ we further have

$$\|w\|_{\beta\rho p^*} \leqslant (C\beta)^{1/\beta}\|w\|_{\beta p^*}.$$

By induction we conclude $w \in \bigcup_{1 \leqslant p \leqslant \infty} L^p(\Omega)$ and by setting $\beta := \rho^m > 1$ for $m \in \mathbb{N}$ we obtain

$$\begin{aligned}
\|w\|_{\rho^{m+1}p^*} &\leqslant (C\rho^m)^{\rho^{-m}}\|w\|_{\rho^m p^*} \\
&\overset{\text{Ind}}{\leqslant} \prod_{k=0}^{m} (C\rho^k)^{\rho^{-k}}\|w\|_{p^*} \\
&\leqslant C^\sigma \rho^\tau \|w\|_{p^*} \qquad (5.3)
\end{aligned}$$

with $\sigma = \sum_{k=0}^\infty \rho^{-k} < \infty$ and $\tau = \sum_{k=0}^\infty k\rho^{-k} < \infty$. Since (5.3) is independent of m we can take the limit $m \to \infty$ to arrive at

$$\sup_\Omega w \leqslant C\|w\|_{p^*},$$

where $C = C(n, \|V\|_p, |\Omega|, p)$. Finally we use the interpolation inequality (Lemma 2.4) with $\lambda = \frac{1}{4}$, $p = 2$ and $q := p^*$ to obtain

$$\begin{aligned}
\|w\|_{p^*} &\leqslant \varepsilon\|w\|_r + \varepsilon^{-\mu}\|w\|_2 \\
&\leqslant \varepsilon\tilde{C}\|w\|_{p^*} + \varepsilon^{-\mu}\|w\|_2
\end{aligned}$$

and by choosing $\varepsilon = \frac{1}{2\tilde{C}}$ we conclude

$$\sup_\Omega w \leqslant \tilde{C}\|w\|_{p^*} \leqslant C\|w\|_2$$

with $C = C(n, \|V\|_p, |\Omega|, p)$. The result now follows with the definition $w := u_+ + k$ and $\|w\|_2 \leqslant \|u_+\|_2 + k|\Omega|^{\frac{1}{2}}$. $\square$

Corollary 5.3. *(Boundedness from below of weak supersolutions)*
Let $\Omega \subset \mathbb{R}^n$, $n \geqslant 3$ be open, bounded and $u \in H^1(\Omega)$ be a weak supersolution of

$$Lu = \Delta u + Vu = f,$$

where V, $f \in L^p(\Omega)$ for some $p > \frac{n}{2}$. Then if $u \geqslant 0$ on $\partial\Omega$ we have

$$\sup_{\Omega} u \leqslant C(\|u_-\|_2 + \|f\|_p),$$

where $C = C(n, \|V\|_p, |\Omega|, p)$.

Proof. By replacing u with $-u$ in the preceding proof. □

Remark 5.4. If $f \equiv 0$ in Lemma 5.2 $C = C(n, \|V\|_p, p)$ is independent of $|\Omega|$.

Theorem 5.5. *(Boundedness of weak solutions)*
Let $\Omega \subset \mathbb{R}^n$, $n \geqslant 3$ be open, bounded and $u \in H^1(\Omega)$ be a weak solution of

$$Lu = \Delta u + Vu = f,$$

where V, $f \in L^p(\Omega)$ for some $p > \frac{n}{2}$. Then if $|u| \leqslant l$ on $\partial\Omega$ for some constant $l > 0$ we have

$$\sup_{\Omega} |u| \leqslant C(\|u\|_2 + \|f\|_p + l),$$

where $C = C(n, \|V\|_p, |\Omega|, p)$.

Proof. Since u is a weak solution it is in particular a weak subsolution. We now consider $\tilde{u} = u - l$ which is a weak subsolution of

$$L\tilde{u} = Lu - Ll = f - lV$$

with $\tilde{u} \leqslant 0$ on $\partial\Omega$. Therefore we can apply Lemma 5.2 with $u \to u - l$ and $k \to k + l\|V\|_p$ to obtain

$$\sup_{\Omega}(u - l) \leqslant C(\|(u-l)_+\|_2 + \|f\|_p + l\|V\|_p)$$

and since $\|(u-l)_+\|_2 \leqslant \|u_+\|_2 \leqslant \|u\|_2$ we have

$$\sup_{\Omega} u \leqslant C(\|u\|_2 + k + l),$$

where $C = C(n, \|V\|_p, |\Omega|, p)$. Also u is a supersolution and $\hat{u} := u + l$ is a weak supersolution of

$$L\hat{u} := f + lV$$

with $\hat{u} \geqslant 0$ on $\partial\Omega$. Thus we can apply Corollary 5.3 with $u \to u+l$ and $k \to k+l\|V\|_p$ and obtain

$$\sup_{\Omega} -u \leqslant C(\|u\|_2 + k + l)$$

□

6 Eigenfunctions

In this chapter we study properties of eigenfunctions of the operator

$$H = -\Delta + V \quad \text{on} \ \ \mathbb{R}^n, \tag{6.1}$$

where V is a real $L^1_{\text{loc}}(\mathbb{R}^n)$ potential and Δ the usual Laplace operator on $\mathbb{R}^n$. This chapter relies on Chapter 11 in [LL01] providing more detailed proofs and relaxing the assumption on V being locally bounded from above in [LL01] Th. 11.8.

6.1 Existence of eigenfunctions

Lemma 6.1. *(Boundedness from below of the kinetic energy)*
Assume $V_- \in L^{\frac{n}{2}}(\mathbb{R}^n) + L^\infty(\mathbb{R}^n)$, $n \geqslant 3$. Define the energy functional $\mathcal{E}$ by

$$\mathcal{E} \colon H^1(\mathbb{R}^n) \to \mathbb{R};\ \psi \mapsto \int |D\psi|^2 + \int V|\psi|^2.$$

Then there exists $C = C(n, V_-)$ s.t.

$$\mathcal{E}(\psi) \geqslant -C\|\psi\|^2.$$

In particular the ground state energy E_0 is bounded from below

$$E_0 := \inf_{\|\psi\|_2=1} \mathcal{E}(\psi) > -\infty$$

and there exists $D = D(n, V_-)$ s.t.

