

Relativizing characterizations of Anosov subgroups, I

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Abstract

We propose several common extensions of the classes of Anosov subgroups and geometrically finite Kleinian groups among discrete subgroups of semisimple Lie groups. We relativize various dynamical and coarse geometric characterizations of Anosov subgroups given in our earlier work, extending the class from intrinsically hyperbolic to relatively hyperbolic subgroups. We prove implications and equivalences between the various relativizations.

Contents

1 Introduction

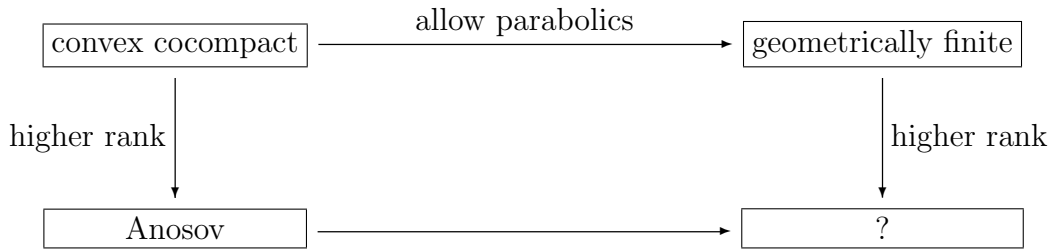
The notion of geometric finiteness was first introduced by Ahlfors [?] in the context of Kleinian group actions on hyperbolic 3-space \mathbb{H}^3 . It was originally defined via the existence of finite-sided convex fundamental polyhedra. A few years later, Beardon and Maskit [?] gave a dynamical characterization of geometrically finite groups in terms the action on the limit set, now called the *Beardon-Maskit condition*. Subsequently, alternative characterizations were given by Marden [?], Thurston [?] and many others, see e.g. [?] and [?].

While Ahlfors' original definition turned out to be unsuitable for hyperbolic space of dimension ≥ 4 , the Beardon-Maskit condition worked well in the context of discrete subgroups of rank one Lie groups and, more generally, of discrete groups of isometries acting on negatively pinched Hadamard manifolds [?], and was shown to be equivalent to a variety of other properties. The Beardon-Maskit condition remains meaningful even in the purely dynamical setting of convergence actions on topological spaces, something which we are exploiting in our work.

A particularly nice subclass of geometrically finite Kleinian groups is formed by convex cocompact subgroups which are distinguished by the absence of parabolic elements. They are intrinsically word hyperbolic, whereas a general geometrically finite Kleinian group inherits a natural structure as a relatively hyperbolic group, the peripheral structure given by the collection of maximal parabolic subgroups.

The notion of convex cocompact Kleinian groups was extended to discrete subgroups of higher rank Lie groups, starting with the notion of *Anosov subgroups* [?], see also [?]. These were originally defined in terms of their dynamics on flag manifolds. We subsequently gave various characterizations of Anosov subgroups in terms of their coarse geometry, dynamics and topology along with a simplification of their original definition [?, ?, ?, ?], see also [?, ?].

As convex cocompact subgroups, Anosov subgroups are intrinsically word hyperbolic, and as the former contain no parabolics, the latter contain no strictly parabolic elements,¹ e.g. no unipotents. Our goal is to find a common extension of the classes of Anosov and geometrically finite subgroups, that is, to complete the diagram:



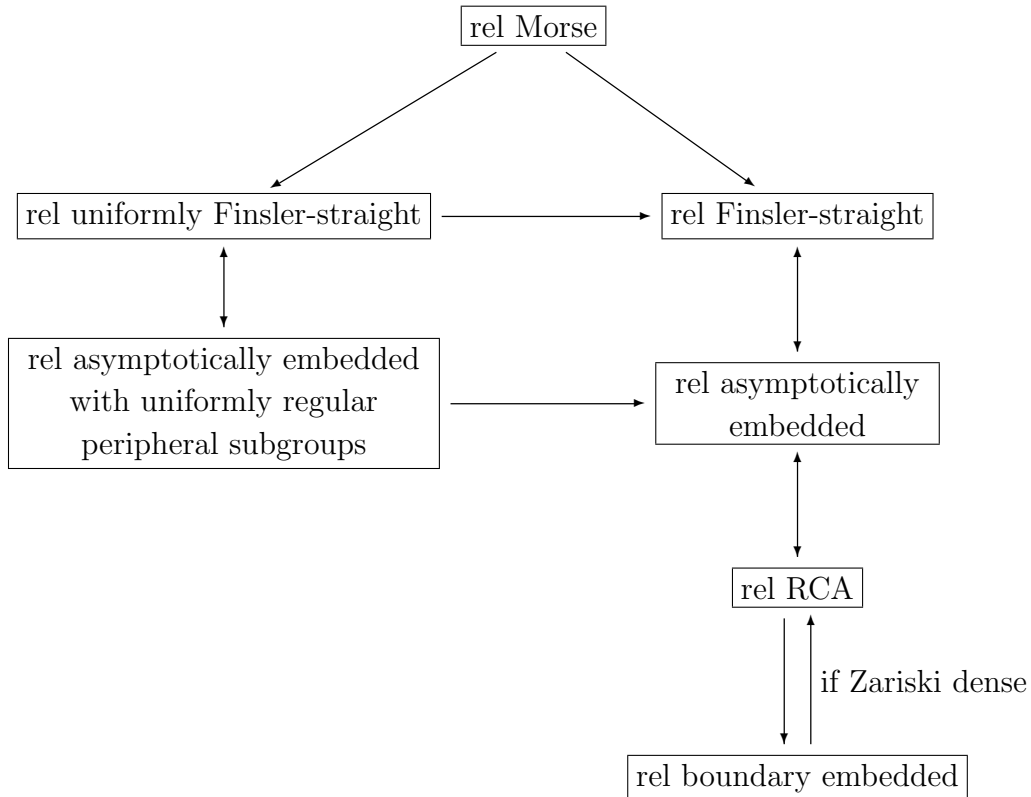
We consider subgroups which are relatively hyperbolic as abstract groups and extend to this more general setting various characterizations of Anosov subgroups studied in our earlier papers.

The Beardon-Maskit condition has the most straightforward generalization, namely by requiring that the discrete subgroup acts on its limit set like a relatively hyperbolic group. This leads to the notions *relatively asymptotically embedded* and *relatively RCA*, see Definitions ?? and ??, which are equivalent in view of Yaman’s dynamical characterization of relatively hyperbolic groups. The intrinsic relatively hyperbolic structure of these discrete subgroups can be read off the dynamics on the limit set and is therefore uniquely determined.

Also our coarse geometric characterization of Anosov subgroups as Morse subgroups, that intrinsic geodesics in the subgroup are extrinsically perturbations of Finsler geodesics in the symmetric space, generalizes naturally. This leads to the notions of *relatively Morse* and *relatively Finsler-straight*, see Definitions ?? and ??.

All these relative notions agree in rank one with geometric finiteness (see Corollary ??). The main result of the paper establishes relations (implications and equivalences) between them in higher rank. It is summarized in Theorem ?? and the diagram:

¹That is, non-elliptic elements with zero infimal displacement.



The most difficult implications are between *relatively Finsler straight* and *relatively asymptotically embedded*, connecting coarse geometry and dynamics, and their analogues in the uniformly regular case. They are proven in section ??, which in turn relies on coarse geometric results about general (non-equivariant) Finsler straight maps established in section ??.

Examples of classes of discrete subgroups satisfying the relative conditions discussed in the paper are:

1. subgroups preserving a rank one symmetric subspace and acting on it in a geometrically finite fashion (Theorem ?? and Example ??)
2. discrete groups of projective transformations acting with finite covolume on strictly convex solids in \mathbb{R}^n (studied in [?])
3. certain families of discrete subgroups of $PGL(3, \mathbb{R})$ not preserving properly convex domains in $\mathbb{R}P^2$ (described in [?] and [?])
4. positive representations (into split semisimple Lie groups) of fundamental groups of punctured surfaces (appearing in [?])
5. certain free products of opposite unipotent subgroups (see [?])
6. small relative deformations (see [?])

The discussion of the relative notions introduced in this paper will be continued in [?].

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2 Preliminaries and notation

2.1 Metric spaces

We will be using the notation xy for a geodesic segment in a metric space connecting points x and y . Similarly, in a geodesic metric space Y which is Gromov hyperbolic or CAT(0), we will use the notation $y\xi$ for a geodesic ray in Y emanating from y and asymptotic to a point ξ in the visual (ideal) boundary $\partial_\infty Y$ of Y . For two distinct ideal boundary points $\eta_\pm \in \partial_\infty Y$ of a Gromov-hyperbolic space Y we will use the notation $\eta_- \eta_+$ for a geodesic in Y asymptotic to η_\pm . A similar notation will be used for *Finsler geodesics* in symmetric spaces: $x\tau$, $\tau_- \tau_+$ will denote a Finsler geodesic ray/line; see section ??.

We will use the notation $B(a, R)$ for the open R -ball with center a in a metric space, and the notation $N_R(A)$ for the open R -neighborhood of subsets A , where $R > 0$. The subsets $N_R(A)$ are called *tubular neighborhoods* of A .

A metric space is called *taut* if every point lies at distance $\leq R$ from a geodesic line for some uniform constant R .

Two subsets in a metric space are called *D-separated* if their infimal distance is $\geq D$.

We call a subset of a metric space *s-spaced* if its distinct points have pairwise distance $\geq s$, and we call a map into a metric space *s-spaced* if it is injective and its image is *s-spaced*.

A sequence (x_n) in a metric space is said to *diverge to infinity* if $\lim_{n \rightarrow \infty} d(x_1, x_n) = \infty$; we will refer to such (x_n) as a *divergent sequence*.

A map between metric spaces is called *metrically proper* if it sends divergent sequences to divergent sequences, equivalently, if the preimages of bounded subsets are bounded.

2.2 Group actions

For an action $\Gamma \curvearrowright X$ of a group Γ on a set X we let $\Gamma_x < \Gamma$ denote the stabilizer of an element $x \in X$. The associated orbit map is defined by

$$o_x : \Gamma \rightarrow X, \quad \gamma \mapsto \gamma x.$$

If $\Gamma \curvearrowright Y$ is another Γ -action and $y \in Y$ is a point such that $\Gamma_y \leq \Gamma_x$, then there is a well-defined Γ -equivariant map of orbits

$$o_{x,y} : \Gamma y \rightarrow \Gamma x, \quad \gamma y \mapsto \gamma x.$$

2.3 Convergence actions

A continuous action $\Gamma \curvearrowright Z$ of a discrete group Γ on a compact metrizable topological space Z is called a (*discrete*) *convergence action* if for each sequence (γ_n) of pairwise distinct elements in Γ there exists a pair of points $z_-, z_+ \in Z$ such that, after extraction, the sequence (γ_n) converges to z_+ uniformly on compacts in $Z - \{z_-\}$. Note that all actions on spaces with at most two points are convergence, except actions of infinite groups on the empty space. Also, all actions of finite groups are convergence.

The *limit set* $\Lambda = \Lambda(\Gamma) \subseteq Z$ consists of all points which occur as such limits z_+ . The limit set is Γ -invariant and compact. If $|\Lambda| \geq 3$, then it is perfect and the action $\Gamma \curvearrowright \Lambda$ has finite kernel and is minimal.² If $|\Gamma| = +\infty$, then $\Lambda \neq \emptyset$, and if $|\Gamma| < +\infty$, then $\Lambda = \emptyset$.

Elements of convergence groups fall into three classes: An element is called *hyperbolic* if it has infinite order and exactly two fixed points, *parabolic* if it has infinite order and exactly one fixed point, and *elliptic* if it has finite order.

A point $z \in Z$ is called a *parabolic fixed point* of Γ if it is the fixed point of some parabolic element in Γ . It then is a limit point of its stabilizer Γ_z , and it turns out that in fact $\Lambda(\Gamma_z) = \{z\}$, see [?, Lemma 2F].

The following types of limit points will be important for this paper (given the nature of the actions of relatively hyperbolic groups on their ideal boundaries):

Definition 2.1. A point $z \in \Lambda(\Gamma)$ is called a

(i) *conical limit point* for Γ if there exists a sequence (γ_n) of distinct elements in Γ and a point $w \in \Lambda - \{z\}$ such that the sequence of pairs $(\gamma_n^{-1}z, \gamma_n^{-1}w)$ does not accumulate at the diagonal of $Z \times Z$.

(ii) *bounded parabolic point* of Γ if its stabilizer $\Gamma_z < \Gamma$ acts on $\Lambda(\Gamma) - \{z\}$ properly discontinuously and cocompactly.

(ii') *bounded parabolic fixed point* of Γ if it is both a bounded parabolic point and a parabolic fixed point.

Note that the stabilizer of a bounded parabolic point z is necessarily infinite and $\Lambda(\Gamma_z) = \{z\}$. Property (ii') is strictly stronger than (ii) because the stabilizer of a bounded parabolic point can be an infinite torsion group.

If $|\Lambda(\Gamma)| = 1$ and Γ is not a torsion group, then the limit point is a bounded parabolic fixed point and not a conical limit point. If $|\Lambda(\Gamma)| = 2$, then both limit points are conical and not bounded parabolic.

²I.e. every orbit is dense.

The importance of convergence actions in our work is due primarily to two reasons:

- If Y is a proper geodesic Gromov-hyperbolic metric space and Γ is a discrete subgroup of the isometry group of Y , then the natural action of Γ on the visual boundary $\partial_\infty Y$ of Y is a convergence action, see [?].
- If Γ is a τ_{mod} -regular antipodal subgroup of the isometry group of a symmetric space of noncompact type, then the natural action of Γ on its τ_{mod} -limit set in the flag-manifold $\text{Flag}_{\tau_{mod}}$ is a convergence action, see [?].

We refer the reader to [?, ?, ?] for in-depth discussions of convergence actions.

3 Some coarse hyperbolic geometry

3.1 Gromov hyperbolic spaces

Background material on hyperbolic spaces can be found in [?], [?], [?], [?] and [?].

Let Y be a proper geodesic metric space which is δ -hyperbolic in the sense of Gromov for some $\delta \geq 0$. We denote by $\bar{Y} = Y \sqcup \partial_\infty Y$ its visual compactification.

Geodesics in Y are roughly unique in the sense that any two geodesic segments with the same endpoints have Hausdorff distance $\leq C\delta$, where C is a uniform constant (depending on the definition of δ -hyperbolicity which is used). The same holds for any two asymptotic geodesic rays with the same initial point, and for any two (at both ends) asymptotic geodesic lines.

A family of geodesics in Y is *bounded* if for some (any) point $y \in Y$ the distance of y to the geodesics in this family is uniformly bounded. The pairs (\bar{y}, \bar{y}') of endpoints of geodesics $\bar{y}\bar{y}'$ in Y lie in the set $(Y \sqcup \partial_\infty Y)^2 - \Delta_{\partial_\infty Y}$. The boundedness of a family of geodesics in a Gromov hyperbolic space is an asymptotic property of its set of pairs of endpoints:

Lemma 3.1. *Let $E_Y \subset (Y \sqcup \partial_\infty Y)^2 - \Delta_{\partial_\infty Y}$. Then the family of all geodesics in Y with pair of endpoints in E_Y is bounded if and only if E_Y is relatively compact in $(Y \sqcup \partial_\infty Y)^2 - \Delta_{\partial_\infty Y}$.*

Proof. Any bounded sequence of geodesics $\bar{y}_n \bar{y}'_n$ in Y subconverges to a geodesic $\bar{y}\bar{y}'$, and the pairs of endpoints (\bar{y}_n, \bar{y}'_n) subconverge to $(\bar{y}, \bar{y}') \in (Y \sqcup \partial_\infty Y)^2 - \Delta_{\partial_\infty Y}$. Thus the sequence of pairs (\bar{y}_n, \bar{y}'_n) does not accumulate at $\Delta_{\partial_\infty Y}$.

On the other hand, if a sequence of geodesics $\bar{y}_n \bar{y}'_n$ diverges, i.e. their distances from some base point $y \in Y$ diverge to infinity, then δ -hyperbolicity implies that there exists points $z_n \in y\bar{y}_n$ and $z'_n \in y\bar{y}'_n$ such that $z_n, z'_n \rightarrow \infty$ and the segments yz_n and yz'_n are $C\delta$ -Hausdorff close. It follows that the pairs (\bar{y}_n, \bar{y}'_n) accumulate at $\Delta_{\partial_\infty Y}$. \square

A sequence (y_n) in Y is said to converge to $\eta \in \partial_\infty Y$ *conically* if $y_n \rightarrow \eta$ and (y_n) is contained in a tubular neighborhood of a ray asymptotic to η . This is independent of the ray since any two asymptotic rays have finite Hausdorff distance. For a subset $A \subset Y$, the *conical accumulation*

set $\partial_\infty^{\text{con}} A \subset \partial_\infty Y$ consists of all points $\eta \in \partial_\infty Y$ for which there exists a sequence (a_n) in A converging to η conically.

Given a discrete isometric group action $\Gamma \curvearrowright Y$, we define its *limit set* $\Lambda = \Lambda_Y$ as the accumulation set $\partial_\infty(\Gamma y)$ in $\partial_\infty Y$ of one (equivalently, every) Γ -orbit in Y . We will use the notation $\Lambda^{\text{con}} = \Lambda_Y^{\text{con}}$ for the *conical limit set* of this action, i.e. the set $\partial_\infty^{\text{con}}(\Gamma y)$ of conical limit points of the group Γ .

Straight triples. We denote by $T(Y) := Y^3$ the space of triples of points in Y and by

$$T(Y, \partial_\infty Y) := (Y \sqcup \partial_\infty Y) \times Y \times (Y \sqcup \partial_\infty Y) \quad (3.2)$$

the space of ideal triples in the visual compactification $\bar{Y} = Y \sqcup \partial_\infty Y$ with middle point in Y .