$$\|D\psi\|_2 \leqslant 2\mathcal{E}(\psi) + D\|\psi\|_2.$$

Proof. By assumption we can write $V_- = \tilde{V}_1 + \tilde{V}_2$ where $\tilde{V}_1 \in L^\infty$ and $\tilde{V}_2 \in L^{\frac{n}{2}}$. For any large M we split

$$\tilde{V}_2 = \tilde{V}_2 \mathbb{1}_{\{\tilde{V}_2 \leqslant M\}} + \tilde{V}_2 \mathbb{1}_{\{\tilde{V}_2 > M\}}$$

and define $V_1 = \tilde{V}_1 + V_2 \mathbb{1}_{\{\tilde{V}_2 \leqslant M\}} \in L^\infty$ and $V_2 = \tilde{V}_2 \mathbb{1}_{\{\tilde{V}_2 > M\}} \in L^{\frac{n}{2}}$. Thus $\|V_2\|_{n/2}$ can be made as small as we please for large M. By the above splitting and using first Hölder's and then Sobolev's inequality we estimate

$$\begin{aligned}\int V_-|\psi|^2 &\leqslant \|V_1\|_\infty\|\psi\|^2 + \int V_2|\psi|^2\\ &\leqslant \|V_1\|_\infty\|\psi\|^2 + \|V_2\|_{\frac{n}{2}}\|\psi\|^2_{\frac{2n}{n-2}}\\ &\leqslant \|V_1\|_\infty\|\psi\|^2 + C^2_{\text{sob}}\|V_2\|_{\frac{n}{2}}\|D\psi\|^2\end{aligned}$$

and thus

$$\mathcal{E}(\psi) \geqslant \int |D\psi|^2 - \int V_-|\psi|^2 \geqslant \|D\psi\|^2 - \|V_1\|_\infty\|\psi\|^2 - C^2_{\text{sob}}\|V_2\|_{\frac{n}{2}}\|D\psi\|^2.$$

Now choosing M large enough s.t. $C^2_{\text{sob}}\|V_2\|_{\frac{n}{2}} \leqslant \frac{1}{2}$ gives

$$\mathcal{E}(\psi) \geqslant \frac{1}{2}\|D\psi\|^2 - \|V_1\|_\infty\|\psi\|^2$$

and concludes the proof of the lemma. □

Theorem 6.2. *(Existence of the ground state)*
Assume $V \in L^{\frac{n}{2}}(\mathbb{R}^n) + L^\infty(\mathbb{R}^n)$, $n \geqslant 3$ with V vanishing at infinity, i.e. the set $\{x \in \mathbb{R}^n : |V(x)| > a\}$ is bounded for all $a > 0$. Let the energy functional $\mathcal{E}$ be defined as before

$$\mathcal{E}(\psi) = \int_{\mathbb{R}^n} |D\psi|^2 + \int_{\mathbb{R}^n} V|\psi|^2.$$

Further assume that

$$E_0 := \inf_{\|\psi\|_2=1} \mathcal{E}(\psi) < 0.$$

Then there exists an energy minimizer $\psi \in H^1(\mathbb{R}^n)$, $\|\psi\|_2 = 1$ s.t. $\mathcal{E}(\psi) = E_0$. Moreover ψ is a weak solution of

$$-\Delta\psi + V\psi = E_0\psi. \tag{6.2}$$

Proof. If not indicated otherwise all integrals will be over $\mathbb{R}^n$ and $\|\cdot\|$ will denote the L^2-norm on $\mathbb{R}^n$. We first proof the second part (6.2) of the theorem. Therefore let $\varphi \in C_0^\infty(\mathbb{R}^n)$, $|\varepsilon| << 1$, $\psi_\varepsilon := \psi + \varepsilon\varphi$ and define

$$R(\varepsilon) = \frac{\mathcal{E}(\psi_\varepsilon)}{\|\psi_\varepsilon\|^2}.$$

Since $\psi = \psi_0$ minimizes the energy we have

$$\left.\frac{\mathrm{d}R(\varepsilon)}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} = 0$$

and by explicit calculation

$$0 = \frac{\mathrm{d}R(\varepsilon)}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} = \frac{\mathrm{d}\mathcal{E}(\psi_\varepsilon)}{\mathrm{d}\varepsilon}\frac{1}{\|\psi_0\|^2} - \mathcal{E}(\psi_0)\frac{\mathrm{d}\|\psi_\varepsilon\|^2}{\mathrm{d}\varepsilon}\frac{1}{\|\psi_\varepsilon\|^4}\bigg|_{\varepsilon=0}$$
$$= \frac{\mathrm{d}\mathcal{E}(\psi_\varepsilon)}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} - E_0\,\frac{\mathrm{d}\|\psi_\varepsilon\|^2}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0}. \tag{6.3}$$

Computing these terms we obtain

$$\frac{\mathrm{d}\mathcal{E}(\psi_\varepsilon)}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0}\int |D\psi + \varepsilon D\varphi|^2 + V|\psi+\varepsilon\varphi|^2$$
$$= 2\mathrm{Re}\int D\bar{\psi}D\varphi + V\bar{\psi}\varphi$$
$$= 2\mathrm{Re}\int -\bar{\psi}\Delta\varphi + V\bar{\psi}\varphi$$

and

$$\frac{\mathrm{d}\|\psi_\varepsilon\|^2}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} = 2\mathrm{Re}\int\bar{\psi}\varphi.$$

Thus along with (6.3) we conclude

$$\mathrm{Re}\int\bar{\psi}(-\Delta\varphi + V\varphi - E_0\varphi) = 0 \qquad \forall\,\varphi\in C_0^\infty(\mathbb{R}^n).$$

Replacing $\varphi \to i\varphi$ gives

$$\mathrm{Im}\int\bar{\psi}(-\Delta\varphi + V\varphi - E_0\varphi) = 0 \qquad \forall\,\varphi\in C_0^\infty(\mathbb{R}^n)$$

and finishes the proof of (6.2).