We first define straightness for (non-ideal) triples in Y :

Definition 3.3 (Straight triple). A triple $(y_-, y, y_+) \in T(Y)$ is called *D-straight*, $D \geq 0$, if the points y_-, y and y_+ are D -close to points y'_-, y' and y'_+ , respectively, which lie in this order on a geodesic (segment).

This notion naturally extends to *ideal* triples in $T(Y, \partial_\infty Y)$: We say that a triple $(y_-, y, \eta_+) \in Y^2 \times \partial_\infty Y$ is D -straight if the points y_- and y are D -close to points y'_- and y' , respectively, such that y' lies on a geodesic ray $y'_-\eta_+$. Analogously for triples $(\eta_-, y, y_+) \in \partial_\infty Y \times Y^2$. Similarly, we say that a triple $(\eta_-, y, \eta_+) \in \partial_\infty Y \times Y \times \partial_\infty Y$ is D -straight if $\eta_- \neq \eta_+$ and the point y lies within distance D of a geodesic line $\eta_-\eta_+$.

Y is taut, if every point y is the middle point of a uniformly straight triple (η_-, y, η_+) .

Straight holey lines. We call a map $q : H \rightarrow Y$ from an arbitrary (“holey”) subset of $H \subset \mathbb{R}$ a *holey line*. If H has a minimal element, we also call q a *holey ray*. A sequence $(y_n)_{n \in \mathbb{N}}$ in Y can be regarded as a holey ray $\mathbb{N} \rightarrow Y$.

We will consider extensions to infinity $\bar{q} : \bar{H} := H \sqcup \{\pm\infty\} \rightarrow \bar{Y} = Y \sqcup \partial_\infty Y$ of holey lines $q : H \rightarrow Y$ by sending $\pm\infty$ to ideal points $\eta_\pm \in \partial_\infty Y$, and refer to \bar{q} as an *extended holey line*. Similarly, for holey rays $q : H_0 \rightarrow Y$, we will consider extensions $\bar{q} : \bar{H}_0 := H_0 \sqcup \{+\infty\} \rightarrow \bar{Y}$ by sending $+\infty$ to an ideal point $\eta \in \partial_\infty Y$, and refer to \bar{q} as an *extended holey ray*.

We carry over the notion of straightness from triples to holey lines by requiring it for all triples in the image:

Definition 3.4 (Straight holey line). A holey line $q : H \rightarrow Y$ is called *D-straight* if the triples $(q(h_-), q(h), q(h_+))$ in Y are D -straight for all $h_- \leq h \leq h_+$ in H .

Similarly, we say that an extended holey line $\bar{q} : \bar{H} \rightarrow \bar{Y}$ is D -straight if the triples $(q(h_-), q(h), q(h_+))$ in \bar{Y} are D -straight for all $h_- \leq h \leq h_+$ in \bar{H} with $h \in H$, and analogously in the ray case.

Straight holey lines are up to bounded perturbation monotonic maps into geodesics. More precisely, for a D -straight holey line $q : H \rightarrow Y$ there exists a geodesic $c \subset Y$ and a monotonic map $\bar{q} : H \rightarrow c$ which is $D'(D)$ -close to q . The holey line $\bar{q} : \bar{H} \rightarrow \bar{Y}$ extended by $\bar{q}(\pm\infty) =$

$\eta_{\pm} := c(\pm\infty)$ is then D' -straight. The ideal points η_{\pm} are unique if q is biinfinite.

Let $I \subseteq \mathbb{R}$ be an interval. We say that a function $f : I \rightarrow \mathbb{R}$

(i) has ϵ -coarsely slope s if

$$|f(t_1) - f(t_2) - s(t_1 - t_2)| \leq \epsilon$$

for all $t_1, t_2 \in I$.

(ii) is ϵ -coarsely convex if

$$\mu_1 f(t_1) + \mu_2 f(t_2) \leq f(\mu_1 t_1 + \mu_2 t_2) + \epsilon$$

for all $t_1, t_2 \in I$ and all $\mu_1, \mu_2 \geq 0$ with $\mu_1 + \mu_2 = 1$.

Note: If there exists $t_0 \in I$ such that $f|_{I \cap (-\infty, t_0]}$ has ϵ -coarsely slope -1 and $f|_{I \cap [t_0, +\infty)}$ has ϵ -coarsely slope $+1$, then f is 2ϵ -coarsely convex.

Transferring these notions, we say that a function $f : Y \rightarrow \mathbb{R}$ has ϵ -coarsely slope s or is ϵ -coarsely convex along a geodesic $c : I \rightarrow Y$ if the composition $f \circ c$ has this property.

Horofunctions and horoballs. Horofunctions coarsely measure relative distances from points at infinity. They arise most naturally as limits of normalized distance functions.

Fix an ideal point $\eta \in \partial_{\infty} Y$. Let (y_n) be a sequence in Y so that $y_n \rightarrow \eta$. After passing to a subsequence, the sequence of distance functions $d(\cdot, y_n)$ converges up to additive constants, i.e. there exists a sequence $a_n \rightarrow \infty$ of real numbers and a function $h : Y \rightarrow \mathbb{R}$ such that

$$d(\cdot, y_n) - a_n \rightarrow h$$

locally uniformly. The function h has the following properties: It is 1-Lipschitz and for every point $y \in Y$ there exists a ray $\rho_y : [0, \infty) \rightarrow Y$ asymptotic to η with initial point y along which h decays with slope $\equiv -1$, i.e. $h \circ \rho_y|_{t_1}^{t_2} = t_1 - t_2$ for all $t_1, t_2 \geq 0$. (Such a ray ρ_y arises as a sublimit of the segments yy_n .) For an arbitrary ray $\rho : [0, \infty) \rightarrow Y$ asymptotic to η , it follows that

$$|(h \circ \rho|_{t_1}^{t_2}) - (t_1 - t_2)| \leq C\delta \tag{3.5}$$

for all $t_1, t_2 \geq 0$ with a uniform constant C . We define a *horofunction at η* as a function $h : Y \rightarrow \mathbb{R}$ which satisfies (??) for all rays ρ asymptotic to η . Any two horofunctions h, h' at η coarsely differ by an additive constant, i.e.

$$|(h(y) - h(y')) - (h'(y) - h'(y'))| = |(h(y) - h'(y)) - (h(y') - h'(y'))| \leq C\delta \tag{3.6}$$

for all $y, y' \in Y$ (with a possibly different uniform constant C).³

Horofunctions are uniformly *coarsely convex*. This is a consequence of the following stronger property: For any horofunction h and segment zz' there exists a division point $y_0 \in yy'$ such

³We will often use the same letter C for a constant with the understanding that the constant may vary from inequality to inequality.

that h has $C\delta$ -coarsely slope 1 along the oriented segments y_0y and y_0y' with a uniform constant C .

We define horoballs as coarse sublevel sets of horofunctions. We say that a subset $Hb \subset Y$ is a *horoball* at $\eta \in \partial_\infty Y$ if there exists a horofunction h at η such that

$$\{h \leq 0\} \subseteq Hb \subseteq \{h \leq 10C\delta\}$$

with the constant C from formula (??). Horoballs are uniformly quasiconvex; in particular, for every $y \in Hb$, the ray $y\eta$ is contained in $N_r(Hb)$ for some uniform constant r . Moreover, the visual boundary of a horoball at η equals $\{\eta\}$. It is mostly these two properties of horoballs which will be used in this paper.

We will call the space Y itself a *horoball* if $\partial_\infty Y$ consists of a single point and the horofunctions are bounded above.

Quasiconvex subsets and hulls. We recall that the *quasiconvex hull* $\text{QCH}(A) \subset Y$ of a subset $A \subset Y$ is the union of all geodesic segments with endpoints in A . The subset A is called *r-quasiconvex* if $\text{QCH}(A) \subset N_r(A)$ and *quasiconvex* if this holds for some $r > 0$. Note that pairs of points in a quasiconvex subset can be connected by uniform quasigeodesics inside it.

As a consequence of the δ -hyperbolicity of Y , quasiconvex hulls are $C\delta$ -quasiconvex subsets, and $\partial_\infty \text{QCH}(A) = \partial_\infty A$. Both properties follow from the fact that any geodesic segment with endpoints in $\text{QCH}(A)$ is contained in the tubular $C'\delta$ -neighborhood of a geodesic segment with endpoints in A . This in turn reduces to the case when A is (at most) a quadruple and follows from the thinness of triangles.

The *quasiconvex hull* $\text{QCH}(B) \subset Y$ of a subset $B \subset \partial_\infty Y$ at infinity is defined accordingly as the union of all geodesic lines $l \subset Y$ asymptotic to (points in) B , $\partial_\infty l \subset B$. It is nonempty unless $|B| \leq 1$, and then again it is $C\delta$ -quasiconvex and $\partial_\infty \text{QCH}(B) = B$, which follows from the fact that any geodesic segment with endpoints in $\text{QCH}(B)$ is contained in the tubular $C'\delta$ -neighborhood of a geodesic line asymptotic to B . An analogous property holds for rays $y\eta$ with $y \in \text{QCH}(B)$ and $\eta \in B$.

3.2 Isometries

For (proper geodesic) Gromov hyperbolic spaces there is a *rough classification of isometries* into three types (elliptic, parabolic and hyperbolic) as in the case of $\text{CAT}(0)$ spaces.

For an isometry ϕ of a Gromov hyperbolic space Y consider the orbit maps $\mathbb{Z} \rightarrow Y, n \mapsto \phi^n y$ of the cyclic group $\langle \phi \rangle$ generated by ϕ . The isometry ϕ is called

elliptic if the orbits are bounded;

hyperbolic if the orbits are quasigeodesics;

parabolic if the orbits are unbounded and distorted.⁴

⁴i.e. the orbit map $n \mapsto \phi^n y$ is not a quasiisometric embedding $\mathbb{Z} \rightarrow Y$

The *asymptotic displacement number* of an isometry ϕ is defined as

$$\tau_\phi := \lim_{n \rightarrow \infty} \frac{1}{n} d(y, \phi^n y).$$

The limit exists (due to the subadditivity of $n \mapsto d(y, \phi^n y)$) and is independent of $y \in Y$. Note that $\tau_\phi > 0$ if ϕ is hyperbolic and $\tau_\phi = 0$ otherwise.

Non-elliptic isometries have unbounded orbits. In particular, they have infinite order and no fixed points in Y . They do have fixed points at infinity:

Proposition 3.7. (i) *If ϕ is hyperbolic, then it has exactly two fixed points on $\partial_\infty Y$, an attractive fixed point η_+ and a repulsive fixed point η_- . It holds that for $y \in Y$, $\phi^n y \rightarrow \eta_\pm$ as $n \rightarrow \pm\infty$.*

(ii) *If ϕ is parabolic, then it has exactly one fixed point η on $\partial_\infty Y$ and $\phi^n y \rightarrow \eta$ as $n \rightarrow \pm\infty$.*

Proof. Suppose that ϕ is not elliptic. Then the sublevel subsets $\{\delta_\phi \leq c\}$ of the displacement function $\delta_\phi(y) = d(y, \phi y)$ are unbounded if non-empty, because they are ϕ -invariant. Their visual boundary is therefore non-empty, $\partial_\infty \{\delta_\phi \leq c\} \neq \emptyset$. On the other hand, it is fixed point-wise by ϕ and therefore can contain at most two ideal points (because ϕ is non-elliptic). The two point case corresponds to ϕ being hyperbolic. Hence, if ϕ is parabolic, then it has a unique fixed point η in $\partial_\infty Y$. Furthermore, the orbits of ϕ are contained in sublevel sets of δ_ϕ and therefore must accumulate at η . \square

As a consequence, parabolic and hyperbolic isometries can be characterized in terms of their action at infinity and in terms of the accumulation of their orbits at infinity.

We turn our attention to the stabilizers of points at infinity. The non-hyperbolic isometries in the stabilizers can shift horofunctions only by a bounded amount:

Lemma 3.8. *Let ϕ be a non-hyperbolic isometry fixing $\eta \in \partial_\infty Y$ and let h be a horofunction at η . Then $|h - h \circ \phi| \leq C\delta$.*

Proof. Fix $\epsilon > 0$. Suppose there exists a point $y \in Y$ such that $|h(y) - h \circ \phi(y)| \geq (1 + \epsilon)C\delta$. We may assume that $h(y) - h \circ \phi(y) \geq (1 + \epsilon)C\delta$. (Otherwise, we replace ϕ with ϕ^{-1} .) In view of (??), it follows that $h - h \circ \phi \geq \epsilon C\delta$ on all of Y . This implies that the orbit path $n \mapsto \phi^n y$ is a quasigeodesic and hence ϕ must be hyperbolic, a contradiction. Letting $\epsilon \searrow 0$ the assertion follows. \square

As a consequence, if $P < \text{Isom}(Y)$ is a subgroup fixing $\eta \in \partial_\infty Y$ which contains no hyperbolic isometry, then P quasi-preserved the horoballs at η . In fact, every horoball at η is uniformly Hausdorff close to a P -invariant one.

A horoball cannot be preserved by a hyperbolic isometry since it contains no quasigeodesic.

Remark 3.9. By [?, Thm. 2G], a discrete group of isometries of Y fixing a point in $\partial_\infty Y$ cannot contain both hyperbolic and parabolic isometries. Hence, the stabilizer of an ideal point then consists either only of non-hyperbolic isometries or only of non-parabolic isometries.

3.3 Relatively hyperbolic groups

3.3.1 Gromov's definition

Since the geometrically finite subgroups and their generalizations considered in this paper will be *relatively hyperbolic* as abstract groups, we need to review the notion of relative hyperbolicity. There are various ways of defining relatively hyperbolic groups (see [?, ?, ?, ?, ?, ?, ?, ?]). We will work essentially with Gromov's original definition [?, §8.6] in terms of actions on hyperbolic spaces. Motivating examples are non-uniform lattices acting on rank one symmetric spaces and, more generally, geometrically finite Kleinian groups.

Definition 3.10. A *relatively hyperbolic (RH) group* is a pair (Γ, \mathcal{P}) consisting of a group Γ and a conjugation invariant collection \mathcal{P} of subgroups $\Pi_i < \Gamma$, $i \in I$, such that there exists a properly discontinuous isometric action $\Gamma \curvearrowright Y$ on a δ -hyperbolic proper geodesic space Y satisfying the following:

- (i) Y is either taut or a horoball.
- (ii) Y is equipped with a Γ -invariant collection $\mathcal{B} = (B_i)_{i \in I}$ of disjoint open horoballs such that the stabilizer of each B_i in Γ is Π_i .
- (iii) The action $\Gamma \curvearrowright Y^{th} := Y - \bigcup_{i \in I} B_i$ is cocompact.
- (iv) The subgroups Π_i are infinite.
- (v) The subgroups Π_i are finitely generated.

The subgroups Π_i in this definition are called the *peripheral subgroups* of Γ and their collection \mathcal{P} the *peripheral structure* on Γ ; the pair (Y, \mathcal{B}) or simply the hyperbolic space Y , respectively, the Γ -action on it is called a *Gromov model* for (Γ, \mathcal{P}) ; the horoballs B_i are called the *peripheral horoballs*, the truncated hyperbolic space Y^{th} is called the *thick part* of Y , and the horospheres $\Sigma_i := \partial B_i$ are called the *boundary* or *peripheral horospheres* of Y^{th} .

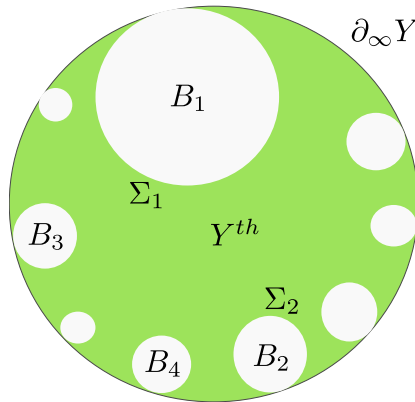


Figure 1: A Gromov model.

For instance, in the case of nonuniform lattices acting on rank one symmetric spaces, the natural Gromov model is the symmetric space itself. For geometrically finite Kleinian groups,

it is the closed convex hull of the limit set.⁵

We call a Gromov model *faithful* if there exists a point $y \in Y^{th}$ with trivial stabilizer in Γ , i.e. so that the action $\Gamma \curvearrowright \Gamma y$ is faithful. Every Gromov model can be modified to a faithful one by a slight enlargement, e.g. by choosing a point $y \in Y^{th}$ and passing to the mapping cone Y' of the orbit map $o_y : \Gamma \rightarrow Y, \gamma \mapsto \gamma y$. The geodesic segments $[\gamma, \gamma(y)]$ in Y' are equipped with metrics as intervals of some fixed length $\lambda > 0$; combined with the original path-metric on Y , this defines a path-metric on Y' such that the natural inclusion map $Y \rightarrow Y'$ is a quasiisometry. The system of horoballs is kept the same.

Remark 3.11. 1. When the peripheral structure is trivial, $\mathcal{P} = \emptyset$, then Γ is word hyperbolic.

2. Y is a horoball if and only if $|\mathcal{P}| = 1$. Then the unique peripheral subgroup is Γ itself.

3. If $|\partial_\infty Y| = 2$, then Γ is virtually cyclic and $\mathcal{P} = \emptyset$. Indeed, by tautness Y is then quasiisometric to a line and, in view of the infiniteness of peripheral subgroups, Γ must be infinite, hence virtually cyclic, and no infinite subgroup preserves a horoball.

4. By passing to subhoroballs, the peripheral horoballs can be made *r-separated* for arbitrary $r > 0$, i.e. we may assume that any two distinct peripheral horoballs have distance $\geq r$.

5. In some treatments of the theory of RH groups, the peripheral subgroups Π_i are not required to be infinite. However, if one omits this condition, then the peripheral horospheres ∂B_i with finite stabilizers Π_i are compact, the horoballs B_i bounded by them are ends of Y Hausdorff close to rays and their centers are isolated points of $\partial_\infty Y$ which do not belong to the limit set of Γ , compare Lemma ?? below whose proof uses only properties (i)-(iii) from our definition of RH groups. We do not want to allow this possibility.

6. Gromov's original definition did not require the finite generation condition (v), only conditions (i-iv), while other definitions discussed in the literature do require finite generation. We added condition (v) because under this assumption all known definitions of RH are equivalent (see [?, ?, ?] for proofs of the equivalences).⁶ Furthermore, finite generation of the peripheral subgroups is a natural assumption in view of Theorem ??.

Several *finiteness* properties can be readily derived from the RH axioms:

The family \mathcal{B} of peripheral horoballs is *locally finite*, i.e. every compact subset of Y is intersected by only finitely many horoballs B_i . This follows from the local compactness of Y and since we can assume the peripheral horoballs to be *r-separated* for some $r > 0$.

The cocompactness of the action $\Gamma \curvearrowright Y^{th}$ further implies that there are *finitely many* Γ -orbits of peripheral horoballs B_i , respectively, conjugacy classes of peripheral subgroups Π_i .

Lemma 3.12. *The actions $\Pi_i \curvearrowright \partial B_i$ are cocompact.*

⁵With two elementary exceptions: For finite groups the Gromov model is a singleton, while for Kleinian groups whose limit set is a single point ζ at infinity, the Gromov model is the intersection of a horoball with a certain convex subset, see the proof of Theorem ??.

⁶Specifically, Propositions 6.12 and 6.13 in Bowditch's paper [?] prove that Definition ?? is equivalent to Definition 1 (and, hence, Definition 2) in [?].

Proof. Fix a base point $y \in Y$. Since the action $\Gamma \curvearrowright Y^{th}$ is cocompact, there exists a subset $S \subset \Gamma$ such that the subset $Sy \subset \Gamma y$ is Hausdorff close to the horosphere ∂B_i . Then the horoballs $\gamma^{-1}B_i$ for $\gamma \in S$ intersect a compact subset and, by the local finiteness of \mathcal{B} , only finitely many of them are different. It follows that S is contained in a finite union of right cosets of Π_i , and hence that $\Pi_i y$ and ∂B_i have finite Hausdorff distance. \square

Together with the finite generation of the peripheral subgroups this further implies:

Lemma 3.13. *Γ is finitely generated.*

Proof. The previous lemma, together with the finite generation of the peripheral subgroups Π_i , implies that the peripheral horospheres ∂B_i are coarsely connected (see [?]). Since Y is path connected, it follows that the thick part Y^{th} is coarsely connected. The cocompactness of the action $\Gamma \curvearrowright Y^{th}$ now implies that Γ is finitely generated. \square

We describe next the *dynamics at infinity*. The action $\Gamma \curvearrowright \partial_\infty Y$ is a *convergence action* with certain characteristic features:

Since the horoballs B_i are disjoint and form a Γ -invariant family, the stabilizers in Γ of the centers $\eta_i \in \partial_\infty Y$ of the horoballs B_i are precisely the peripheral subgroups Π_i . We can regard \mathcal{P} as a subset of $\partial_\infty Y$ via the natural embedding $\Pi_i \mapsto \eta_i$. The points η_i are limit points of Γ due to our condition (iv) that the Π_i are infinite, however they are not conical. They are called the *parabolic points* of Γ in $\partial_\infty Y$ and their stabilizers Π_i the *maximal parabolic subgroups* of Γ .

All other points in $\partial_\infty Y$ are *conical limit points* of Γ as a consequence of the cocompactness of the action $\Gamma \curvearrowright Y^{th}$. (Recall that an ideal point $\eta \in \partial_\infty Y$ is a *conical limit point* in the dynamical sense of Definition ?? if and only if the following geometric property is satisfied: A(ny) geodesic ray in Y asymptotic to η has a tubular neighborhood which contains infinitely many points of a Γ -orbit.) In particular, the limit set of $\Gamma \curvearrowright Y$ is the entire $\partial_\infty Y$.

Note that the peripheral structure \mathcal{P} can be read off the action $\Gamma \curvearrowright \partial_\infty Y$ as the set of non-conical limit points and their stabilizers.

If $|\partial_\infty Y| \geq 3$, then $\partial_\infty Y$ is a perfect metrizable compact topological space and the action $\Gamma \curvearrowright \partial_\infty Y$ is a *minimal convergence action*.

The cocompactness of the actions $\Pi_i \curvearrowright \partial B_i$ implies that also the actions $\Pi_i \curvearrowright \partial_\infty Y - \{\eta_i\}$ are properly discontinuous and cocompact, i.e. the η_i are *bounded parabolic points*, cf. Definition ?. Thus, the convergence action $\Gamma \curvearrowright \partial_\infty Y$ is *geometrically finite* in the dynamical sense of Beardon-Maskit:

Proposition 3.14. *All points in $\partial_\infty Y$ are either conical limit points or bounded parabolic points for the action of Γ .*

In particular, Γ is relatively hyperbolic in the sense of Bowditch's first definition in [?] which is formulated in terms of the dynamics at infinity of a properly discontinuous isometric action on a Gromov hyperbolic space.

The Gromov model Y is *not canonical* in the sense that its quasiisometry type is not

determined by the pair (Γ, \mathcal{P}) . This is because the action $\Gamma \curvearrowright Y$ is cocompact only on the thick part Y^{th} and the geometry of the peripheral horoballs is not controlled by the group. Nevertheless, the *asymptotic* geometry of the Gromov models *is* determined. By a remarkable result due to Bowditch [?, Thm. 9.4], for any two Gromov models $\Gamma \curvearrowright Y_1$ and $\Gamma \curvearrowright Y_2$ there is a Γ -equivariant homeomorphism $\partial_\infty Y_1 \xrightarrow{\cong} \partial_\infty Y_2$. If $|\partial_\infty Y_1| \geq 3$, since the Γ -actions on $\partial_\infty Y_1, \partial_\infty Y_2$ are minimal, this homeomorphism is necessarily unique.

It follows that if the Gromov model Y_1 is faithful and the point $y_1 \in Y_1$ has trivial stabilizer in Γ , then the Γ -equivariant map of orbits $\Gamma y_1 \rightarrow \Gamma y_2, \gamma y_1 \mapsto \gamma y_2$ extends, by a homeomorphism at infinity, to an equivariant continuous map

$$\Gamma y_1 \sqcup \partial_\infty Y_1 \xrightarrow{\cong} \Gamma y_2 \sqcup \partial_\infty Y_2 \tag{3.15}$$

of the orbit closures inside the visual compactifications of the Gromov models. Indeed, it can be read off the convergence dynamics of the action $\Gamma \curvearrowright \partial_\infty Y_i$ whether a sequence $(\gamma_n y_i)$ in the orbit Γy_i converges in \bar{Y}_i and, if yes, to which ideal point in $\partial_\infty Y_i$. If also the point y_2 has trivial stabilizer in Γ , then (??) is a homeomorphism.

This enables one to define a boundary at infinity and a compactification of an RH group:

Definition 3.16 (Ideal boundary). The *ideal boundary* $\partial_\infty \Gamma$ of an RH group (Γ, \mathcal{P}) is defined as the visual boundary $\partial_\infty Y$ of a Gromov model Y . The *compactification*

$$\bar{\Gamma} = \Gamma \sqcup \partial_\infty \Gamma$$

of Γ is topologized at infinity by embedding it into the visual compactification $\bar{Y} = Y \sqcup \partial_\infty Y$ using an injective orbit map $\Gamma \rightarrow Y$, after enlarging Y to a faithful Gromov model.

Both $\partial_\infty \Gamma$ and $\bar{\Gamma}$ do not depend on the choice of the Gromov model and the orbit inside it. To simplify notation, we will suppress the peripheral structure.

Remark 3.17. Bowditch also constructed in [?] a “canonical” Gromov model, unique up to (equivariant) quasiisometry, with uniform *strict* exponential distortion of the peripheral horospheres.

The natural action $\Gamma \curvearrowright \partial_\infty \Gamma$ at infinity for an RH group Γ is a minimal convergence action with finite kernel (unless $1 \leq |\partial_\infty \Gamma| \leq 2$) satisfying the Beardon-Maskit condition that every point $\eta \in \partial_\infty \Gamma$ is either a conical limit point or a bounded parabolic fixed point. The stabilizers of the latter ones are the peripheral subgroups of Γ .

Yaman showed that, conversely, the existence of an action with this kind of dynamics characterizes RH groups, thereby generalizing Bowditch’s dynamical characterization of (absolutely) hyperbolic groups:

Theorem 3.18 (Yaman [?]). *Let $\Gamma \curvearrowright Z$ be a convergence action on a nonempty perfect metrizable compact topological space Z . Suppose that*

- (i) *each point in Z is either a conical limit point or a bounded parabolic fixed point;*

(ii) there are finitely many Γ -orbits of bounded parabolic fixed points and their Γ -stabilizers are finitely generated.

Then the family \mathcal{P} of these stabilizers forms an RH structure on Γ , and Z is equivariantly homeomorphic to $\partial_\infty(\Gamma, \mathcal{P})$.

Remark 3.19. 1. As Yaman points out herself [?, pp. 41-42], the assumption that there are finitely many Γ -orbits of bounded parabolic fixed points can be dropped as a consequence of a result by Tukia [?, Thm 1B].

2. The finite generation of the Γ -stabilizers of bounded parabolic fixed points is needed in Yaman's paper [?] only indirectly, namely, because she verifies Definition 2 in [?] and the latter requires finite generation of peripheral subgroups.

The following result on peripheral subgroups is also relevant for our paper. Here, a space is said to have *coarsely bounded geometry* if there exists a scale $R_0 > 0$ and a function $\psi : [R_0, \infty) \rightarrow \mathbb{N}$ such that for all $R \geq R_0$ every R -ball in the space can be covered by at most $\psi(R)$ R_0 -balls.

Theorem 3.20 (Dahmani, Yaman [?]). *If an RH group admits a Gromov model with coarsely bounded geometry, then all peripheral subgroups have polynomial growth.*

Consequently, by Gromov's theorem, the peripheral subgroups then are virtually nilpotent.

Remark 3.21. Dahmani and Yaman work with a stricter notion of bounded geometry: They put $R_0 = 1$ and also require on the small scale that every 1-ball can be covered by at most $\psi(R)$ balls of the radius $\frac{1}{R}$. However, their proof only uses the assumption of coarsely bounded geometry.

We observe that the property of coarsely bounded geometry behaves well under quasiisometric embeddings: A space has coarsely bounded geometry as soon as it quasiisometrically embeds into a space with this property, for instance, into a symmetric space. Therefore:

Corollary 3.22. *If an RH group admits a Gromov model which quasiisometrically embeds into a symmetric space, then all peripheral subgroups are virtually nilpotent.*

3.3.2 Straight triples

We carry over the notion of straightness of triples (defined in section ??) from Gromov hyperbolic spaces to RH groups as follows.

For an RH group (Γ, \mathcal{P}) we consider the spaces of pairs

$$(\Gamma \sqcup \partial_\infty \Gamma)^2 - \Delta_{\partial_\infty \Gamma} \subset (\Gamma \sqcup \partial_\infty \Gamma)^2$$

and triples

$$T(\Gamma, \partial_\infty \Gamma) := (\Gamma \sqcup \partial_\infty \Gamma) \times \Gamma \times (\Gamma \sqcup \partial_\infty \Gamma)$$

in $\bar{\Gamma}$, compare (??). If (Y, \mathcal{B}) is a Gromov model for (Γ, \mathcal{P}) and $y \in Y$ is a point, then the orbit map σ_y induces natural continuous maps $\bar{\Gamma}^2 \rightarrow \bar{Y}^2$ and $\bar{\Gamma}^3 \rightarrow \bar{Y}^3$ of pairs and triples which restrict to maps $(\Gamma \sqcup \partial_\infty \Gamma)^2 - \Delta_{\partial_\infty \Gamma} \rightarrow (Y \sqcup \partial_\infty Y)^2 - \Delta_{\partial_\infty Y}$ and $T(\Gamma, \partial_\infty \Gamma) \rightarrow T(Y, \partial_\infty Y)$. Whether a family of triples in $\bar{\Gamma}$ projects to a uniformly straight (in the sense of Definition ??) family of triples in \bar{Y} , depends only on its asymptotics at infinity and is therefore independent of the Gromov model Y and the point y :

Lemma 3.23 (Straightness is independent of Gromov model). *The image in $T(Y, \partial_\infty Y)$ of a subset $T \subset T(\Gamma, \partial_\infty \Gamma)$ consists of D -straight triples for some uniform $D \geq 0$ if and only if the subset of pairs $\{(\gamma_2^{-1}\bar{\gamma}_1, \gamma_2^{-1}\bar{\gamma}_3) : (\bar{\gamma}_1, \gamma_2, \bar{\gamma}_3) \in T\}$ is contained in $(\Gamma \sqcup \partial_\infty \Gamma)^2 - \Delta_{\partial_\infty \Gamma}$ as a relatively compact subset.*

Proof. By Γ -equivariance, the triples in the image of T in $T(Y, \partial_\infty Y)$ are D -straight if and only if the triples in the image of the family $E := \{(\gamma_2^{-1}\bar{\gamma}_1, e, \gamma_2^{-1}\bar{\gamma}_3) : (\bar{\gamma}_1, \gamma_2, \bar{\gamma}_3) \in T\}$ are. The latter holds for some uniform $D \geq 0$ if and only if the family of all geodesics in Y with pair of endpoints in the image E_Y of E under the natural map $(\Gamma \sqcup \partial_\infty \Gamma)^2 - \Delta_{\partial_\infty \Gamma} \rightarrow (Y \sqcup \partial_\infty Y)^2 - \Delta_{\partial_\infty Y}$ is bounded (in the sense defined in section ??). This condition (by Lemma ??) holds if and only if E_Y is relatively compact in $(Y \sqcup \partial_\infty Y)^2 - \Delta_{\partial_\infty Y}$, so in view of the continuity of the map $\bar{\Gamma}^2 \rightarrow \bar{Y}^2$, if and only if E is relatively compact in $(\Gamma \sqcup \partial_\infty \Gamma)^2 - \Delta_{\partial_\infty \Gamma}$. \square

It therefore makes sense to call a family of triples in $T(\Gamma, \partial_\infty \Gamma)$ *straight* if its image in $T(Y, \partial_\infty Y)$ consists of D -straight triples for some fixed data (D, Y, y) .

Note that straightness is a useful concept for triples in RH groups Γ only if $|\partial_\infty \Gamma| > 1$. If $\partial_\infty \Gamma$ is a singleton, then a family of triples $(\gamma_1, \gamma_2, \gamma_3)$ in Γ^3 is straight if and only if the corresponding subsets $\{\gamma_2^{-1}\gamma_1, \gamma_2^{-1}\gamma_3\}$ intersect some finite subset of Γ .

4 Some geometry of higher rank symmetric spaces

4.1 Basic notions and standing notation

In this section we briefly discuss some basic definitions pertaining to symmetric spaces X of noncompact type. (We will call them simply *symmetric spaces*.) We refer the reader to the books [?, ?] and to [?] for the foundational material, and to our earlier papers [?, ?, ?, ?, ?, ?] for more specialized aspects of the theory, developed specifically to study the asymptotic geometry and discrete isometry groups of symmetric spaces.

The visual boundary $\partial_\infty X$ of a symmetric space X admits a structure as a thick spherical building (the Tits building of X). Throughout the paper we will use the notation σ_{mod} for the model spherical chamber of this building, Δ for the model euclidean Weyl chamber of X and $\theta : \partial_\infty X \rightarrow \sigma_{mod}$ for the *type projection*. The full isometry group of X acts on σ_{mod} isometrically; the map θ is equivariant with respect to this action. We will denote by $G < \text{Isom}(X)$ the kernel of this action, i.e. the subgroup of type preserving isometries. It is a semisimple Lie group and has finite index in $\text{Isom}(X)$.

We will freely use the notions introduced in our earlier papers, such as the opposition involution ι of σ_{mod} , a type $\bar{\theta} \in \sigma_{mod}$, the face types $\tau_{mod} \subseteq \sigma_{mod}$ [?, §2.2.2], the associated τ_{mod} -flag manifolds $\text{Flag}_{\tau_{mod}}$ [?, §2.2.2, 2.2.3], the open Schubert cells $C(\tau) \subset \text{Flag}_{\tau_{mod}}$ [?, §2.4], the τ_{mod} -boundary $\partial_{\tau_{mod}}\Delta$ of Δ [?, §2.5.2], the Δ -valued distance d_{Δ} on X [?, §2.6], Θ -regular geodesic segments [?, §2.5.3], parallel sets $P(\tau_-, \tau_+)$, stars $\text{st}(\tau)$, open stars $\text{ost}(\tau)$, Θ -stars $\text{st}_{\Theta}(\tau)$, Weyl cones $V(x, \text{st}(\tau))$ and Θ -cones $V(x, \text{st}_{\Theta}(\tau))$, diamonds $\diamond_{\tau_{mod}}(x, y)$ and Θ -diamonds $\diamond_{\Theta}(x, y)$ [?, §2.5], τ_{mod} -regular sequences and subgroups [?, §4.2], uniformly τ_{mod} -regular sequences and subgroups [?, §4.6], τ_{mod} -convergence subgroups, flag-convergence, the Finsler interpretation of flag-convergence, see [?, §4.5 and 5.2] and [?], τ_{mod} -limit sets $\Lambda_{X, \tau_{mod}} = \Lambda_{\tau_{mod}} = \Lambda_{\tau_{mod}}(\Gamma) \subset \text{Flag}_{\tau_{mod}}$ [?, §4.5], visual limit sets [?, p. 4], Morse subgroups [?, §5.4], Morse quasigeodesics and Morse maps [?, Defs. 5.31, 5.33], antipodal limit sets [?, Def. 5.1] and antipodal maps to flag manifolds [?, Def. 6.11], to name a few. We review some of this material in sections ?? and ??.