We now turn back to the first part of the theorem. Let $\{\psi_n\}_{n\in\mathbb{N}}$, $\|\psi_n\| = 1$ be a minimizing sequence with $\mathcal{E}(\psi_n) \to E_0$ as $n\to\infty$. By Lemma 6.1 we know that there exists $C = C(V)$ s.t.

$$\int |D\psi_n| \leqslant 2\mathcal{E}(\psi_n) + C\|\psi_n\|^2.$$

Thus we have $\psi_n \in H^1(\mathbb{R}^n)$ and $\sup_{n\in\mathbb{N}}\|\psi_n\|_{H^1(\mathbb{R}^n)} < \infty$ and by Theorem 2.17 we can select a $\|\cdot\|_{H^1}$-weakly convergent subsequence

$$\psi_{n_j} \rightharpoonup \psi \qquad \text{in}\ \ H^1(\mathbb{R}^n).$$

After renaming $\psi_j := \psi_{n_j}$ we now claim the following:

$$
\begin{aligned}
(i) \quad & \int |D\psi|^2 \leqslant \liminf_{j\to\infty} \int |D\psi_j|^2, \\
(ii) \quad & \int V|\psi|^2 = \lim_{j\to\infty} \int V|\psi_j|^2, \\
(iii) \quad & \|\psi\| = 1.
\end{aligned}
$$

Assuming $(i) - (ii)$ we obtain

$$\mathcal{E}(\psi) \leqslant \liminf_{j\to\infty} \mathcal{E}(\psi_j) = E_0. \tag{6.4}$$

But by (iii) we have $\|\psi\| = 1$ and moreover for any ψ

$$\mathcal{E}(\psi) \geqslant E_0 = \inf_{\|\psi\|_2=1} \mathcal{E}(\psi)$$

and thus $\mathcal{E}(\psi) = E_0$. This completes the proof of the theorem.

It remains to show the validity of $(i) - (iii)$:

Proof of (iii)

Assume (i) and (ii), then by (6.4) and by passing to another subsequence we have

$$E_0 = \lim_{k\to\infty} \mathcal{E}(\psi_{j_k}) \geqslant \mathcal{E}(\psi) \geqslant E_0\|\psi\|^2,$$

and since $E_0 < 0$ we obtain $\|\psi\| \geqslant 1$. On the other hand $\|\psi\| \leqslant \liminf_{k\to\infty} \|\psi_{j_k}\| = 1$ and therefore $\|\psi\| = 1$. #

Proof of (i)

By (2.5) we have

$$\|D\psi\| = \sup\{|\langle D\psi, \mathbf{g}\rangle| \colon g^i \in C_0^\infty(\mathbb{R}^n),\ i = 1,\dots,n;\ \|\mathbf{g}\| = 1\}.$$

By weak convergence of ψ_j in $H^1(\mathbb{R}^n)$ we obtain

$$\|D\psi\| = \sup\{\lim_{j\to\infty} |\langle D\psi_j, \mathbf{g}\rangle| \colon g^i \in C_0^\infty(\mathbb{R}^n), \dots ; \|\mathbf{g}\| = 1\}.$$

Interchanging sup and lim inf gives

$$\|D\psi\| \leqslant \liminf_{j\to\infty} \sup\{|\langle D\psi_j, \mathbf{g}\rangle| \colon g^i \in C_0^\infty(\mathbb{R}^n), \dots ; \|\mathbf{g}\| = 1\} = \liminf_{j\to\infty} \|D\psi_j\|$$

and proves (i). #

Proof of (ii)

We first define a cut-off function $V_M(x) := V(x)\mathbb{1}_{\{x:|V(x)|\leqslant M\}}$ and write

$$V = V_M + (V - V_M).$$

For sufficiently large M we then have $(V - V_M) \in L^{\frac{n}{2}}(\mathbb{R}^n)$ with $V - V_M \overset{M\to\infty}{\longrightarrow} 0$ in $L^{\frac{n}{2}}(\mathbb{R}^n)$. With this splitting we obtain

$$\lim_{j\to\infty} \int V|\psi_j|^2 - \int V|\psi|^2 = \lim_{j\to\infty} \int (V - V_M)|\psi_j|^2 - \int (V - V_M)|\psi|^2 \tag{6.5}$$

$$+ \lim_{j\to\infty} \int V_M|\psi_j|^2 - \int V_M|\psi|^2. \tag{6.6}$$

By Hölder's and Sobolev's inequality we can estimate (6.5) by

$$\begin{aligned}\left|\int (V - V_M)|\psi_j|^2\right| &\leqslant \|V - V_M\|_{\frac{n}{2}} \|\psi_j\|_{2\frac{n}{n-2}} \\ &\leqslant \|V - V_M\|_{\frac{n}{2}} \|D\psi_j\|_2 \overset{M\to\infty}{\longrightarrow} 0 \quad \text{unif. in } j,\end{aligned}$$

since $\|D\psi_j\|_2 \leqslant \infty$ uniformly in j. Likewise we estimate the minuend in (6.5).
It remains to estimate (6.6). We therefore show that for any fixed M we have

$$\lim_{j\to\infty} \int V_M|\psi_j|^2 = \int V_M|\psi|^2.$$

Now for M fixed choose $\varepsilon > 0$ and define A_ε as the ball containing the set $\{x : |V(x)| \geqslant \varepsilon\}$ s.t. $|A_\varepsilon| < \infty$ (by assumption $\{x : |V(x)| \geqslant \varepsilon\} < \infty$) to obtain another splitting

$$\int V_M|\psi_j|^2 = \int_{A_\varepsilon} V_M|\psi_j|^2 + \int_{\mathbb{R}^n\setminus A_\varepsilon} V_M|\psi_j|^2,$$

where the second summand can be bounded by $\varepsilon\|\psi_j\|^2 = \varepsilon$ uniformly in j. Thus it remains to show that for any fixed $\varepsilon > 0$ and M

$$\lim_{j\to\infty} \int_{A_\varepsilon} V_M|\psi_j|^2 = \int_{A_\varepsilon} V_M|\psi|^2.$$

Let us therefore consider the difference

$$\begin{aligned}\left|\int_{A_\varepsilon} V_M|\psi_j|^2 - \int_{A_\varepsilon} V_M|\psi|^2\right| &\leqslant M \int_{A_\varepsilon} \left||\psi_j|^2 - |\psi|^2\right| \\ &\leqslant M \int_{A_\varepsilon} |\psi_j - \psi| \left(|\psi_j| + |\psi|\right) \\ &\leqslant M \left(\int_{A_\varepsilon} |\psi_j - \psi|^2\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} 2\left(|\psi_j|^2 + |\psi|^2\right)\right)^{\frac{1}{2}},\end{aligned}$$

where we first used the triangle- and then Cauchy-Schwarz's inequality. The second factor can be bounded by $\sqrt{2}(\|\psi_j\| + \|\psi\|)^{1/2} \leqslant 2$ uniformly in j. Since A_ε is a bounded set and $\psi_{n_j} \rightharpoonup \psi$ in $H^1(\mathbb{R}^n)$ it follows by Rellich Kondrachov (Theorem 2.18) that

$$\int_{A_\varepsilon} |\psi_j - \psi|^2 \overset{j\to\infty}{\longrightarrow} 0.$$