We will use the following conventions and standing notation.

Throughout the paper, X will denote a symmetric space of noncompact type. We will denote by Θ an ι -invariant, compact, Weyl-convex (see [?, Def. 2.7]) subset of the open star $\text{ost}(\tau_{mod}) \subset \sigma_{mod}$. For pairs $\Theta, \Theta' \subset \text{ost}(\tau_{mod})$ of such subsets, we will always assume that $\Theta \subset \text{int}(\Theta')$. Similarly, for pairs of positive constants d, d' we will always assume that $d < d'$. Note that, when X has rank one, the data τ_{mod}, Θ are obsolete. In this case, we also have that $\partial\sigma_{mod} = \emptyset$ and $\Theta = \text{int}(\sigma_{mod}) = \sigma_{mod}$ is clopen; in particular, $\Theta' = \text{int}(\Theta) = \Theta$.

4.2 Finsler geometric notions

In [?], see also [?], we considered a certain class of G -invariant ‘‘polyhedral’’ Finsler metrics on X . Their geometric and asymptotic properties turned out to be well adapted to the study of geometric and dynamical properties of regular subgroups. They provide a Finsler geodesic *combing* of X which is, in many ways, more suitable for analyzing the asymptotic geometry of X than the geodesic combing given by the standard Riemannian metric on X . These Finsler metrics also play a basic role in the present paper. We briefly recall their definition and some basic properties, and refer to [?, §5.1] for more details.

Let $\bar{\theta} \in \text{int}(\tau_{mod})$ be a type spanning the face type τ_{mod} . The $\bar{\theta}$ -Finsler distance $d^{\bar{\theta}}$ on X is the G -invariant pseudo-metric defined by

$$d^{\bar{\theta}}(x, y) := \max_{\theta(\xi)=\bar{\theta}} (b_{\xi}(x) - b_{\xi}(y))$$

for $x, y \in X$, where the maximum is taken over all ideal points $\xi \in \partial_{\infty}X$ with type $\theta(\xi) = \bar{\theta}$. It is positive, i.e. a (non-symmetric) metric, if and only if the radius of σ_{mod} with respect to $\bar{\theta}$ is $< \frac{\pi}{2}$. This is in turn equivalent to $\bar{\theta}$ not being contained in a factor of a nontrivial spherical join decomposition of σ_{mod} , and is always satisfied e.g. if X is irreducible.

If $d^{\bar{\theta}}$ is positive, it is equivalent to the Riemannian metric. In general, if it is only a pseudo-metric, it is still equivalent to the Riemannian metric d on uniformly regular pairs of points.

More precisely, if the pair of points x, y is Θ -regular, then

$$L^{-1}d(x, y) \leq d^{\bar{\theta}}(x, y) \leq Ld(x, y)$$

with a constant $L = L(\Theta) \geq 1$.

Regarding symmetry of the Finsler distance, one has the identity

$$d^{\bar{\theta}}(y, x) = d^{\bar{\theta}}(x, y)$$

and hence $d^{\bar{\theta}}$ is symmetric if and only if $\iota\bar{\theta} = \bar{\theta}$. We refer to $d^{\bar{\theta}}$ as a Finsler metric of type τ_{mod} .

The $d^{\bar{\theta}}$ -balls in X are convex but not strictly convex. (Their intersections with flats through their centers are polyhedra.) Accordingly, $d^{\bar{\theta}}$ -geodesics connecting two given points x, y are not unique. To simplify notation, xy will stand for *some* $d^{\bar{\theta}}$ -geodesic connecting x and y . The union of all $d^{\bar{\theta}}$ -geodesic xy equals the τ_{mod} -diamond $\diamond_{\tau_{mod}}(x, y)$, that is, a point lies on a $d^{\bar{\theta}}$ -geodesic xy if and only if it is contained in $\diamond_{\tau_{mod}}(x, y)$, see [?]. Finsler geometry thus provides an alternative description of diamonds. Note that with this description, the diamond $\diamond_{\tau_{mod}}(x, y)$ is also defined when the segment xy is not τ_{mod} -regular. Such a *degenerate* τ_{mod} -diamond is contained in a smaller totally-geodesic subspace, namely in the intersection of all τ_{mod} -parallel sets containing the points x, y . The description of geodesics and diamonds also implies that the unparameterized $d^{\bar{\theta}}$ -geodesics depend only on the face type τ_{mod} , and not on $\bar{\theta}$. We will refer to $d^{\bar{\theta}}$ -geodesics as τ_{mod} -Finsler geodesics. Note that Riemannian geodesics are Finsler geodesics.

We will call a Θ -regular τ_{mod} -Finsler geodesic a Θ -Finsler geodesic. If xy is a Θ -regular (Riemannian) segment, then the union of Θ -Finsler geodesics xy equals the Θ -diamond $\diamond_{\Theta}(x, y)$.

Every τ_{mod} -Finsler ray in X is contained in a τ_{mod} -Weyl cone, and we will use the notation $x\tau$ for a τ_{mod} -Finsler ray contained $V(x, \text{st}(\tau))$. Similarly, every τ_{mod} -Finsler line is contained in a τ_{mod} -parallel set, and we denote by $\tau_-\tau_+$ an oriented τ_{mod} -Finsler line forward/backward asymptotic to two antipodal simplices $\tau_{\pm} \in \text{Flag}_{\tau_{mod}}$ and contained in $P(\tau_-, \tau_+)$.

4.3 Types of isometries

Let $g \in \text{Isom}(X)$. The function $\delta_g(x) = d(x, gx)$ on X is called the *displacement function* of g and the number

$$m_g := \inf_X \delta_g$$

is called the *infimal displacement* or *translation number* of g .

The isometry g is called *semisimple* if δ_g attains its infimum. The semisimple isometries split into two subclasses: A semisimple isometry g is called

(i) *elliptic* if $m_g = 0$, i.e. if it fixes a point in X . Equivalently, the orbits in X of the cyclic group $\langle g \rangle$ are bounded.

(ii) *axial* or *hyperbolic* if $m_g > 0$. In this case, the minimum set $\text{Min}(g)$ of δ_g is the union of the *axes* of g , i.e. of the g -invariant geodesic lines. On each axis, g acts as a translation by m_g . The subset $\text{Min}(g)$ is a symmetric subspace of X and splits metrically as $\text{Min}(g) \cong \mathbb{R} \times CS$, the lines $\mathbb{R} \times pt$ being the axes of g and the cross section CS being a symmetric (sub)space.

A hyperbolic isometry g is a *transvection* if it preserves the parallel vector fields along some (and hence any) axis. Then every line parallel to an axis of g is itself an axis, i.e. the minimum set is the full parallel set of a line. The transvections are the isometries of X which can be written as the product of two distinct point reflections.

The isometries g for which δ_g does not attain its infimum are called *parabolic*. A parabolic isometry g has at least one fixed point in the visual boundary. To see this, consider a sequence (x_n) in X such that $\delta_g(x_n) \searrow m_g$. Then $x_n \rightarrow \infty$ and the accumulation points of (x_n) in $\partial_\infty X$ are fixed by g . Moreover, at some of the fixed points at infinity also the horoballs are preserved by g . Namely, choose a sequence (x_n) more carefully, by picking a base point $o \in X$ and a sequence $\epsilon_n \searrow 0$ and letting x_n be the nearest point projection of o to $\{\delta_g \leq \epsilon_n\}$. Then for any accumulation point $\xi \in \partial_\infty X$ of (x_n) the horoballs centered at ξ are g -invariant, see e.g. [?].

The parabolic isometries break up into several subclasses. We will call a parabolic isometry g *strictly parabolic* if $m_g = 0$ and non-strictly parabolic otherwise. If $\text{rank } X = 1$ then all parabolic isometries are strictly parabolic, but non-strictly parabolic isometries occur if $\text{rank } X \geq 2$.

An isometry $g \neq \text{id}_X$ is called *unipotent* if the closure of its conjugacy class in $\text{Isom}(X)$ contains id_X , i.e. if there exists a sequence of isometries $h_n \rightarrow \infty$ such that $h_n g h_n^{-1} \rightarrow \text{id}_X$. In this case, there exists a transvection h such that

$$\lim_{n \rightarrow \infty} h^n g h^{-n} = \text{id}_X .$$

Unipotent isometries are strictly parabolic.

Every isometry g of X has a unique *Jordan decomposition*

$$g = g_s g_u = g_t g_e g_u \tag{4.1}$$

where $g_s = g_t g_e$ and g_t, g_e, g_u are commuting isometries which are, respectively, a transvection, elliptic and unipotent. The factor g_s is semisimple. Note that

$$m_g = m_{g_s} = m_{g_t}$$

and $\text{Min}(g_s)$ is preserved by g_u .

If g is non-strictly parabolic, equivalently, if g_t and g_u are nontrivial, then g_u preserves the cross sections $\{t\} \times CS$ of $\text{Min}(g_t) \cong \mathbb{R} \times CS$ and acts on them as a strictly parabolic isometry.

We refer the reader to [?] for further discussion.

If $u \in G$ is a unipotent isometry, then it is of the form $u = \exp(n)$ with $n \in \mathfrak{g}$ nilpotent. (Here \mathfrak{g} is the Lie algebra of G .) According to the Morozov-Jacobson theorem regarding nilpotent elements in semisimple Lie algebras, see [?], n belongs to a 3-dimensional simple Lie subalgebra $\mathfrak{g}' \cong \mathfrak{sl}(2, \mathbb{R})$. Correspondingly, u lies in a rank one Lie subgroup $G' < G$ locally isomorphic to $SL(2, \mathbb{R})$. The subgroup G' preserves a totally-geodesic hyperbolic plane $X' \subset X$ and u acts on it as a parabolic element. Consequently, u fixes a unique ideal point $\xi \in \partial_\infty X'$ and preserves the horocycles in X' centered at ξ . It follows that its orbits accumulate in \overline{X} at ξ , $\Lambda(\langle u \rangle) = \{\xi\}$.

More generally, if $g \in G$ is strictly parabolic, then g_s is elliptic, $g_t = \text{id}_X$, and g has the Jordan decomposition $g = g_e g_u$ with $g_u \neq \text{id}_X$. The latter implies that the g_u -invariant fixed subspace $\text{Fix}(g_e) \subset X$ is noncompact. As above, it contains a g_u -invariant hyperbolic plane X' on which g_u acts as a parabolic isometry. The g -orbits in X have bounded distance from the g_u -orbits, and it follows that they accumulate in \overline{X} at a unique ideal point $\xi \in \partial_\infty X' \subseteq \partial_\infty \text{Fix}(g_e)$,

$$\Lambda(\langle g \rangle) = \{\xi\}. \quad (4.2)$$

The hyperbolic plane X' and the horocycles in it centered at ξ are g -invariant.

We define the *type* of the strictly parabolic isometry g as $\theta(g) := \theta(\xi) \in \sigma_{mod}$ and its *face type* $\tau_{mod}(g) \subseteq \sigma_{mod}$ as the face of σ_{mod} spanned by its type. Note that both are ι -invariant because the points in the visual boundary of a rank one symmetric subspace of X (such as X') are pairwise antipodal and therefore have the same ι -invariant type.

Let g be an isometry which fixes an ideal point $\xi \in \partial_\infty X$. Then g induces an isometry g_ξ on the space X_ξ of strong asymptote classes at ξ , cf. e.g. [?, ?]. If g preserves also the horoballs at ξ , then

$$m_{g_\xi} = m_g \quad (4.3)$$

For every isometry g of X it holds that

$$m_{g^n} = n m_g \quad (4.4)$$

for $n \in \mathbb{N}_0$. This is clear for semisimple isometries. If g is parabolic, it follows by induction on the rank of X using (??), or from the Jordan decomposition.

As in the case of Gromov hyperbolic spaces, one can relate the rough classification of isometries to the *distortion* of their *orbit paths*. For an isometry g of X consider the orbit paths $\mathbb{Z} \rightarrow X, n \mapsto g^n x$ of the cyclic group $\langle g \rangle$ generated by it. One has the following:

If $m_g = 0$, equivalently, if g is elliptic or strictly parabolic, then $n \mapsto \delta_{g^n}(x)$ grows sublinearly as $n \rightarrow \infty$ and the orbit paths are distorted (not quasiisometrically embedded). On the other hand, if $m_g > 0$, equivalently, if g is hyperbolic or non-strictly parabolic, then (??) or the Jordan decomposition implies that the orbit paths are undistorted. Thus the orbits of g are undistorted if and only if $m_g > 0$.

We obtain a more precise picture by applying the Jordan decomposition:

If g is a strictly parabolic isometry, then, as we noted earlier, g preserves a hyperbolic plane $X' \subset X$ and acts on X' as a parabolic isometry. In particular, the orbits of $\langle g \rangle$ in X are *logarithmically distorted*, $\delta_{g^n}(x) = O(\log |n|)$.

It follows for an arbitrary isometry g that its orbit paths deviate sublinearly, in fact logarithmically, from the orbit paths of its semisimple part g_s ,

$$d(g^n x, g_s^n x) = O(\log |n|) \quad (4.5)$$

Thus, if $m_g > 0$ and l is an oriented axis of g_s , then

$$g^n x \rightarrow l(\pm\infty) \in \partial_\infty X$$

in the visual compactification as $n \rightarrow \pm\infty$.

We conclude:

Proposition 4.6 (Distortion of orbits of isometries). *Let g be an isometry of X .*

If g is elliptic, then its orbits are bounded.

If g is strictly parabolic, then its orbits are unbounded, but logarithmically distorted. They accumulate in \overline{X} at a single ideal point in $\partial_\infty X$. (It lies in the visual boundary of a g_u -invariant totally geodesic hyperbolic plane.)

If g is hyperbolic, then its orbits are undistorted. They are Hausdorff close to an (y) axis l of g and accumulate in \overline{X} at the pair of antipodes $\partial_\infty l \subset \partial_\infty X$.

If g is non-strictly parabolic, then its orbits are undistorted. They deviate sublinearly, in fact, logarithmically, from an (y) axis l of the semisimple part g_s but they are not Hausdorff close to any line. They accumulate in \overline{X} at the pair of antipodes $\partial_\infty l \subset \partial_\infty X$.

In particular, the vanishing resp. positivity of m_g can be read off *coarse* properties of the $\langle g \rangle$ -orbits: $m_g > 0$ if and only if each $\langle g \rangle$ -orbit is undistorted in X .

4.4 Regularity

4.4.1 Notions of regularity and limit sets

As in our earlier papers, we will be imposing certain *regularity assumptions* on discrete subgroups $\Gamma < G$. In this section, we go through some variations of the notion of regularity.

Remark 4.7. It is imperative to note here that notions of τ_{mod} -regularity and τ_{mod} -limit sets, as well as the relation to convergence-type dynamics and many other indispensable related concepts and theorems have their origin in the foundational paper by Yves Benoist [?], in fact, even earlier in the work of Tits [?] and Guivarc'h [?].

A subset of X is called *τ_{mod} -regular* if all divergent sequences in it are τ_{mod} -regular. A map into X is called *τ_{mod} -regular* if its image is τ_{mod} -regular.

The following strengthening of regularity occurs naturally in equivariant settings:

Definition 4.8 (Weakly uniformly regular). We say that an (unbounded) subset $W \subset X$ is *(τ_{mod}, ϕ) -regular* if for $x, x' \in W$

$$d(d_\Delta(x, x'), \partial_{\tau_{mod}} \Delta) \geq \phi(d(x, x'))$$

where $\phi : [0, +\infty) \rightarrow \mathbb{R}$ is a monotonic function with $\lim_{d \rightarrow +\infty} \phi(d) = +\infty$. We say that a subset $W \subset X$ is *weakly uniformly τ_{mod} -regular* if it is (τ_{mod}, ϕ) -regular for some ϕ .

Accordingly, we say that a map into X is *(τ_{mod}, ϕ) -regular* or *weakly uniformly τ_{mod} -regular* if its image in X is.

The orbits $\Gamma x \subset X$ of τ_{mod} -regular actions $\Gamma \curvearrowright X$ are weakly uniformly τ_{mod} -regular subsets.

Weak uniform regularity is stable under bounded perturbation: If $W' \subset X$ is d -Hausdorff close to $W \subset X$ and W is (τ_{mod}, ϕ) -regular, then W' is $(\tau_{mod}, \phi(\cdot - 2d) - 2d)$ -regular, as follows from the Δ -triangle inequality.

Note that a subset of X is *uniformly τ_{mod} -regular* (see [?, §4.6]) if and only if it is (τ_{mod}, ϕ) -regular for some (affine) linear function ϕ .

For a τ_{mod} -regular subset $W \subset X$, we define $\partial_{\infty}^{\tau_{mod}} W \subset \text{Flag}_{\tau_{mod}}$ as its τ_{mod} -*accumulation set*. Similarly, we define the τ_{mod} -*conical accumulation set* $\partial_{\infty}^{\tau_{mod}, con} W \subset \text{Flag}_{\tau_{mod}}$ as the set of conical τ_{mod} -limits of sequences in W (see [?, Def. 5.33]).

For a τ_{mod} -regular subgroup $\Gamma < G$, besides the limit set $\Lambda_{\tau_{mod}} = \Lambda_{X, \tau_{mod}} = \partial_{\infty}^{\tau_{mod}}(\Gamma x)$ we will also consider the *conical τ_{mod} -limit set*

$$\Lambda_{X, \tau_{mod}}^{con} := \partial_{\infty}^{\tau_{mod}, con}(\Gamma x) \subset \Lambda_{X, \tau_{mod}}.$$

A τ_{mod} -regular subgroup $\Gamma < G$ is said to be τ_{mod} -*antipodal* if its limit set $\Lambda_{\tau_{mod}}$ is antipodal, i.e. if any two distinct points in $\Lambda_{\tau_{mod}}$ are antipodal. A τ_{mod} -regular subgroup $\Gamma < G$ is called τ_{mod} -*elementary* if $|\Lambda_{\tau_{mod}}| \leq 2$.