Since this estimate holds independently of ε and M, by letting $\varepsilon \to 0$ and $M \to \infty$ in the previous equations we conclude (ii). □

Definition 6.3. (Excited states)
Let $H = -\Delta + V$ with $V \in L^{\frac{n}{2}}(\mathbb{R}^n) + L^\infty(\mathbb{R}^n)$ vanishing at infinity. Further define $\mathcal{E}(\psi)$ as before and denote the ground state by ψ_0 and the ground state energy by E_0. Now define higher energy eigenfunctions ψ_k ("excited states") successively for $k = 1, 2, ...$ as follows:

Assume we already defined ψ_j and E_j for $j = 0, 1, \dots, k-1$, then the k-th eigenvalue E_k is defined as

$$E_k := \inf\{\mathcal{E}(\psi) \colon \psi \in H^1(\mathbb{R}^n),\ \|\psi\| = 1,\ \langle \psi, \psi_j \rangle = 0,\ j = 0, 1, \dots, k-1\} \tag{6.7}$$

and ψ_k as the corresponding minimizer (if it exists).

Theorem 6.4. *(Existence of excited states)*
Under the assumptions of Theorem 6.2 assume that $\psi_0, \dots, \psi_{k-1}$ exist and $E_k < 0$. Then also the k-th eigenfunction $\psi_k \in H^1(\mathbb{R}^n)$ exists as a minimizer and is a weak solution of

$$(-\Delta + V)\psi_k = E_k \psi_k \tag{6.8}$$

Proof. We start with the proof of (6.8). The proof is similar to the previous proof, where now ψ_k plays the role of ψ. As before let $\varphi \in C_0^\infty(\mathbb{R}^n)$, $|\varepsilon| << 1$, $\psi_\varepsilon = \psi_k + \varepsilon\varphi$ and define

$$R(\varepsilon) = \frac{\mathcal{E}(\psi_\varepsilon}{\|\psi_\varepsilon\|^2}.$$

But now φ is not an arbitrary perturbation but also has to satisfy

$$\langle \varphi, \psi_j \rangle = 0, \quad j = 0, \dots, k-1.$$

The very same calculation as before then shows that

$$\int \overline{\psi_k}(-\Delta\varphi + V\varphi - E_k\varphi) = 0 \tag{6.9}$$

for all $\varphi \in C_0^\infty(\mathbb{R}^n)$ s.t. $\langle \varphi, \psi_j \rangle = 0$, $j = 1, \dots, k-1$. The claim would follow if we can show (6.9) for all $\varphi \in C_0^\infty(\mathbb{R}^n)$ by the definition of weak solutions. In other words defining $T := (-\Delta + V - E_k)\psi_k \in \mathcal{D}'(\mathbb{R}^n)$ as a distribution we know that

$$T(\varphi) = 0 \quad \forall\, \varphi \in C_0^\infty(\mathbb{R}^n) \text{ s.t. } \langle \varphi, \psi_j \rangle = 0,\ j = 1, \dots, k-1$$

and we want to show that $T \equiv 0$. By Lemma 2.14 there exist $c_0, \dots, c_{k-1} \in \mathbb{C}$ s.t.

$$T = \sum_{i=0}^{k-1} c_i \psi_i$$

and thus it remains to show that $c_i = 0$ for $i = 0, \dots, k-1$. The proof relies on testing T against $\overline{\psi_i}$, $i < k$ first and then testing the weak equation for ψ_i against $\overline{\psi_k}$. Although ψ_i, ψ_k are not valid test functions, we will outline this idea first and then make it rigorous.

Formally

By orthogonality of the ψ_i's we have $T(\overline{\psi_i}) = c_i$ and thus

$$c_i = \int \overline{\psi_i}(-\Delta + V - E_k)\psi_k = \int \overline{D\psi_i} D\psi_k + \int V\overline{\psi_i}\psi_k - E_k \cdot 0.$$

On the otherhand by assumption ψ_i fulfills $(-\Delta + V - E_i)\psi_i$ weakly and therefore

$$0 = \int \overline{\psi_k}(-\Delta + V - E_i)\psi_i = \int \overline{D\psi_k} D\psi_i + \int V\overline{\psi_k}\psi_i - E_i \cdot 0$$

and after taking a complex conjugate we conclude $c_i = 0$. #

Rigorously

The proof is obtained as before, but now in place of the two equations above we approximate ψ_i by a sequence $\psi_i^{(n)} \in C_0^\infty(\mathbb{R}^n)$ with $\|\psi_i^{(n)}\| = 1$ and show that for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ s.t. for all $n \geqslant N$

$$|T(\overline{\psi}_i^{(n)}) - c_i| < \varepsilon \text{ and } |T(\overline{\psi}_i^{(n)})| < \varepsilon \quad \text{for} \quad i = 0, \dots, k-1$$

and thus the claim for $\varepsilon \to 0$.

For the first part, by writing $\psi_i^{(n)} = \psi_i + (\psi_i^{(n)} - \psi_i)$ and using Cauchy-Schwarz's inequality we have

$$\begin{aligned} |T(\overline{\psi}_i^{(n)}) - c_i| &= |\sum_{j=0}^{k-1} c_j \langle \psi_i^{(n)}, \psi_j \rangle - c_i| \\ &= |c_i + \sum_{j=0}^{k-1} c_j \langle \psi_i^{(n)} - \psi_i, \psi_j \rangle - c_i| \\ &\leqslant k \max_{l=0,\dots,k-1} |c_l| \|\psi_j\| \|\psi_i^{(n)} - \psi_i\| < \varepsilon \end{aligned}$$

for N large enough.

For the second part we approximate ψ_k by $\psi_k^{(n)} \in C_0^\infty(\mathbb{R}^n)$ s.t. $\|\psi_k^{(n)}\| = 1$ and obtain

$$\left|T(\overline{\psi}_i^{(n)})\right| = \left|T(\overline{\psi}_i^{(n)}) - \overline{\langle \psi_k^{(n)}, (-\Delta + V - E_i)\psi_i \rangle}\right|,$$

where we used that ψ_i is a weak solution of $(-\Delta + V - E_i)\varphi = 0$. Therefore

$$\begin{aligned}\left|T(\overline{\psi}_i^{(n)})\right| &= \left|\int \overline{\psi}_i^{(n)}(-\Delta + V - E_k)\psi_k - \psi_k^{(n)}(-\Delta + V - E_i)\overline{\psi}_i\right| \\ &\leqslant \underbrace{\left|\int \overline{D\psi}_i^{(n)} D\psi_k - D\psi_k^{(n)}\overline{D\psi}_i\right|}_{(I)} + \underbrace{\left|\int V\left(\overline{\psi}_i^{(n)}\psi_k - \psi_k^{(n)}\overline{\psi}_i\right)\right|}_{(II)} \\ &\quad + \underbrace{\left|E_k\int \overline{\psi}_i^{(n)}\psi_k - E_i\int \psi_k^{(n)}\overline{\psi}_i\right|}_{(III)}.\end{aligned}$$

Rewriting the first summand (I) and using the Cauchy-Schwarz inequality we obtain

$$\begin{aligned}|I| &= \left|\int (\overline{D\psi}_i^{(n)} - \overline{D\psi}_i)D\psi_k - (\psi_k^{(n)} - D\psi_k)\overline{D\psi}_i\right| \\ &\leqslant \|\psi_i^{(n)} - \psi_i\|_{H^1}\|\psi_k\|_{H^1} + \|\psi_k^{(n)} - \psi_k\|_{H^1}\|\psi_i\|_{H^1}.\end{aligned}$$

For the second summand (II) we use the same method as in claim (ii) of the previous theorem. Thus we split $V = V_M + (V - V_M)$, where $V_M(x) := V(x)\mathbb{1}_{\{x:|V(x)|\leqslant M\}}$, and for M large enough we have $V - V_M \in L^{\frac{n}{2}}(\mathbb{R}^n)$ with $V - V_M \overset{M\to\infty}{\longrightarrow} 0$ in $L^{\frac{n}{2}}(\mathbb{R}^n)$. Therefore we can estimate

$$\begin{aligned}|II| &\leqslant |\int V(\psi_k^{(n)} - \psi_k)\overline{\psi}_i| + |\int V(\overline{\psi}_i^{(n)} - \overline{\psi}_i)\psi_k| \\ &\leqslant M\|\psi_k^{(n)} - \psi_k\|\|\psi_i\| + (i \leftrightarrow k) \\ &\quad + \left|\int (V - V_M)\left(|\psi_i^{(n)} - \psi_i|^2 + |\psi_k|^2\right)\right| + (i \leftrightarrow k),\end{aligned}$$

where $(i \leftrightarrow k)$ stands for the same expression with i and k interchanged. By Sobolev's inequality the second summand can be bounded by

$$\begin{aligned}\left|\int (V - V_M)\left(|\psi_i^{(n)} - \psi_i|^2 + |\psi_k|^2\right)\right| &\leqslant \|V - V_M\|_{\frac{n}{2}}(\|D\psi_i^{(n)} - D\psi_i\|^2 + \|D\psi_k\|^2) \\ &\leqslant C\|V - V_M\|_{\frac{n}{2}},\end{aligned}$$

where C is a fixed finite constant depending on $\|\psi_k\|_{H^1}$ and $\|\psi_i\|_{H^1}$. Now for any given $\varepsilon > 0$ we choose M large enough s.t. $C\|V - V_M\|_{\frac{n}{2}} \leqslant \varepsilon$ and then N large enough s.t. $M\|\psi_k^{(n)} - \psi_k\| \leqslant \varepsilon$ for all $n \geqslant N$.

For the third summand (III) we estimate

$$\begin{aligned}|III| &= |E_k\langle\psi_i^{(n)} - \psi_i + \psi_i, \psi_k\rangle - E_i\langle\psi_i, \psi_k^{(n)} - \psi_k + \psi_k\rangle| \\ &\leqslant |E_k\langle\psi_i^{(n)} - \psi_i, \psi_k\rangle - E_i\langle\psi_i, \psi_k^{(n)} - \psi_k\rangle| \\ &\leqslant |E_k|\|\psi_i^{(n)} - \psi_i\|\|\psi_k\| + |E_i|\|\psi_i\|\|\psi_k^{(n)} - \psi_k\| < \varepsilon\end{aligned}$$

for N large enough. Finally letting $\varepsilon \to 0$ completes the proof of (6.8). #

For the first part of the theorem we apply the same method as in the existence of the ground state, where now ψ_k plays the role of ψ. However we now have to choose our energy minimizing sequence from the proper space, i.e. we choose a minimizing sequence $\psi_k^{(n)} \in H^1(\mathbb{R}^n)$ with $\|\psi_k^{(n)}\| = 1$ and $\langle \psi_k^{(n)}, \psi_i \rangle = 0$ for $i = 0, \dots, k-1$ s.t. $\mathcal{E}(\psi_k^{(n)}) \to E_k$ as $n \to \infty$. The properties of this sequence are as in the previous theorem and we can select a weakly convergent subsequence. Thereby we note that the condition $\langle \psi_k^n, \psi_i \rangle = 0$ survives the weak limit, i.e.

$$\psi_k^{(n)} \rightharpoonup \psi_k,\ \langle \psi_k^{(n)}, \psi_i \rangle = 0 \quad \Longrightarrow \quad \langle \psi_k, \psi_i \rangle = 0.$$

The proof proceeds analogously and shows the existence of ψ_k as a minimizer with $\psi_k \perp \psi_0, \dots, \psi_{k-1}$ by the above note. # □

6.2 Uniqueness of eigenfunctions/eigenspaces

Lemma 6.5. *(Convexity inequality for gradients)*
Assume $f \in H^1(\mathbb{R}^n)$ then also $|f| \in H^1(\mathbb{R}^n)$ and further

$$\int |D|f||^2 \leqslant \int |Df|^2 \tag{6.10}$$

and if $|f| > 0$, equality holds iff there exists $\lambda \in \mathbb{C}^*$ s.t. $f = \lambda |f|$.