It is a basic fact connecting the theory of regular discrete subgroups of G to the classical theory of Kleinian groups, that each τ_{mod} -regular antipodal subgroup $\Gamma < G$ acts as a convergence group on its τ_{mod} -limit set, see [?, §5.1] or [?, Corollary 3.16]. In particular, for a nonelementary τ_{mod} -regular antipodal subgroup $\Gamma < G$, its τ_{mod} -limit set $\Lambda_{\tau_{mod}}$ is perfect and every Γ -orbit is dense in $\Lambda_{\tau_{mod}}$.

Example 4.9. Let $G_1 < G$ be a connected rank one simple Lie subgroup. By the Karpelevich-Mostow theorem, there exists a rank one symmetric subspace $X_1 \subset X$ which is a G_1 -orbit. Its visual boundary $\partial_{\infty} X_1 \subset \partial_{\infty} X$ is an antipodal subset. Hence, it consists of ideal points of the same ι -invariant type $\bar{\xi} \in \sigma_{mod}$, $\theta(\partial_{\infty} X_1) = \{\bar{\xi}\}$. We call $\theta(\partial_{\infty} X_1) := \bar{\xi}$ the *type* of the rank one subspace X_1 , and the ι -invariant face $\tau_{mod}(X_1) := \tau_{mod}(\bar{\xi}) \subseteq \sigma_{mod}$ spanned by $\bar{\xi}$ its *face type*. All non-degenerate segments in X_1 have type $\bar{\xi}$. We thus have a map

$$\partial_{\infty} X_1 \rightarrow \text{Flag}_{\tau_{mod}(X_1)}(X)$$

sending $\xi_1 \in \partial_{\infty} X_1$ to the simplex $\tau_{\xi_1} \in \text{Flag}_{\tau_{mod}(X_1)}$ spanned by ξ_1 . Also, for every ι -invariant face $\tau_{mod} \subseteq \tau_{mod}(X_1)$, by composing this map with the projection $\text{Flag}_{\tau_{mod}(X_1)} \rightarrow \text{Flag}_{\tau_{mod}}$, one obtains a natural antipodal embedding $\beta : \partial_{\infty} X_1 \rightarrow \text{Flag}_{\tau_{mod}}$. All divergent sequences in X_1 are uniformly τ_{mod} -regular and the τ_{mod} -accumulation set of X_1 in $\text{Flag}_{\tau_{mod}}$ equals $\beta(\partial_{\infty} X_1)$. Every discrete subgroup $\Gamma_1 < G_1$ is uniformly τ_{mod} -regular as a subgroup of G . Moreover, $\Lambda_{\tau_{mod}}(\Gamma_1) = \beta(\Lambda(\Gamma_1))$, where $\Lambda(\Gamma_1) \subset \partial_{\infty} X_1$ is the visual limit set of Γ_1 .

4.4.2 Zariski dense subgroups

In general, verifying the (uniform) regularity of a subgroup is not an easy task. However, it is simpler for Zariski dense subgroups (see our paper [?, Theorem 9.6]):

Theorem 4.10. *Let $\Gamma < G$ be Zariski dense. Suppose that Z is a compact metrizable space, $\Gamma \curvearrowright Z$ is a convergence action and $\beta : Z \rightarrow \text{Flag}_{\tau_{\text{mod}}}$ is a Γ -equivariant antipodal continuous map. Then Γ is τ_{mod} -regular.*

Moreover, we can say about the relation of the image to the limit set of Γ :

Addendum 4.11. *If, in addition, the action $\Gamma \curvearrowright Z$ is minimal, then $\beta(Z) = \Lambda_{\tau_{\text{mod}}}$.*

Proof. Let $\lambda_+ \in \Lambda_{\tau_{\text{mod}}}$. Then, since regular subgroups act on flag manifolds as discrete convergence groups (see [?, Lemma 4.19]), there exist a sequence (γ_n) in Γ and a point $\lambda_- \in \text{Flag}_{\tau_{\text{mod}}}$ such that $\gamma_n|_{C(\lambda_-)} \rightarrow \lambda_+$ uniformly on compacts ($C(\lambda_-) \subset \text{Flag}_{\tau_{\text{mod}}}$ being the open Schubert cell). The complement $\text{Flag}_{\tau_{\text{mod}}} - C(\lambda_-)$ is a proper subvariety of $\text{Flag}_{\tau_{\text{mod}}}$. Hence, by the Zariski density, $\beta(Z) \cap C(\lambda_-) \neq \emptyset$. For any point τ in the intersection, it holds that $\gamma_n(\tau) \rightarrow \lambda_+$. Since $\beta(Z)$ is closed and Γ -invariant, it follows that $\lambda_+ \in \beta(Z)$. Thus $\Lambda_{\tau_{\text{mod}}} \subseteq \beta(Z)$. The minimality of the action $\Gamma \curvearrowright Z$ implies equality. \square

4.4.3 Accumulation sets of regular sequences

We collect some facts needed later in the paper.

Lemma 4.12. *Suppose that (x_n) and (y_n) are uniformly τ_{mod} -regular sequences in X such that $\frac{d(x_n, y_n)}{d(x_n, o)} \rightarrow 0$, where $o \in X$ is a base point. Then their τ_{mod} -accumulation sets in $\text{Flag}_{\tau_{\text{mod}}}$ coincide.*

Proof. It suffices to consider the case when (x_n) τ_{mod} -flag converges, $x_n \rightarrow \tau \in \text{Flag}_{\tau_{\text{mod}}}$, and to show that then also $y_n \rightarrow \tau$.

We extend the Riemannian segments ox_n and oy_n to Riemannian rays $o\xi_n$ and $o\eta_n$. By uniform regularity, we may assume that the types $\theta(\xi_n), \theta(\eta_n)$ of the ideal points $\xi_n, \eta_n \in \partial_\infty X$ are contained in a compact subset $\Theta \subset \text{ost}(\tau_{\text{mod}})$. Let $\tau_{\xi_n}, \tau_{\eta_n} \in \text{Flag}_{\tau_{\text{mod}}}$ denote the simplices in $\partial_\infty X$ spanned by them. Then $\tau_{\xi_n} \rightarrow \tau$ in $\text{Flag}_{\tau_{\text{mod}}}$ and we must show that also $\tau_{\eta_n} \rightarrow \tau$.

After extraction, we may assume that also (ξ_n) converges in $\partial_\infty X$, $\xi_n \rightarrow \xi$. Then $\xi \in \tau$. By our assumption, $\angle_o(\xi_n, \eta_n) = \angle_o(x_n, y_n) \rightarrow 0$. It follows that also $\eta_n \rightarrow \xi$. In view of uniform regularity, $\theta(\eta_n) \in \Theta$, this implies that $\tau_{\eta_n} \rightarrow \tau$. \square

Let $K < G$ denote a maximal compact subgroup. We denote by $o \in X$ its fixed point.

Lemma 4.13. *Let $x_n \rightarrow \infty$ be a uniformly τ_{mod} -regular sequence in X which flag converges, $x_n \rightarrow \tau \in \text{Flag}_{\tau_{\text{mod}}}$. Let (k_n) be a sequence in K such that $d(x_n, k_n x_n)$ is uniformly bounded.*

Then (k_n) accumulates at $\text{Stab}_K(\tau) < K$.

Proof. Let $o \in X$ be the fixed point of K . After passing to a subsequence, we may assume that the Riemannian segments ox_n converge to a Θ -regular ray $o\xi$, $\xi \in \text{ost}(\tau)$, and that $k_n \rightarrow k$. Since $d(x_n, k_n x_n)$ is uniformly bounded, k fixes ρ and hence τ . (Compare Lemma ??.) \square

The visual and flag accumulation sets of regular sequences are related as follows:

Lemma 4.14. *Let (x_n) be a τ_{mod} -regular sequence in X which accumulates in \overline{X} at the (compact) subset $A \subseteq \partial_\infty X$. Then the accumulation set of (x_n) in $\text{Flag}_{\tau_{mod}}$ consists only of simplices which are faces of chambers containing a point of A .*

Proof. We may assume that A consists only of one ideal point ξ . We fix a base point $o \in X$ and extend the segments ox_n to rays $o\xi_n$. Then $\xi_n \rightarrow \xi$ in $\partial_\infty X$. Let $\sigma_n \subset \partial_\infty X$ be chambers containing the ideal points ξ_n and let $\tau_n \in \text{Flag}_{\tau_{mod}}$ be their faces of type τ_{mod} . By the definition of flag convergence, the accumulation set of the τ_{mod} -regular sequence (x_n) in $\text{Flag}_{\tau_{mod}}$ equals the accumulation set of the sequence (τ_n) . Its elements are faces of chambers in the accumulation set of the sequence (σ_n) in $\text{Flag}_{\sigma_{mod}}$. The chambers in the latter accumulation set contain ξ . \square

4.4.4 A continuity property for Weyl cones

From Lemma ??, we deduce a continuity property for Weyl cones. Let again $K < G$ denote a maximal compact subgroup and $o \in X$ its fixed point.

Lemma 4.15. *For Θ, d, r, ϵ there exists R such that the following holds.*

Let $\tau, \tau' \in \text{Flag}_{\tau_{mod}}$ and let $x \in V(o, \text{st}(\tau))$ and $x' \in V(o, \text{st}(\tau'))$ be points such that the pairs (o, x) and (o, x') are Θ -regular with distance $\geq R$. Suppose that $d(x, x') \leq d$.

Then $V(o, \text{st}(\tau)) \cap B(o, r)$ and $V(o, \text{st}(\tau')) \cap B(o, r)$ have Hausdorff distance $\leq \epsilon$.

Proof. We can write $\tau' = k\tau$ with $k \in K$ so that the points kx and x' lie in the same euclidean Weyl chamber with tip at o . Then $d(kx, x') \leq d(x, x') \leq d$ and hence $d(x, kx) \leq 2d$.

The elements $\tilde{k} \in K$, for which $V(o, \text{st}(\tau)) \cap B(o, r)$ and $V(o, \text{st}(\tilde{k}\tau)) \cap B(o, r)$ have Hausdorff distance $\leq \epsilon$, form a neighborhood U of $\text{Stab}_K(\tau)$. Lemma ?? implies that, if R is sufficiently large, k must lie in U . \square

5 Elementary and unipotent subgroups

In our relativizations of the Anosov condition a prominent role is played by the stabilizers of bounded parabolic limit points. They are the peripheral subgroups for an induced relatively hyperbolic structure. It is the presence of such subgroups that distinguishes the *relative* from the *absolute* case. They are regular subgroups with a unique limit point.

In this section we collect geometric and algebraic information about and discuss some examples of subgroups with a unique limit point. We will see that they tend to consist of elements with zero infimal displacement. This leads us to also discussing subgroups with zero infimal displacement, in particular unipotent subgroups.

5.1 Cyclic subgroups

We first establish some properties of cyclic subgroups and their limit sets.

Let $g \in G$ be non-elliptic and consider the (discrete and free) cyclic subgroup $\langle g \rangle < G$.

We first look at the case $m_g > 0$ and let l be an oriented axis of the semisimple part g_s (see section ??). The orbits of $\langle g \rangle$ deviate sublinearly from l and their visual limit set is $\Lambda(\langle g \rangle) = \partial_\infty l$, cf. (??) and Proposition ??. If l is τ_{mod} -regular, then $l(\pm\infty) \in \text{ost}(\tau_\pm)$ for a pair of antipodal simplices $\tau_\pm \in \text{Flag}_{\tau_{mod}}$ and $\partial_\infty^{\tau_{mod}} l = \{\tau_-, \tau_+\}$.

Lemma 5.1. *Suppose that $m_g > 0$ and l is an axis of g_s . Then:*

(i) $\langle g \rangle$ is uniformly τ_{mod} -regular if and only if l is τ_{mod} -regular. In this case, $\Lambda_{\tau_{mod}}(\langle g \rangle) = \partial_\infty^{\tau_{mod}} l$ is a pair of antipodes.

(ii) If g is hyperbolic and $\langle g \rangle$ is τ_{mod} -regular, then $\langle g \rangle$ is uniformly τ_{mod} -regular.

Proof. (i) The equivalence follows from (??), since l contains g_s -orbits. If the g - and g_s -orbits are uniformly τ_{mod} -regular, then in view of Lemma ?? they have the same τ_{mod} -flag limits, $g^{\pm n} \rightarrow \tau_\pm$ and $g_s^{\pm n} \rightarrow \tau_\pm$, and thus $\Lambda_{\tau_{mod}}(\langle g \rangle) = \Lambda_{\tau_{mod}}(\langle g_s \rangle) = \partial_\infty^{\tau_{mod}} l$.

(ii) If g is hyperbolic, then l is an axis of g . Hence, if $\langle g \rangle$ is τ_{mod} -regular, then so is l . \square

If g is non-strictly parabolic, τ_{mod} -regularity of $\langle g \rangle$ does not imply uniform τ_{mod} -regularity. If $\langle g \rangle$ is non-uniformly τ_{mod} -regular, then the axis l of g_s is not τ_{mod} -regular. By Lemma ??, the limit set $\Lambda_{\tau_{mod}}(\langle g \rangle)$ then consists of simplices τ in $\text{Flag}_{\tau_{mod}}$ which are faces of chambers containing one of the ideal points $l(\pm\infty)$; both points $l(\pm\infty)$ must be covered.

In the σ_{mod} -regular case we obtain, supplementing the previous lemma:

Lemma 5.2. *If g is non-strictly parabolic and $\langle g \rangle$ is σ_{mod} -regular, then $|\Lambda_{\sigma_{mod}}(\langle g \rangle)| \geq 2$.*

Proof. By Lemma ??, both ideal points $l(\pm\infty)$ lie in a chamber contained in $\Lambda_{\sigma_{mod}}(\langle g \rangle)$. \square

We are left with the case when $m_g = 0$ and g is strictly parabolic. Then, according to the discussion in section ?? leading to (??), the orbits of $\langle g \rangle$ are Hausdorff close to horocycles in a totally-geodesic hyperbolic plane $X' \subset X$ and $|\Lambda(\langle g \rangle)| = 1$. We also had defined there the type $\theta(g) \in \sigma_{mod}$ and the face type $\tau_{mod}(g)$ of g ; both are ι -invariant.

Lemma 5.3. *If $g \in G$ is strictly parabolic, then the subgroup $\langle g \rangle$ is uniformly τ_{mod} -regular precisely for the face types $\tau_{mod} \subseteq \tau_{mod}(g)$, and $|\Lambda_{\tau_{mod}}(\langle g \rangle)| = 1$ for these τ_{mod} .*

Proof. The unique visual limit point ξ of $\langle g \rangle$ spans a simplex $\tau(g)$ of type $\tau_{mod}(g)$ and therefore is τ_{mod} -regular if and only if $\tau_{mod} \subseteq \tau_{mod}(g)$. Hence $\langle g \rangle$ is uniformly τ_{mod} -regular precisely for these τ_{mod} . From the definition of flag convergence it follows that $\Lambda_{\tau_{mod}}(\langle g \rangle)$ consists of the type τ_{mod} face of $\tau(g)$. \square

5.2 Elementary subgroups

Recall that an antipodal τ_{mod} -regular subgroup $\Gamma < G$ is τ_{mod} -elementary if $|\Lambda_{\tau_{mod}}(\Gamma)| \leq 2$. Since Γ is discrete, it holds that $\Lambda_{\tau_{mod}}(\Gamma) = \emptyset$ if and only if Γ is finite.

For cyclic subgroups, we saw in section ??:

Example 5.4. Uniformly τ_{mod} -regular cyclic subgroups $\langle g \rangle < G$ are τ_{mod} -antipodal elementary. Moreover, $|\Lambda_{\tau_{mod}}(\langle g \rangle)| = 1$ if and only if g is strictly parabolic.

Further examples are provided by rank one symmetric subspaces and products of rank one symmetric spaces:

Example 5.5. Let $G_1 < G$ and $X_1 \subset X$ be as in Example ???. Suppose that $\Gamma_1 < G_1$ is a discrete subgroup which consists entirely of parabolic and elliptic elements, equivalently, which preserves a horosphere $HS_1 \subset X_1$, whose center we denote by $\zeta_1 \in \partial_\infty X_1 \subset \partial_\infty X$. Then Γ_1 has visual limit set $\Lambda(\Gamma_1) = \{\zeta_1\}$ and is uniformly τ_{mod} -regular with $\Lambda_{\tau_{mod}}(\Gamma) = \{\tau_{\zeta_1}\}$ for the face types $\tau_{mod} \subseteq \tau_{mod}(X_1)$.

Example 5.6. Consider the product space $X = X_1 \times X_2 = \mathbb{H}^2 \times \mathbb{H}^2$. In this case, σ_{mod} is an arc of length $\frac{\pi}{2}$, and we denote by τ_{mod}^i the vertex of σ_{mod} corresponding to the hyperbolic plane factor X_i . Then $\text{Flag}_{\sigma_{mod}} \cong \text{Flag}_{\tau_{mod}^1} \times \text{Flag}_{\tau_{mod}^2}$ with $\text{Flag}_{\tau_{mod}^i} = \partial_\infty X_i \cong S^1$.