Theorem 6.6. *(Uniqueness of the ground state)*
Under the assumptions of Theorem 6.2 let $\psi_0 \in H^1(\mathbb{R}^n)$ be the ground state. Assume further that $V_+ \in L^p_{loc}(\mathbb{R}^n)$ for $p > \frac{n}{2}$. Then the ground state is unique up to a constant phase and can be chosen strictly positive.

Proof. Recalling the definition of the energy functional $\mathcal{E}$ we have by the previous lemma

$$\mathcal{E}(|\psi|) = \int_{\mathbb{R}^n} |D|\psi||^2 + \int_{\mathbb{R}^n} V|\psi|^2 \leqslant \mathcal{E}(\psi)$$

for all $\psi \in H^1(\mathbb{R}^n)$. Thus if ψ, $\varphi \in H^1(\mathbb{R}^n)$ are two linear independent ground states, then $|\psi| \geqslant 0$ and $|\varphi| \geqslant 0$ are also ground states. We may assume w.l.o.g. $\psi \perp \varphi$.

We now claim that even $|\psi|, |\varphi| > 0$. This along with the uniqueness shows that the ground state can be chosen strictly positive. Given the claim we use the case of equality in (6.10) to conclude

$$\psi = \lambda|\psi|,\ \varphi = \mu|\varphi| \quad \text{for some} \quad \lambda,\, \mu \neq 0$$

and thus

$$0 = \langle \psi, \psi \rangle = \int \overline{\psi}\varphi = \bar{\lambda}\mu \int |\psi||\varphi| \neq 0,$$

which is a contradiction and shows the uniqueness of the ground state up to a phase.

Proof of the claim

We want to apply the weak Harnack inequality (Theorem 4.9) to the eigenfunction equation to receive a lower bound on $\inf|\psi|$. Since $u := |\psi|$ minimizes the energy it fulfills the eigenfunction equation weakly, i.e.

$$\int DvDu + \int (V_+ - V_- - E_0)uv = 0 \quad \forall\, v \in C_0^1(\mathbb{R}^n),$$

the proof is as in (6.2). In particular for any compact subset $\Omega \Subset \mathbb{R}^n$ we have

$$\int DvDu + \int (V_+ - V_- - E_0)uv = 0 \quad \text{for all } 0 \leqslant v \in C_0^1(\Omega)$$

and since $u := |\psi| \geqslant 0$ in particular

$$\int DvDu + \int (V_+ - E_0)uv \geqslant 0 \quad \text{for all } 0 \leqslant v \in C_0^1(\Omega).$$

Thus u is a supersolution of $Lu := -\Delta u + cu = 0$ with $c = V_+ - E_0 \in L^p(\Omega)$, $p > \frac{n}{2}$ by assumption. Since $|\psi| \not\equiv 0$ we can find a point $y_0 \in \mathbb{R}^n$ and a ball $B_{2R}(y_0) \subset \mathbb{R}^n$ s.t. $\|\psi\|_{1;B_{2R}(y_0)} > 0$ and thus by Theorem 4.9

$$\inf_{B_R(y_0)} |\psi| \geqslant \frac{1}{CR^n}\|\psi\|_{1;B_{2R}(y_0)} > 0.$$

We now choose a second point $y_1 \in \partial B_{2R}(y_0)$ and a ball $B_{2R}(y_1) \subset \mathbb{R}^n$. Since it contains $B_R(y_0)$ also $\|\psi\|_{1;B_{2R}(y_1)} > 0$. Iterating this process shows $|\psi| > 0$ in the sense of Definition 2.6. □

Remark 6.7. Eigenfunctions with the same eigenvalue E form a linear subspace. Further the above recursion stops only when $E_k = 0$ and thus eigenvalues may only accumulate at 0 and each eigenvalue $E_k < 0$ has finite multiplicity.

Proof. The first part follows immediately from the linearity of $H = -\Delta + V$. For the second part let us assume the contrary, i.e. that there exists $E_k < 0$ with infinite multiplicity. Then we can find a L^2-orthonormal sequence $\psi_k, \psi_{k+1}, \dots$ with

$$(-\Delta + V)\psi_j = E_k\psi_j \quad \forall\, j \geqslant k.$$

By Lemma 6.1 we infer $\sup_{j\geqslant k}\|\psi_j\|_{H^1} < \infty$ and thus by Theorem 2.18 we can select a weakly convergent subsequence

$$\psi_{j_l} \rightharpoonup \psi \text{ in } H^1, \quad \psi_{j_l} \rightharpoonup \psi \text{ in } L^2$$

and thus $\psi = 0$, since it is the weak limit of an orthonormal sequence. Further by claim (*ii*) of Theorem 6.2 we have $\int V|\psi|^2 = \lim_{l\to\infty} \int V|\psi_{j_l}|^2$ and therefore

$$E_k = \lim_{l\to\infty} \mathcal{E}(\psi_{j_l}) = \lim_{l\to\infty} \int |D\psi_{j_l}|^2 + \int V|\psi_{j_l}|^2 \geqslant 0,$$

which contradicts our assumption $E_k < 0$. □

7 Regularity of eigenfunctions

In this chapter we develop a splitting of eigenfunctions into a more regular and a singular part. The idea of this splitting is obtained from [HHØ01], where it was performed for the potential $V(x) = |x|^{-1}$ and in a many-body setup.

Theorem 7.1. *(Hölder continuity of weak solutions)*
Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geqslant 3$, $\Omega' \Subset \Omega$ a subdomain and let $u \in H^1(\Omega)$ be a weak solution of

$$\Delta u + b_i D_i u + cu = g \quad in \quad \Omega.$$

Then the following holds:

(a) If $\mathbf{b}$, c, $g \in L^\infty(\Omega)$ then $u \in C^{1,\alpha}(\Omega')$ for all $\alpha \in (0,1)$ and we have

$$|u|_{1,\alpha;\Omega'} \leqslant C(|u|_{0;\Omega} + |g|_{0;\Omega}),$$

where $C = C(n,p,M,\operatorname{dist}(\Omega',\partial\Omega))$ with $\max\{1, \|\mathbf{b}\|_{\infty;\Omega}, \|c\|_{\infty;\Omega}\} \leqslant M$.