A *non-strictly parabolic isometry* of X has, up to switching the factors, the form $g = (g_1, g_2)$ with $g_1 \in \text{Isom}(X_1)$ hyperbolic (with two ideal fixed points $\lambda_\pm \in \partial_\infty X_1$) and $g_2 \in \text{Isom}(X_2)$ parabolic (with unique ideal fixed point $\mu \in \partial_\infty X_2$). The subgroup $\langle g \rangle < G$ is τ_{mod} -regular and τ_{mod} -elementary for all face types $\tau_{mod} \subseteq \sigma_{mod}$, but its uniformity and antipodality depend on τ_{mod} : It is non-uniformly σ_{mod} -regular with $\Lambda_{\sigma_{mod}}(\langle g \rangle) = \{(\lambda_-, \mu), (\lambda_+, \mu)\}$ and hence *not* σ_{mod} -antipodal. It is uniformly τ_{mod}^1 -regular with $\Lambda_{\tau_{mod}^1}(\langle g \rangle) = \{\lambda_-, \lambda_+\}$ and hence τ_{mod}^1 -antipodal. And it is non-uniformly τ_{mod}^2 -regular with $\Lambda_{\tau_{mod}^2}(\langle g \rangle) = \{\mu\}$ and hence also τ_{mod}^2 -antipodal.

In the case of two antipodal limit points and $\tau_{mod} = \sigma_{mod}$, we can say in general:

Proposition 5.7. *If $\Gamma < G$ is σ_{mod} -regular antipodal with $|\Lambda_{\sigma_{mod}}| = 2$. Then Γ is virtually cyclic and contains only semisimple elements.*

Proof. There exists a Γ -invariant maximal flat $F \subset X$ on which Γ acts by translations. Since $|\Lambda_{\sigma_{mod}}| = 2$, Γ must be virtually cyclic. \square

In the τ_{mod} -regular case, the algebraic conclusion (virtually cyclic) still holds and Γ must be uniformly τ_{mod} -regular, see [?, Lemma 5.45].

We now turn to discussing subgroups with a unique limit point.

5.3 Unique limit point versus zero infimal displacement

We begin with a geometric property of subgroups with a unique limit point:

Lemma 5.8. *(i) If $\Gamma < G$ is σ_{mod} -regular and $|\Lambda_{\sigma_{mod}}(\Gamma)| = 1$, then all elements of Γ have zero infimal displacement number, equivalently, are elliptic or strictly parabolic.*

(ii) If $\Gamma < G$ is uniformly τ_{mod} -regular and $|\Lambda_{\tau_{mod}}(\Gamma)| = 1$, then the same conclusion holds.

Proof. This is a direct consequence of Lemmas ??? and ???. \square

We now discuss consequences of zero infimal displacement.

For arbitrary subgroups with a fixed point on $\text{Flag}_{\sigma_{\text{mod}}}$, we can say the following. We recall that the stabilizer of a chamber $\sigma \in \text{Flag}_{\sigma_{\text{mod}}}$ is a minimal parabolic subgroup $P_\sigma < G$.

Proposition 5.9. *If a (not necessarily discrete) subgroup $\Gamma < P_\sigma$ consists of elements with zero infimal displacement, then it preserves every horocycle based at σ , i.e. $\Gamma < N_\sigma$.*

We note that the common stabilizer of all horocycles at σ is the horocyclic subgroup $N_\sigma \triangleleft P_\sigma$. It decomposes as the semidirect product $N_\sigma = U_\sigma \rtimes K_{\sigma, \hat{\sigma}}$, where $U_\sigma \triangleleft P_\sigma$ is the unipotent radical of P_σ , $\hat{\sigma}$ is a chamber opposite to σ , and $K_{\sigma, \hat{\sigma}}$ is the pointwise stabilizer in G of the maximal flat $F \subset X$ spanned by (asymptotic to) σ and $\hat{\sigma}$. (Compare e.g. the discussion in [?, secs 2.10 and 2.11, rem 2.42].)

Proof. P_σ preserves the (transversely Riemannian) foliation of X by horocycles based at σ . Every maximal flat $F \subset X$ asymptotic to σ is a cross section to this foliation and hence is naturally isometric to the leaf space. The action of P_σ on the leaf space is by translations. It follows that elements with zero infimal displacement number act trivially on it, i.e. preserve every horocycle at σ . \square

This has the following algebraic consequence for discrete subgroups:

Corollary 5.10. *If in addition Γ is discrete, then it is finitely generated and virtually nilpotent.*

Proof. This follows from Auslander's theorem (see Theorem ?? in the appendix). \square

We conclude for σ_{mod} -regular subgroups with unique limit point:

Corollary 5.11. *If $\Gamma < G$ is σ_{mod} -regular and $\Lambda_{\sigma_{\text{mod}}}(\Gamma) = \{\sigma\}$, then $\Gamma < N_\sigma$ and therefore Γ is finitely generated and virtually nilpotent.*

More generally, without a fixed point assumption, one can deduce from the work of Prasad [?] or already from Tits [?]:

Theorem 5.12. *Suppose that $\Gamma < G$ is a subgroup consisting only of elements with zero infimal displacement. Then there exists a unipotent Lie subgroup $N < G$ and a compact subgroup $K_N < G$ normalizing N , such that Γ is contained in $N \rtimes K_N$.*

Proof. To relate our condition of zero infimal displacement to the one used by Prasad, we note that $m_g = 0$ for $g \in G$ if and only if the transvection component g_t in the Jordan decomposition (??) is trivial, equivalently, if the adjoint action of g has all eigenvalues in S^1 .

Consider now the Zariski closure $\bar{\Gamma} < G$ of Γ . Let N denote the unipotent radical of the identity component $\bar{\Gamma}_0$ of $\bar{\Gamma}$. Then the projection Γ' of Γ to $G' = \bar{\Gamma}/N$ still consists only of elements of zero displacement and is Zariski dense in G' .

We claim that G' is compact. If not, then a theorem by Prasad [?] implies that Γ' contains elements g whose adjoint action has eigenvalues outside the unit circle, a contradiction.

Hence, G' is compact and we obtain that $\Gamma < N \rtimes K_N$ with $K_N \cong G'$. \square

We conclude for uniformly τ_{mod} -regular subgroups with unique limit point:

Corollary 5.13. *If $\Gamma < G$ is τ_{mod} -uniformly regular and $|\Lambda_{\tau_{mod}}(\Gamma)| = 1$, then there exists a unipotent Lie subgroup $N < G$ and a compact subgroup $K_N < G$ normalizing N , such that Γ is contained in $N \rtimes K_N$. In particular, Γ is finitely generated and virtually nilpotent.*

5.4 Unipotent subgroups

A natural class of zero displacement subgroups is provided by unipotent subgroups. We now look at their limit sets in some examples.

In rank one, unipotent subgroups always have a unique limit point. We also know that, in higher rank, unipotent *one-parameter* subgroups are τ_{mod} -regular⁷ for some type τ_{mod} depending on the subgroup and have a single τ_{mod} -limit point. In contrast, as we will see, unipotent subgroups of dimension ≥ 2 in higher rank are not necessarily τ_{mod} -regular for any τ_{mod} , and even if they are uniformly τ_{mod} -regular, they may fail to be τ_{mod} -elementary.

We now discuss this in the case of $G = SL(3, \mathbb{R})$.

We begin with one-parameter unipotent subgroups. There are two conjugacy classes of such subgroups. Each subgroup of either type is contained in a Lie subgroup locally isomorphic to $SL(2, \mathbb{R})$ and preserves a totally geodesic hyperbolic plane of θ -type equal to the midpoint $\bar{\mu}$ of the Weyl arc σ_{mod} . The subgroups conjugate to the group U_1 consisting of the elements

$$\begin{pmatrix} 1 & & t \\ & 1 & \\ & & 1 \end{pmatrix}$$

are contained in $SL(2, \mathbb{R}) \subset SL(3, \mathbb{R})$ (reducibly embedded). The subgroups conjugate to the group V_1 consisting of the elements

$$\begin{pmatrix} 1 & t & \frac{t^2}{2} \\ & 1 & t \\ & & 1 \end{pmatrix}$$

are contained in $SO(2, 1) \subset SL(3, \mathbb{R})$ (irreducibly embedded).

The unique limit flags of these subgroups can be determined as follows. Consider the unipotent subgroup $\exp(\mathbb{R} \cdot n)$ for a nilpotent element $n \in \mathfrak{sl}(3, \mathbb{R})$. If $\text{rank}(n) = 1$, then the limit flag equals $\text{im}(n) \subset \ker(n)$, and if $\text{rank}(n) = 2$, it equals $\text{im}(n^2) \subset \ker(n^2)$.

The geometry of the U_1 -orbit foliation of X is particularly nice: Since the normalizer of U_1 contains a minimal parabolic subgroup and therefore acts transitively on X , this foliation is homogeneous, i.e. any two U_1 -orbits are congruent. As a consequence, the Δ -distance of any pair of points in any U_1 -orbit has type $\bar{\mu}$, i.e. lies on the bisector of Δ . The V_1 -orbit foliation does not have either of these properties.

⁷I.e. their orbits are τ_{mod} -regular subsets.

Now we turn to two-parameter unipotent subgroups. There are three conjugacy classes represented by the subgroups U_2^\pm and V_2 consisting of the elements

$$\begin{pmatrix} 1 & \star & \star \\ & 1 & \\ & & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & & \star \\ & 1 & \star \\ & & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & t & s \\ & 1 & t \\ & & 1 \end{pmatrix},$$

respectively. Note that U_2^\pm are conjugate inside the full isometry group of X .

Again the foliations of X by U_2^\pm -orbits have nice geometry, even though they are no longer homogeneous: Since all one-parameter subgroups of U_2^\pm are conjugate to U_1 , the Δ -distance of any pair of points in any U_2^\pm -orbit still has type $\bar{\mu}$. In particular, U_2^\pm is *uniformly σ_{mod} -regular*.

However, the subgroups U_2^\pm have *large limit sets*: One verifies that they consist of the limit points of their one-parameter subgroups. In the case of U_2^+ these are the flags of the form $\langle e_1 \rangle \subset E^2$, and in the case of U_2^- the flags of the form $L^1 \subset \langle e_1, e_2 \rangle$.

We note that the same discussion applies to unipotent subgroups of $SL(n, \mathbb{R})$ of the form

$$\begin{pmatrix} 1 & \star & \dots & \star \\ & \ddots & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & & \star \\ & \ddots & \vdots \\ & & 1 & \star \\ & & & 1 \end{pmatrix}.$$

Returning to $SL(3, \mathbb{R})$, in contrast, the subgroup V_2 is *not σ_{mod} -regular* (and hence neither are its lattices). This is a consequence of the following fact about the non-regularity of sequences in the full horocyclic subgroup: Any diverging sequence of elements

$$\begin{pmatrix} 1 & x_n \\ & 1 & y_n \\ & & 1 \end{pmatrix}$$

where $0 < c \leq \frac{|x_n|}{|y_n|} \leq C$ is not σ_{mod} -regular, as one can see from its dynamics on $\mathbb{R}P^2$.

In conclusion, $SL(3, \mathbb{R})$ contains no non-cyclic discrete σ_{mod} -regular elementary unipotent subgroups.

6 Finsler-straight paths and maps in symmetric spaces

In this section we introduce a notion of Finsler-straightness which adapts the notion of straightness in Gromov hyperbolic spaces discussed earlier in section ?? to the geometry of higher rank symmetric spaces. This notion of straightness can be regarded as a regularity condition and is implicit in our earlier work on Morse quasigeodesics [?]. We will use it later on to define relative versions of our notions of Morse (equivalently, Anosov) subgroups, namely the notions of *relatively Morse*, see section ??, and *relatively straight* subgroups, see section ??.

The main results of this sections are Propositions ?? and ?? dealing with extensions of straight maps to infinity. They will be used in section ?? to construct boundary embeddings for straight subgroups.

6.1 Triples

We denote by $T(X) := X^3$ the space of triples of points in X and by

$$T(X, \text{Flag}_{\tau_{\text{mod}}}) := (X \sqcup \text{Flag}_{\tau_{\text{mod}}}) \times X \times (X \sqcup \text{Flag}_{\tau_{\text{mod}}})$$

the space of ideal triples in the Finsler bordification $X \sqcup \text{Flag}_{\tau_{\text{mod}}}$ with middle point in X .

We first define straightness for (non-ideal) triples in X :

Definition 6.1 (Finsler-straight triple). A triple $(x_-, x, x_+) \in T(X)$ is called

- (i) (Θ, d) -*straight*, $d \geq 0$, if the points x_-, x and x_+ are d -close to points x'_-, x' and x'_+ , respectively, which lie in this order on a Θ -Finsler geodesic.
- (ii) (τ_{mod}, d) -*straight* if the same property holds with Θ replaced by τ_{mod} .

In particular, a triple (x_-, x, x_+) is $(\Theta, 0)$ -straight if and only if the points x_-, x, x_+ lie in this order on a Θ -Finsler geodesic.

Finsler-straightness is stable under perturbation: Any triple $(\hat{x}_-, \hat{x}, \hat{x}_+)$ which is r -close to a (Θ, d) -straight triple (x_-, x, x_+) , i.e. $d(x_-, \hat{x}_-), d(x, \hat{x}), d(x_+, \hat{x}_+) \leq r$, is $(\Theta, d + r)$ -straight.

It is useful to note that (modulo doubling the constant d) the nearby Finsler geodesic in the definition can be chosen through one of the endpoints of the triple:

Lemma 6.2. *If (x_-, x, x_+) is (Θ, d) -straight, then the points x and x_+ are $2d$ -close to points x'' and x''_+ , respectively, so that x_-, x'' and x''_+ lie in this order on a Θ -Finsler geodesic.*

The same assertion holds with Θ replaced by τ_{mod} .

Proof: The points x' and x'_+ in the definition of Finsler-straightness are contained in a τ_{mod} -Weyl cone $V(x'_-, \text{st}(\tau_+))$. The Weyl cone $V(x_-, \text{st}(\tau_+))$ asymptotic to it has Hausdorff distance $\leq d(x_-, x'_-) \leq d$. It can be represented as the image $V(x_-, \text{st}(\tau_+)) = gV(x'_-, \text{st}(\tau_+))$ by an isometry $g \in G$ fixing τ at infinity and mapping $x'_- \mapsto x_-$. The isometry g has displacement $\leq d$ on the entire cone $V(x'_-, \text{st}(\tau_+))$. We put $x'' = gx'$, $x''_+ = gx'_+$ and choose the Θ -Finsler geodesic through x_- as the g -image of the one through x'_- given by the definition.⁸ \square

Note that the Finsler geodesic in the conclusion of the lemma can be chosen as a Finsler segment $x_-x''_+$ through x'' and is then contained in a Weyl cone $V(x_-, \text{st}(\tau_+))$, $\tau_+ \in \text{Flag}_{\tau_{\text{mod}}}$.

To show that the nearby Finsler geodesic can be chosen through both endpoints of the triple and to control its distance from the middle point, takes more effort.⁹

The notion of Finsler-straightness naturally extends to *ideal* triples in $T(X, \text{Flag}_{\tau_{\text{mod}}})$:

We say that a triple $(x_-, x, \tau_+) \in X^2 \times \text{Flag}_{\tau_{\text{mod}}}$ is (Θ, d) -straight if the points x_- and x are d -close to points x'_- and x' , respectively, such that x' lies on a Θ -Finsler ray $x'_-\tau_+$. This ray

⁸The points x'' and x''_+ can also be described as follows: The point x' lies on a Riemannian ray $x'_-\xi$ asymptotic to $\xi \in \text{st}(\tau_+)$. We choose $x'' \in x_-\xi$ with $d(x_-, x'') = d(x'_-, x')$. The point x''_+ is constructed similarly.

⁹Weyl cones vary 1-Lipschitz continuously with their tips, whereas we do not have such a result for diamonds at our disposal in full generality.

is then d -Hausdorff close to a Θ -Finsler ray $x_- \tau_+$, compare the proof of Lemma ??, and the latter passes within distance $2d$ from x . We say that (x_-, x, τ_+) is (τ_{mod}, d) -straight if the same property holds with Θ replaced by τ_{mod} . Analogously for triples $(\tau_-, x, x_+) \in \text{Flag}_{\tau_{mod}} \times X^2$.

Similarly, we say that a triple $(\tau_-, x, \tau_+) \in \text{Flag}_{\tau_{mod}} \times X \times \text{Flag}_{\tau_{mod}}$ is (τ_{mod}, d) -straight if the simplices τ_{\pm} are antipodal and x lies within distance d of a τ_{mod} -Finsler line $\tau_- \tau_+$. Note that (Θ, d) -straightness would be an equivalent property for triples (τ_-, x, τ_+) , because $\tau_- \tau_+$ can be chosen Θ -regular, which is why we do not introduce it.

6.2 Paths

6.2.1 Holey rays and lines

As for Gromov hyperbolic spaces, we call a map $q : H \rightarrow X$ from a subset of $H \subset \mathbb{R}$ a *holey line*. If H has a minimal element, we also call q a *holey ray*. (The domains of holey rays will usually be denoted H_0 below). A *sequence* $(x_n)_{n \in \mathbb{N}}$ in X can be regarded as a holey ray $\mathbb{N} \rightarrow X$.

We will consider extensions to infinity

$$\bar{q} : \bar{H} := H \sqcup \{\pm\infty\} \rightarrow X \sqcup \text{Flag}_{\tau_{mod}}$$

of holey lines $q : H \rightarrow X$ by sending $\pm\infty$ to simplices $\tau_{\pm} \in \text{Flag}_{\tau_{mod}}$, and refer to \bar{q} as an *extended holey line*. In the case of holey rays $q : H_0 \rightarrow X$, we consider extensions $\bar{q} : \bar{H}_0 := H_0 \sqcup \{+\infty\} \rightarrow X \sqcup \text{Flag}_{\tau_{mod}}$ by sending $+\infty$ to a simplex $\tau \in \text{Flag}_{\tau_{mod}}$, and refer to \bar{q} as an *extended holey ray*.

We carry over the notion of Finsler-straightness from triples to holey lines by requiring it for all triples in the image:

Definition 6.3 (Finsler-straight holey line). A holey line $q : H \rightarrow X$ is called

- (i) (Θ, d) -*straight* if the triples $(q(h_-), q(h), q(h_+))$ are (Θ, d) -straight for all $h_- \leq h \leq h_+$.
- (ii) (τ_{mod}, d) -*straight* if the same property holds with Θ replaced by τ_{mod} .

We say that q is Θ -*straight* if it is (Θ, d) -straight for some d , analogously for τ_{mod} -*straight*, and that q is *uniformly* τ_{mod} -*straight* if it is Θ -straight for some Θ .

Remark 6.4. (i) Finsler-straightness is preserved under restriction to subsets of H .

(ii) Finsler-straightness is stable under perturbation: If two holey lines $q, q' : H \rightarrow X$ are r -close, $d(q(h), q'(h)) \leq r$ for all $h \in H$, and q is (Θ, d) -straight, then q' is $(\Theta, d+r)$ -straight.

(iii) A holey line $q : H \rightarrow X$ is $(\Theta, 0)$ -straight if and only if q maps monotonically into a Θ -Finsler geodesic.

Similarly, we say that an extended holey line $\bar{q} : \bar{H} \rightarrow X \sqcup \text{Flag}_{\tau_{mod}}$ is (Θ, d) -*straight* if all triples $(\bar{q}(h_-), q(h), \bar{q}(h_+))$ in $X \sqcup \text{Flag}_{\tau_{mod}}$ for $-\infty \leq h_- \leq h \leq h_+ \leq +\infty$ in \bar{H} with $h \in H$ are (Θ, d) -straight, and analogously in the ray case. The properties (τ_{mod}, d) -*straight*, Θ -*straight*, τ_{mod} -*straight* and *uniformly* τ_{mod} -*straight* are then defined in the obvious way.

The (τ_{mod-}) straightness of an extended holey ray $\bar{q} : \bar{H}_0 \rightarrow X \sqcup \text{Flag}_{\tau_{mod}}$ implies that $q(H_0)$ lies within distance $2d$ of the Weyl cone $V(x_-, \text{st}(\tau_+))$ with $x_- = q(\min H_0)$ and $\tau_+ = \bar{q}(+\infty)$, however it does not imply flag-convergence $q(h) \rightarrow \tau_+$ as $h \rightarrow \sup H_0$ due to possible lack of regularity. If also q is τ_{mod} -regular, then $q(h) \rightarrow \tau_+$ conically. The straightness of an extended holey line $\bar{q} : \bar{H} \rightarrow X \sqcup \text{Flag}_{\tau_{mod}}$ implies that the simplices $\tau_{\pm} = \bar{q}(\pm\infty) \in \text{Flag}_{\tau_{mod}}$ are antipodal and the image $q(H)$ lies within distance d from the parallel set $P(\tau_-, \tau_+)$.

6.2.2 Asymptotics at infinity

We now discuss the *convergence at infinity* of Finsler straight holey rays.

To obtain flag-convergence, one needs to impose in addition regularity.

Lemma 6.5. *Let $x_n, x'_n \rightarrow \infty$ be τ_{mod} -regular sequences in X so that the triples (x, x_n, x'_n) are (τ_{mod}, d) -straight for some base point $x \in X$ and $d \geq 0$.*

Then the sequences (x_n) and (x'_n) have the same accumulation set in $\text{Flag}_{\tau_{mod}}$.

Proof. By straightness, there exists a sequence (τ_n) in $\text{Flag}_{\tau_{mod}}$ so that the points x_n, x'_n are contained in the $2d$ -neighborhood of the Weyl cone $V(x, \text{st}(\tau_n))$ for all n , see Lemma ???. It follows that the τ_{mod} -flag accumulation sets of both sequences (x_n) and (x'_n) in $\text{Flag}_{\tau_{mod}}$ coincide with the accumulation set of (τ_n) . \square

It follows that regular Finsler-straight holey rays converge at infinity, as long as they are unbounded. (Note that we allow “infinite holes” and put no restriction on the “speed”.) Here, we call a holey ray or line τ_{mod} -regular if its image in X is a τ_{mod} -regular subset.

Corollary 6.6. *If $q : H_0 \rightarrow X$ is a τ_{mod} -regular τ_{mod} -straight holey ray with unbounded image $q(H_0)$, then it τ_{mod} -flag converges at infinity, $q(h) \rightarrow \tau \in \text{Flag}_{\tau_{mod}}$ as $h \rightarrow \sup H_0$.*

Proof. Since $q(H_0)$ is unbounded, there exists a sequence $h_n \nearrow \sup H_0$ in H_0 so that the sequence $(q(h_n))$ in X diverges and hence is τ_{mod} -regular. By the compactness of $\text{Flag}_{\tau_{mod}}$, after passing to a subsequence, it flag converges, $q(h_n) \rightarrow \tau \in \text{Flag}_{\tau_{mod}}$.

If $h'_n \rightarrow \sup H_0$ is another sequence in H_0 , there exists a sequence of indices $m_n \rightarrow +\infty$ in \mathbb{N} growing slowly enough so that $h_{m_n} \leq h'_n$ for large n . The triples $(x, q(h_{m_n}), q(h'_n))$ are then (τ_{mod}, d) -straight for some base point $x \in X$ and $d > 0$. By Lemma ???, also $q(h'_n) \rightarrow \tau$. \square

In the *uniformly* regular case, we can show that the flag-convergence is *conical*:

Lemma 6.7. *If $q : H_0 \rightarrow X$ is a (Θ, d) -straight holey ray with unbounded image $q(H_0)$, then it conically τ_{mod} -flag converges at infinity, $q(h) \rightarrow \tau \in \text{Flag}_{\tau_{mod}}$ as $h \rightarrow \sup H_0$. More precisely, the extended holey ray $\bar{q} : \bar{H}_0 \rightarrow X \sqcup \text{Flag}_{\tau_{mod}}$ with $\bar{q}(+\infty) = \tau$ is still $(\Theta, 2d)$ -straight.*

Proof. Let $h_0 = \min H_0$ and $o = q(h_0)$.

By straightness, for any $h < h'$ in H_0 there exists a τ_{mod} -Weyl cone with tip at o which intersects both balls $\bar{B}(q(h), 2d)$ and $\bar{B}(q(h'), 2d)$.

With Lemma ?? it follows that for every $\epsilon > 0$ and every $h_1 \in H_0$ there exists $h_2 > h_1$ in H_0 with the property: If $h \leq h_1 < h_2 \leq h'$, then every τ_{mod} -Weyl cone with tip at o , which intersects $\overline{B}(q(h'), 2d)$, also intersects $B(q(h), 2d + \epsilon)$.

Now we take a sequence $h'_n \rightarrow \sup H_0$ and let $\tau_n \in \text{Flag}_{\tau_{\text{mod}}}$ so that $q(h'_n) \in V(o, \text{st}(\tau_n))$. Then for every $h \in H_0$ and $\epsilon > 0$ the Weyl cone $V(o, \text{st}(\tau_n))$ intersects $B(q(h), 2d + \epsilon)$ for all sufficiently large n . It follows that (τ_n) converges, $\tau_n \rightarrow \tau \in \text{Flag}_{\tau_{\text{mod}}}$, and that $q(H_0)$ is contained in $\overline{N}_{2d}(V(o, \text{st}(\tau)))$. \square

For *extended* Finsler-straight holey rays, already a weaker uniformity assumption implies conical flag-convergence, more precisely, closeness to a Finsler geodesic:

Claim 6.8. *For d, ϕ there exists r such that:*

If $q : H_0 \rightarrow X$ is a holey ray which admits a (τ_{mod}, d) -straight extension \bar{q} and is $(\tau_{\text{mod}}, \phi)$ -regular, then there exists a τ_{mod} -Finsler ray $q(0)\tau$ and a monotonic map $q' : H_0 \rightarrow q(0)\tau$ which is r -close to q , where $\tau = \bar{q}(+\infty)$.

Proof. By straightness, $q(H_0) \subset \overline{N}_{2d}(V(q(0), \text{st}(\tau)))$.

Let $q'' : H_0 \rightarrow V(q(0), \text{st}(\tau))$ be a map $2d$ -Hausdorff close to q , e.g. the nearest point projection to the Weyl cone. We extend q'' to infinity by $\bar{q}''(+\infty) := \tau$. Then \bar{q}'' is $(\tau_{\text{mod}}, 3d)$ -straight and $(\tau_{\text{mod}}, \phi - 4d)$ -regular.

The straightness of \bar{q}'' implies that, for $h_1 < h_2$ in H_0 , the point $q''(h_2)$ lies within distance $6d$ of the subcone $V(q''(h_1), \text{st}(\tau)) \subset V(q(0), \text{st}(\tau))$. We wish to show that it is contained in the subcone, provided that its distance from the tip $q''(h_1)$ is sufficiently large. To do so, let $s > 0$ so that $\phi(s) > 10d$. Then, if $d(q(h_1), q(h_2)) > s$, it holds that $d(d_\Delta(q''(h_1), q''(h_2)), \partial_\infty^{\tau_{\text{mod}}} \Delta) \geq \phi(s) - 4d > 6d$. It follows that $q''(h_2)$ has distance $> 6d$ from the boundary of the cone $V(q''(h_1), \text{st}(\tau))$ which forces it to lie inside it,

$$q''(h_2) \in V(q''(h_1), \text{st}(\tau)).$$

Now let $H_0^s \subset H_0$ be a maximal subset containing 0 so that $q(H_0^s)$ is s -spaced. Then $q(H_0)$ is s -Hausdorff close to $q(H_0^s)$. By the above, for $h_1 < h_2$ in H_0^s , it holds that $q''(h_2) \in V(q''(h_1), \text{st}(\tau))$. It follows that $q''|_{H_0^s}$ maps monotonically into some τ_{mod} -Finsler ray $q(0)\tau \subset V(q(0), \text{st}(\tau))$. By interpolation, it can be extended to a monotonic map $q' : H_0 \rightarrow q(0)\tau$ which is $(2d + s)$ -close to q . Thus, the assertion holds with $r = 2d + s$. \square

A corresponding result for holey lines $q : H \rightarrow X$ is readily derived:

Addendum 6.9. *For d, ϕ there exists r such that:*

If $q : H \rightarrow X$ is a holey line which admits a (τ_{mod}, d) -straight extension \bar{q} and is $(\tau_{\text{mod}}, \phi)$ -regular, then there exists a τ_{mod} -Finsler line $\tau_- \tau_+$ and a monotonic map $q' : H \rightarrow \tau_- \tau_+$ which is r -close to q , where $\tau_\pm = \bar{q}(\pm\infty)$.

Proof. Pick some $h_0 \in H$ and a point $q'(h_0) \in P(\tau_-, \tau_+)$ within distance d from $q(h_0)$. Applying Claim ?? to the two holey subrays of q starting in $q(h_0)$ yields monotonic maps into suitable

τ_{mod} -Finsler rays $q(h_0)\tau_{\pm}$. The latter are d -Hausdorff close to two τ_{mod} -Finsler rays $q'(h_0)\tau_{\pm}$ which together form a τ_{mod} -Finsler line $\tau_-\tau_+$ with the desired property. \square

When there are no arbitrarily large holes, regularity can be promoted to *uniform* regularity.

We say that a holey line $q : H \rightarrow X$ is *coarsely l -connected*, $l \geq 0$, if for any $h < \tilde{h}$ in H there exists a sequence $h_0 = h < h_1 < \dots < h_n = \tilde{h}$ in H so that $d(q(h_{i-1}), q(h_i)) \leq l$ for all i .

Claim 6.10. *For d, ϕ, l there exist Θ, r such that:*

If q is as in Addendum ?? and moreover coarsely l -connected, then the τ_{mod} -Finsler line $\tau_-\tau_+$ can be chosen to be Θ -regular.

Proof. Take $r = r(d, \phi)$ as in Addendum ??, and choose $s, a > 0$ so that $\phi(s) \geq 2r + a$. Then for $h < \tilde{h}$ in H with $d(q(h), q(\tilde{h})) \geq s$, the vector $d_{\Delta}(q'(h), q'(\tilde{h}))$ has distance $\geq a$ from $\partial_{\infty}^{\tau_{mod}} \Delta$.

Let $H_s \subset H$ be maximal so that $q(H_s)$ is s -spaced. Then $q|_{H_s}$ is coarsely $(2s + l)$ -connected, and consequently $q'|_{H_s}$ is coarsely $(2s + l + 2r)$ -connected. It follows that any pair of consecutive points in $q'(H_s)$ is Θ -regular for some $\Theta = \Theta(d, \phi, l)$. Since q' maps monotonically into a τ_{mod} -Finsler line $\tau_-\tau_+$ as in Addendum ??, it follows further that $q'|_{H_s}$ maps monotonically into a (different) Θ -Finsler line $\tau_-\tau_+$. The map $q'|_{H_s}$ can, by interpolation, be extended to a monotonic map $q' : H \rightarrow \tau_-\tau_+$ which, after suitably increasing $r = r(d, \phi, l)$, is r -close to q . \square

If one allows arbitrarily large holes, one needs an extra assumption to ensure uniform regularity. We will consider the following condition:

Definition 6.11 (Uniformly regular large holes). A holey line $q : H \rightarrow X$ with $H \subset \mathbb{R}$ closed and discrete is said to have $(\hat{\Theta}, l)$ -regular large holes if for any two consecutive elements $h < \tilde{h}$ in H with $d(q(h), q(\tilde{h})) > l$, the pair $(q(h), q(\tilde{h}))$ is $\hat{\Theta}$ -regular. We say that q has *uniformly τ_{mod} -regular large holes* if it has $(\hat{\Theta}, l)$ -regular large holes for some data $\hat{\Theta}, l$.

Then a very similar argument as for Claim ?? yields that uniformly regular large holes imply uniform regularity for the holey lines under consideration:

Claim 6.12. *For $d, \phi, l, \hat{\Theta}$ there exist Θ, r such that:*

Let q be as in Addendum ?? and suppose moreover that it has $(\hat{\Theta}, l)$ -regular large holes. Then the τ_{mod} -Finsler line $\tau_-\tau_+$ can be chosen to be Θ -regular.

6.2.3 Morse quasigeodesics

Morse quasigeodesics are a particular class of uniformly Finsler-straight holey lines (with “bounded holes”) which play a prominent role in our earlier work, see [?, ?, ?].

Definition 6.13 (Morse quasigeodesic). A (Θ, d, L, A) -Morse quasigeodesic in X is a (Θ, d) -straight holey line $q : I \rightarrow X$ which is defined on an interval $I \subseteq \mathbb{R}$ and is an (L, A) -quasiisometric embedding.

We will call a (Θ, d, L, A) -Morse quasigeodesic also briefly a τ_{mod} -Morse quasigeodesic.

One can show that τ_{mod} -Morse quasigeodesics are, up to quasiisometric reparameterization, uniformly close to τ_{mod} -Finsler geodesics. For quasirays and quasilines, this is a consequence of Claim ?? and Addendum ?. The definition therefore agrees with the definition given in our earlier papers.

The main result of [?] is that uniformly τ_{mod} -regular quasigeodesics are τ_{mod} -Morse. (Note that the converse holds trivially.)

6.3 Maps

6.3.1 Straight maps

We now generalize the notion of Finsler-straight holey line and introduce and study a class of maps from subsets of Gromov hyperbolic spaces into symmetric spaces which preserve straightness. Here we use in the hyperbolic spaces the notion of straightness in terms of closeness to geodesics (cf. Definition ??) and in the symmetric spaces the notion of straightness in terms of closeness to *Finsler* geodesics (cf. Definition ??) which is well-adapted to the higher rank geometry. Straight maps are coarse analogues of *projective maps* in Riemannian geometry which are smooth maps sending unparameterized geodesics to unparameterized geodesics.

Let Y be a δ -hyperbolic proper geodesic space and X a symmetric space. In the following,

$$f : A \rightarrow X$$

will always denote a *metrically proper* map defined on a subset $A \subset Y$.

Definition 6.14 (Finsler-straight map). The map $f : A \rightarrow X$ is called

- (i) Θ -*straight* if for every D there exists $d = d(\Theta, D)$ such that f sends D -straight triples in A to (Θ, d) -straight triples in X .
- (ii) τ_{mod} -*straight* if the same property holds with Θ replaced by τ_{mod} .
- (iii) *uniformly* τ_{mod} -*straight* if it is Θ -straight for some Θ .

Note that uniform Finsler-straightness implies (coarse) uniform regularity.

Finsler-straight maps carry straight holey lines to Finsler-straight holey lines.

Uniformly regular quasiisometric embeddings $Y \rightarrow X$ are uniformly straight [?].

We will use the notion of straightness also for *extensions* of maps to infinity: If $\beta : B \rightarrow \text{Flag}_{\tau_{mod}}$ is a map defined on a subset $B \subset \partial_{\infty} A$, we say that the combined map

$$\bar{f} = f \sqcup \beta : A \sqcup B \rightarrow X \sqcup \text{Flag}_{\tau_{mod}}$$

is Θ -*straight* if for every D there exists $d = d(\Theta, D)$ such that the induced map on triples

$$T(f \sqcup \beta) : T(A, B) \rightarrow T(X, \text{Flag}_{\tau_{mod}})$$

sends D -straight triples to (Θ, d) -straight triples, where we use the notation (cf. section ??)

$$T(A, B) := (A \sqcup B) \times A \times (A \sqcup B).$$

The properties τ_{mod} -straight and *uniformly* τ_{mod} -straight are defined accordingly.

6.3.2 Morse quasiisometric embeddings

Especially important are straight maps which are quasiisometric embeddings.