(b) If $p > n$ and $\mathbf{b}$, c, $g \in L^p(\Omega)$ then $u \in C^{1,\alpha}(\Omega')$ with $\alpha = 1 - \frac{n}{p}$ and we have

$$|u|_{1,\alpha;\Omega'} \leqslant C(|u|_{0;\Omega} + \|g\|_{p;\Omega}),$$

where $C = C(n,p,M,\operatorname{dist}(\Omega',\partial\Omega))$ with $\max\{1, \|\mathbf{b}\|_{p;\Omega}, \|c\|_{p;\Omega}\} \leqslant M$.

(c) If $n \geqslant p > \frac{n}{2}$ and c, $g \in L^p(\Omega)$, $\mathbf{b} \in L^{2p}(\Omega)$ then $u \in C^{0,\alpha}(\Omega')$ and

$$|u|_{0,\alpha;\Omega'} \leqslant C(|u|_{0;\Omega} + \|g\|_{p;\Omega}),$$

with

i. for $\mathbf{b} \equiv 0$: $\alpha = 2 - \frac{n}{p}$ and $C = C(n,p,M,dist(\Omega',\partial\Omega))$ with $\max\{1, \|c\|_{p;\Omega}\} \leqslant M$.

ii. for $\mathbf{b} \not\equiv 0$: $\alpha = \alpha(n,p,M,dist(\Omega',\partial\Omega)) > 0$ and $C = C(n,p,M,dist(\Omega',\partial\Omega))$ with $\max\{1, \|\mathbf{b}\|_{2p;\Omega}, \|c\|_{p;\Omega}\} \leqslant M$.

(d) If $\mathbf{b}$, g, $c \in C^{0,\alpha}(\Omega)$ then $u \in C^{2,\alpha}(\Omega')$ and we have

$$|u|_{2,\alpha;\Omega'} \leqslant C(|u|_{0;\Omega} + |g|_{0,\alpha;\Omega}),$$

where $C = C(n,p,M,dist(\Omega',\partial\Omega))$ with $\max\{1, \|\mathbf{b}\|_{0,\alpha;\Omega}, \|c\|_{0,\alpha;\Omega}\} \leqslant M$.

Proof. By Theorem 5.1 we may assume that u is locally bounded. The proofs of (b) and (c) i. are given in Theorem 4.6 and Theorem 4.7 respectively. The proof of (c) ii. is given in Theorem 4.13. #

Proof of (a)
Choose any $0 < \alpha < 1$ and set $p := \frac{n}{1-\alpha}$. Since Ω is bounded we have $\mathbf{b}$, c, $g \in L^p(\Omega)$, where the norms can be controlled explicitly by their corresponding $|\cdot|_{0;\Omega}$-norms. Thus we can apply (b) to arrive at the desired result. #

Proof of (d)
By rewriting the PDE to

$$\Delta u = g - b_i D_i u - cu$$

we infer that $u \in C^{2,\alpha}(\Omega')$ since by part (a) the RHS is in $C^\alpha(\Omega')$. The explicit bound is a consequence of Th. 6.2 in [GT01], where we estimate $|u|^*_{2,\alpha;\Omega}$ from below and $|g|^{(2)}_{0,\alpha;\Omega}$ from above according to Remark 4.4. # □

Theorem 7.2. *(Hölder continuity of eigenfunctions)*
Let $\psi \in H^1(\mathbb{R}^n)$, $n \geqslant 3$ be a weak solution of the eigenfunction equation

$$-\Delta\psi - \frac{1}{|x|^s}\psi = E\psi, \quad E < 0. \tag{7.1}$$

Define

$$F\colon \mathbb{R}^n \to \mathbb{R},\ x \mapsto -C(s,n)|x|^{2-s}.$$

Then ψ can be represented as

$$\psi = e^F \varphi$$

with

(a) If $0 < s < 1$: $\varphi \in C^{2,\alpha}_{loc}(\mathbb{R}^n)$ and $\psi \in C^{1,\alpha}_{loc}(\mathbb{R}^n)$ for all $\alpha \in (0, 1-s)$.

(b) If $s = 1$: $\varphi \in C^{1,\alpha}_{loc}(\mathbb{R}^n)$ for all $\alpha \in (0,1)$ and $\psi \in C^{0,1}_{loc}(\mathbb{R}^n)$.

(c) If $1 < s < 3/2$: $\varphi \in C^{1,\alpha}_{loc}(\mathbb{R}^n)$ for all $\alpha \in (0, 3-2s)$ and $\psi \in C^{0,2-s}_{loc}(\mathbb{R}^n)$.

(d) If $3/2 \leqslant s < 2$: $\varphi \in C^{0,\alpha}_{loc}(\mathbb{R}^n)$ for an $\alpha(n,s,\Omega) \geqslant 2-s$ depending on $\Omega \Subset \mathbb{R}^n$ and further we have $\psi \in C^{0,2-s}_{loc}(\mathbb{R}^n)$.

Proof. We set $C(s,n) = \frac{1}{(2-s)(n-s)}$ and $V(x) := -|x|^{-s}$. Therefore $V \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ for all $p < \frac{n}{s}$ and by a simple computation

$$\Delta F = \sum_{i=1}^n \partial^2_{x_i} F = -\frac{n(2-s)|x|^{-s} - (2-s)s|x|^{-s-2}|x|^2}{(2-s)(n-s)} = V.$$

Proof of (*a*)
Since $\frac{n}{s} > n$ we can apply Theorem 7.1 (b) and obtain $F \in C^{1,\alpha}$ with $\alpha < 1-s$ and

thus $D_iF \in C^\alpha$. By substituting $\psi = e^F\varphi$ and using the eigenfunction equation (7.1) we have

$$V\psi - E\psi = \Delta\psi = \Delta F\psi + e^F|DF|^2\varphi + 2e^F D_iFD_i\varphi + e^F\Delta\varphi.$$

After using $\Delta F = V$ and multiplying by e^F we obtain a PDE for φ

$$\Delta\varphi + 2D_iFD_i\varphi + \left(|DF|^2 - E\right)\varphi = 0. \tag{7.2}$$

Applying Theorem 7.1(d) now shows that $\varphi \in C^{2,\alpha}_{\text{loc}}$ for all $\alpha < 1-s$. Finally $\psi \in C^{1,\alpha}_{\text{loc}}$ follows from Theorem 7.1(b) with $c \equiv V$, $\mathbf{b} \equiv 0$. #

Proof of (c)

Again we substitute $\psi = e^F\varphi$ and then we want to apply Theorem 7.1(b) to (7.2). I.e. we have to find the maximal q s.t. $D_iF \in L^q_{\text{loc}}(\mathbb{R}^n)$ and $|DF|^2 \in L^q_{\text{loc}}(\mathbb{R}^n)$. Therefore let $0 \in \Omega \Subset \mathbb{R}^n$. Then

$$\begin{aligned}\int_\Omega |D_iF|^{2q} &= (2-s)^{2q}\int_\Omega |x_i|^{2q}|x|^{-2qs}\,\mathrm{dx}\\ &\leqslant C\int_\Omega |x|^{2q(1-s)}\,\mathrm{dx}\\ &< \infty \Leftrightarrow\ q < \frac{n}{2(s-1)}.\end{aligned}$$

Thus, since $s < \frac{3}{2}$ and therefore $q > n$ we conclude by Theorem 7.1(b) that $\varphi \in C^{1,\alpha}_{\text{loc}}(\mathbb{R}^n)$ with

$$\alpha = 1 - \frac{n}{q} < 1 - 2(s-1) = 3 - 2s.$$

Expanding $e^F = 1 + \mathcal{O}(|x|^{2-s})$ shows that e^F and therefore also $\psi \in C^{0,2-s}_{\text{loc}}(\mathbb{R}^n)$. #

Proof of (b)

The proof is analogous to (c), where we now apply Theorem 7.1(a) in place of Theorem 7.1(b). #

Proof of (d)

The proof goes analogously to (c), except that now we apply Theorem 7.1(c) *ii.* to (7.2). The estimate for ψ is as above. □

Remark 7.3. The previous theorem is only of interest for a neighborhood of $x_0 = 0$, since away from zero the potential and thus the eigenfunction is even real analytic.

Proof. Away from $x_0 = 0$ the potential is certainly a $C^{0,\alpha}$-function for any $\alpha > 0$. Thus by Theorem 7.1(d) we have $u \in C^{2,\alpha}_{\text{loc}}$ and by iterating this procedure we can eventually conclude $u \in C^\infty_{\text{loc}}$. For the full details we refer to e.g. Ch. 10 in [LL01] or Th. 6.3.3 in [Eva10]. □

We now want to generalize the preceding theorem to an arbitrary potential $V \in L^p_{\text{loc}}$.

Theorem 7.4. *(General Hölder continuity of eigenfunctions)*
Let $\psi \in H^1(\mathbb{R}^n)$, $n \geqslant 3$ be a weak solution of the eigenfunction equation

$$-\Delta\psi + V\psi = E\psi, \quad E < 0, \tag{7.3}$$

where $V \in L^p_{loc}(\mathbb{R}^n)$. Let F be the distributional solution of

$$\Delta F = V.$$

Then ψ can be represented as

$$\psi = e^F\varphi$$

with

(a) If $p > n$: $\varphi \in C^{2,\alpha}_{loc}(\mathbb{R}^n)$ *and* $\psi \in C^{1,\alpha}_{loc}(\mathbb{R}^n)$ *for all* $\alpha \in (0, 1 - \frac{n}{p}]$.

(b) If $p = n$: $\varphi \in C^{1,\alpha}_{loc}(\mathbb{R}^n)$ *and* $\psi \in C^{\alpha}_{loc}(\mathbb{R}^n)$ *for all* $\alpha \in (0, 1)$.

(c) If $\frac{2}{3}n < p \leqslant n$: $\varphi \in C^{1,\alpha}_{loc}(\mathbb{R}^n)$ *for all* $\alpha \in (0, 3 - 2n/p]$ *and* $\psi \in C^{0,2-\frac{n}{p}}_{loc}(\mathbb{R}^n)$.

(d) If $\frac{n}{2} < p \leqslant \frac{2}{3}n$: $\varphi \in C^{0,\alpha}_{loc}(\mathbb{R}^n)$ *for an* $\alpha(n, s, \|V\|_{p;\Omega}) \geqslant 2 - s$ *depending on local properties of V on $\Omega \Subset \mathbb{R}^n$ and further we have* $\psi \in C^{0,2-\frac{n}{p}}_{loc}(\mathbb{R}^n)$.

Proof. The proof proceeds mostly analogously to the proof of Theorem 7.2.

Proof of (a)
Since F is the solution of $\Delta F = V$ we know by Theorem 7.1(b) that $F \in C^{1,\alpha}$ with $\alpha = 1 - \frac{n}{p}$ and thus $D_iF \in C^\alpha$. By substituting $\psi = e^F\varphi$ and using the eigenfunction equation (7.3) we obtain

$$V\psi - E\psi = \Delta\psi = \Delta F\psi + e^F|DF|^2\varphi + 2e^F D_iF D_i\varphi + e^F\Delta\varphi.$$

After using $\Delta F = V$ and multiplying by e^F we obtain a PDE for φ

$$\Delta\varphi + 2D_iFD_i\varphi + \left(|DF|^2 - E\right)\varphi = 0. \tag{7.4}$$

Applying Theorem 7.1(d) now shows that $\varphi \in C^{2,\alpha}_{\text{loc}}$ for all $\alpha = 1 - \frac{n}{p}$. Finally $\psi \in C^{1,\alpha}_{\text{loc}}$ follows from Thereom 7.1(b) with $c \equiv V$, $\mathbf{b} \equiv 0$. #

Proof of (c)
Again we substitute $\psi = e^F\varphi$ and by Lemma 2.30 we have $|D_iF|^2 \in L^{p'/2}_{\text{loc}}(\mathbb{R}^n)$ with $\frac{p'}{2} = \frac{1}{2}\frac{np}{n-p} > n$. Thus we can apply Theorem 7.1(b) to (7.4) and obtain $\alpha = 1 - 2n/p' = 3 - 2n/p$. Further from Theorem 7.1(c) *i.* we conclude that $\psi \in C^{0,2-\frac{n}{p}}_{\text{loc}}(\mathbb{R}^n)$. #

The proof of (b) is analogous.

Proof of (d)

The proof goes analogously to (c), except that we now apply Theorem 7.1(c) *ii.* to (7.4). The estimate for ψ is as above. $\square$

Remark 7.5. According to Theorem 6.4, the assumptions of Theorem 7.4 can be fulfilled by assuming $V \in L^{\frac{n}{2}}(\mathbb{R}^n) + L^\infty(\mathbb{R}^n)$ vanishing at infinity.

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