Definition 6.15 (Morse quasiisometric embedding). A map $Y \rightarrow X$ from a Gromov hyperbolic geodesic metric space Y is called a τ_{mod} -Morse quasiisometric embedding if it sends geodesics to uniform τ_{mod} -Morse quasigeodesics.

This is equivalent to the definitions given in our earlier papers (see [?, Def. 7.23] and [?, Def. 5.29]). Note that there we allowed more generally for quasigeodesic metric spaces as domains. However, we showed in [?, Thm. 6.13] that such domains are necessarily Gromov hyperbolic.

Reformulating the above definition, a quasiisometric embedding $Y \rightarrow X$ is τ_{mod} -Morse if and only if it is uniformly τ_{mod} -straight.

The main result of [?] implies that a quasiisometric embedding $Y \rightarrow X$ is τ_{mod} -Morse if and only if it is uniformly τ_{mod} -regular.

6.3.3 Asymptotics at infinity

It is plausible that straightness is related to good asymptotic behavior and the existence of boundary maps.

We first address *continuity at infinity*. We assume in the following that the map $f : A \rightarrow X$ is τ_{mod} -regular and τ_{mod} -straight. Our tool is the following direct consequence of Lemma ??:

Lemma 6.16. *Suppose that $y_n, y'_n \rightarrow \infty$ are divergent sequences contained in A so that the triples (y, y'_n, y_n) are D -straight for some base point $y \in Y$ and $D \geq 0$.*

Then the sequences $(f(y_n))$ and $(f(y'_n))$ in X have the same accumulation set in $\text{Flag}_{\tau_{mod}}$.

Definition 6.17 (Shadowing). A subset $S \subset A$ is *shadowing* a subset $\Sigma \subset \partial_\infty A$ at infinity (in A) if for every sequence $y_n \rightarrow \infty$ in A accumulating at Σ there exists a sequence $y'_n \rightarrow \infty$ in S such that the triples (y, y'_n, y_n) are D -straight for some base point $y \in Y$ and $D \geq 0$.

The last lemma immediately yields:

Lemma 6.18. *Suppose that $S \subset A$ is shadowing $\Sigma \subset \partial_\infty A$ in A . Then for every subset $W \subset A$ with $\partial_\infty W \subset \Sigma$ it holds that $\partial_\infty^{\tau_{mod}}(f(W)) \subseteq \partial_\infty^{\tau_{mod}}(f(S))$.*

In particular:

Corollary 6.19. *Suppose that $S, S' \subset A$ have the same visual accumulation set $\Sigma \subset \partial_\infty A$, and they are both shadowing Σ in A , then $\partial_\infty^{\tau_{mod}}(f(S)) = \partial_\infty^{\tau_{mod}}(f(S'))$.*

We will apply the last lemma in the following way:

Corollary 6.20. *If $S \subset A$ is shadowing the ideal point $\eta \in \partial_\infty A$ in A and $\partial_\infty^{\tau_{mod}}(f(S))$ consists of a single simplex τ , then f is continuously extended to η by mapping $\eta \mapsto \tau$.*

Example 6.21. (i) A conical accumulation point $\eta \in \partial_\infty^{con} A$ is shadowed (in Y) by a sequence in A conically converging to it.

(ii) Suppose that A is disjoint from a horoball $B \subset Y$ centered at $\zeta \in \partial_\infty Y$ and contains a subset S which has finite Hausdorff distance from the horosphere ∂B . Then $\zeta \in \partial_\infty A$ is shadowed (in A) by S .

Our second observation concerns *antipodality* at infinity.

Lemma 6.22. *Suppose that $y_n^\pm \rightarrow \infty$ are sequences in A whose accumulation sets in $\partial_\infty A$ are disjoint. Then the accumulation sets of the sequences $(f(y_n^\pm))$ in $\text{Flag}_{\tau_{mod}}$ are antipodal.¹⁰*

Proof. We may assume that the sequences $(f(y_n^+))$ τ_{mod} -converge, $(f(y_n^+)) \rightarrow \tau_\pm$. Let $y \in A$ be a base point. The assumption implies that the triples (y_n^-, y, y_n^+) in Y are D -straight for some D , cf. Lemma ???. By the Finsler-straightness of f , the triples $(f(y_n^-), f(y), f(y_n^+))$ in X are then (τ_{mod}, d) -straight for some d . This means that there exists a bounded sequence of τ_{mod} -parallel sets $P(\tau_n^-, \tau_n^+)$ such that $f(y_n^\pm)$ has uniformly bounded distance from $V(f(y), \text{st}(\tau_n^\pm))$. Then $\tau_n^\pm \rightarrow \tau_\pm$ by the definition of flag convergence. The antipodality of τ_\pm follows from the boundedness of the sequence of parallel sets. \square

We next apply these observations to show the existence of a *partial boundary map* at infinity.

For a map $\beta : B \rightarrow \text{Flag}_{\tau_{mod}}$ defined on a subset $B \subseteq \partial_\infty A$ we say that the combined map

$$f \sqcup \beta : A \sqcup B \rightarrow X \sqcup \text{Flag}_{\tau_{mod}}$$

is *continuous at infinity* if for every sequence (y_n) in A with $y_n \rightarrow \eta \in B$ it holds that $f(y_n) \rightarrow \beta(\eta)$ in the sense of flag convergence. Note that then β must necessarily be continuous.

We obtain that the map $f : A \rightarrow X$ extends continuously to the conical accumulation set:

Proposition 6.23. *There exists an antipodal continuous map $\partial_\infty^{con} f : \partial_\infty^{con} A \rightarrow \partial_\infty^{\tau_{mod}}(f(A)) \subset \text{Flag}_{\tau_{mod}}$ such that the extended map*

$$f \sqcup \partial_\infty^{con} f : A \sqcup \partial_\infty^{con} A \rightarrow X \sqcup \text{Flag}_{\tau_{mod}}$$

is continuous at infinity.

If f is uniformly τ_{mod} -straight, then $\partial_\infty^{con} f(\partial_\infty^{con} A) \subseteq \partial_\infty^{\tau_{mod}, con}(f(A))$.

Proof. Given a point $\eta \in \partial_\infty^{con} A$, we pick a sequence (y_n) in A converging to η conically. After extraction, this sequence moves to infinity “monotonically” in the sense that it is D -straight for some D . Hence, the image sequence $(f(y_n))$ in X is (τ_{mod}, d) -straight for some d . (It is also

¹⁰I.e. every accumulation point of $(f(y_n^+))$ is antipodal to every accumulation point of $(f(y_n^-))$.

τ_{mod} -regular due to our assumption that f is τ_{mod} -regular.) By Corollary ??, it τ_{mod} -converges at infinity,

$$f(y_n) \rightarrow \tau \in \partial_\infty^{\tau_{mod}}(f(A)) \subset \text{Flag}_{\tau_{mod}}.$$

Since the sequence (y_n) converges to η conically, it is shadowing η (in Y). Corollary ?? therefore implies that f is continuously extended to η by mapping $\eta \mapsto \tau$. This shows that there exists a well-defined map at infinity $\partial_\infty^{con} f : \partial_\infty^{con} A \rightarrow \partial_\infty^{\tau_{mod}}(f(A)) \subset \text{Flag}_{\tau_{mod}}$ so that the extension $f \sqcup \partial_\infty^{con} f$ is continuous at infinity.

The antipodality of $\partial_\infty^{con} f$ is a consequence of Lemma ??.

The last part follows from Lemma ??. □

In particular, we recover:

Corollary 6.24 (see [?, Theorem 6.14]). *For every Morse quasiisometric embedding $f : Y \rightarrow X$ there exists an antipodal continuous map $\partial_\infty f : \partial_\infty Y \rightarrow \partial_\infty^{\tau_{mod}, con}(f(Y)) \subset \text{Flag}_{\tau_{mod}}$ such that the extended map*

$$f \sqcup \partial_\infty f : Y \sqcup \partial_\infty Y \rightarrow X \sqcup \text{Flag}_{\tau_{mod}}$$

is continuous at infinity.

We now specialize the discussion to a setting motivated by RH groups. Here we can show the existence of *full* boundary maps:

Proposition 6.25. *Suppose that*

(i) *$A \subset Y$ has finite Hausdorff distance from the complement of a family $\mathcal{B} = (B_i)_{i \in I}$ of disjoint open horoballs;*

(ii) *for some, equivalently, every subset $S_i \subset A$ which has finite Hausdorff distance from a horosphere ∂B_i , the τ_{mod} -accumulation set $\partial_\infty^{\tau_{mod}}(f(S_i))$ consists of a single simplex τ_i .*

Then there exists an antipodal continuous map $\partial_\infty f : \partial_\infty A \rightarrow \partial_\infty^{\tau_{mod}}(f(A)) \subset \text{Flag}_{\tau_{mod}}$, sending the center $\zeta_i \in \partial_\infty A$ of each horoball B_i to τ_i , such that the combined map

$$f \sqcup \partial_\infty f : A \sqcup \partial_\infty A \rightarrow X \sqcup \text{Flag}_{\tau_{mod}}$$

is continuous at infinity.

If f is uniformly τ_{mod} -straight, then $\partial_\infty f(\partial_\infty^{con} A) \subseteq \partial_\infty^{\tau_{mod}, con}(f(A))$.

Regarding condition (ii), note that according to Corollary ?? the simplex τ_i is independent of the choice of S_i .

Proof. We continue the argument in the proof of the last proposition.

In order to further extend the boundary map $\partial_\infty^{con} f$ to the non-conical part $(\partial_\infty - \partial_\infty^{con})A$ of the accumulation set, we note that the latter consists of the centers ζ_i of the horoballs $B_i \in \mathcal{B}$.

Subsets $S_i \subset A$ as in hypothesis (ii) exist by hypothesis (i). Each subset S_i is shadowing the ideal point ζ_i (in A). We can therefore apply Corollary ?? again to obtain the desired continuous extension $\partial_\infty f$ of $\partial_\infty^{con} f$ by mapping $\zeta_i \mapsto \tau_i$ for all $i \in I$.

The antipodality of $\partial_\infty f$ follows again from Lemma ??.

□

7 Asymptotic conditions for subgroups

7.1 Relative asymptotic and boundary embeddedness

We start with characterizations of Anosov subgroups in terms of their topological dynamics on associated flag manifolds. The first such notion given in [?, Def. 5.12] is *asymptotic embeddedness*. The relative version is as follows:¹¹

Definition 7.1 (Relatively asymptotically embedded). A subgroup $\Gamma < G$ is called *relatively τ_{mod} -asymptotically embedded* if it is τ_{mod} -regular, antipodal and admits a structure as a relatively hyperbolic group (Γ, \mathcal{P}) such that there exist a Γ -equivariant homeomorphism

$$\alpha : \partial_\infty \Gamma \xrightarrow{\cong} \Lambda_{\tau_{mod}} \subset \text{Flag}_{\tau_{mod}}$$

from its ideal boundary to its τ_{mod} -limit set.

The definition can be phrased purely dynamically in terms of the Γ -action on $\text{Flag}_{\tau_{mod}}$ by replacing the τ_{mod} -regularity with the τ_{mod} -convergence condition, see [?]. Note that the peripheral structure is uniquely determined because it can be read off the dynamics on the limit set; the peripheral subgroups are the maximal ones with exactly one limit point in $\Lambda_{\tau_{mod}}$.

Since relatively hyperbolic groups act as convergence groups on their ideal boundaries, so do asymptotically embedded subgroups on their limit sets, and notions from the theory of abstract convergence groups apply to our setting, such as conical limit points, bounded parabolic points and bounded parabolic fixed points, see Definition ?. As explained in section ??, every limit point is either conical or bounded parabolic. The peripheral subgroups $\Pi_i < \Gamma$ are precisely the stabilizers of the bounded parabolic points τ_i , and $\Lambda(\Pi_i) = \{\tau_i\}$. For general relatively hyperbolic groups, the Π_i can be infinite torsion groups, however for asymptotically embedded subgroups this cannot occur, because they are linear, as follows from Schur's theorem or Selberg's lemma. Thus all bounded parabolic points are bounded parabolic fixed points.

Remark 7.2. For antipodal τ_{mod} -regular subgroups $\Gamma < G$ with at least two limit points, intrinsic conicality (defined in terms of the dynamics of $\Gamma \curvearrowright \Lambda_{\tau_{mod}}$) is equivalent to *extrinsic conicality* (defined in terms of the conical convergence of sequences in $\Gamma x \subset X$), see [?, Proposition 5.41 and Lemma 5.38].

For relatively asymptotically embedded subgroups, the orbit maps extend continuously to infinity by an asymptotic embedding (which is unique if the limit set has at least three points):

Lemma 7.3. *If $\Gamma < G$ is relatively τ_{mod} -asymptotically embedded and $x \in X$, then there is a continuous extension*

$$\bar{o}_x = o_x \sqcup \alpha : \bar{\Gamma} = \Gamma \sqcup \partial_\infty \Gamma \longrightarrow X \sqcup \Lambda_{\tau_{mod}}$$

¹¹It will be extended further beyond geometrically finite subgroups in Definition ??.

of the orbit map o_x by an asymptotic embedding α .

Proof. Suppose first that $|\partial_\infty\Gamma| \geq 3$. Let (γ_n) be a sequence in Γ converging to $\xi_+ \in \partial_\infty\Gamma$ in $\bar{\Gamma}$. It follows that there exists a point $\xi_- \in \partial_\infty\Gamma$ such that (γ_n) converges, as a sequence of maps, to ξ_+ uniformly on compacts in $\partial_\infty\Gamma - \{\xi_-\}$. We claim that (γ_n) flag-converges to $\alpha(\xi_+)$. Assume that this is not the case: In view of the τ_{mod} -regularity of Γ , there exist $\lambda_\pm \in \Lambda_{\tau_{mod}}$ such that, after extraction, (γ_n) converges to λ_+ uniformly on compacts in the open Schubert cell $C(\lambda_-)$, where $\lambda_+ \neq \alpha(\xi_+)$. Since $\Lambda_{\tau_{mod}}$ is antipodal, it follows that (γ_n) flag-converges to λ_+ uniformly on compacts in $\Lambda_{\tau_{mod}} - \{\lambda_-\}$. Since the limit set contains a third point beyond $\alpha(\xi_-)$ and λ_- , we conclude that $\tau = \lambda_+$, a contradiction.

If $|\partial_\infty\Gamma| \leq 1$, there is nothing to prove. If $|\partial_\infty\Gamma| = 2$ then $\mathcal{P} = \emptyset$, Γ is virtually cyclic (see Remark ??) and the claim follows from [?, Lemma 5.38]. \square

The following related condition is weaker than relative asymptotic embeddedness, but easier to verify, since there is no need to check regularity and to identify the limit set:

Definition 7.4 (Relatively boundary embedded). A discrete subgroup $\Gamma < G$ is called *relatively τ_{mod} -boundary embedded* if it admits a structure as a relatively hyperbolic group (Γ, \mathcal{P}) such that there exist an antipodal Γ -equivariant embedding

$$\beta : \partial_\infty\Gamma \longrightarrow \text{Flag}_{\tau_{mod}}$$

called a *boundary embedding*.

In the *Zariski dense* case, relative boundary embeddedness implies relative asymptotic boundary embeddedness (see [?, Corollary 5.14] for the absolute case):

Theorem 7.5. *If $\Gamma < G$ is relatively τ_{mod} -boundary embedded and Zariski dense, then it is relatively τ_{mod} -asymptotically embedded.*

Proof. By Zariski density, $\partial_\infty\Gamma$ is infinite and hence the action $\Gamma \curvearrowright \partial_\infty\Gamma$ is minimal. Applying Theorem ?? and Addendum ?? to the given boundary embedding, it follows that Γ is τ_{mod} -regular and $\Lambda_{\tau_{mod}} = \beta(\partial_\infty\Gamma)$. \square

7.2 A higher rank Beardon-Maskit condition

The actions of relatively hyperbolic groups on their ideal boundaries are convergence actions characterized by the Beardon-Maskit property [?]. We use this characterization to translate relative asymptotic embeddedness into a higher rank Beardon-Maskit condition for the action on the limit set. We formulate this condition for antipodal regular subgroups because their actions on the limit set are convergence:

Definition 7.6 (Relatively RCA). An antipodal τ_{mod} -regular subgroup $\Gamma < G$ is called *relatively τ_{mod} -RCA* if each τ_{mod} -limit point is either a conical limit point or a bounded parabolic

point (for the action $\Gamma \curvearrowright \Lambda_{\tau_{mod}}$) and, moreover, the stabilizers of the bounded parabolic points are finitely generated.

We recall, see section ??, that for the stabilizer $\Gamma_\tau < \Gamma$ of a bounded parabolic point $\tau \in \Lambda_{\tau_{mod}}(\Gamma)$ it holds that $\Lambda_{\tau_{mod}}(\Gamma_\tau) = \{\tau\}$.

All finitely generated regular subgroups with one point limit set are relatively RCA. In general, we know little about the structure of such subgroups. On the other hand, for *uniformly* regular subgroups with one point limit set we have the following information on their geometric and algebraic properties, which are well-known in rank one:

Lemma 7.7. *A uniformly τ_{mod} -regular subgroup of G with one point τ_{mod} -limit set consists of elements with zero translation number, is finitely generated and virtually nilpotent.*

Proof. This follows from Lemma ?? and Corollary ?. □

Clearly, relative asymptotic embeddedness implies relatively RCA. The converse is a consequence of Yaman's theorem. Thus:

Theorem 7.8. *A subgroup $\Gamma < G$ is relatively τ_{mod} -RCA if and only if it is relatively τ_{mod} -asymptotically embedded.*

In fact, Yaman's theorem applies only if $|\Lambda_{\tau_{mod}}| \geq 3$. If $|\Lambda_{\tau_{mod}}| = 2$, then $\mathcal{P} = \emptyset$ and Γ is (absolutely) τ_{mod} -asymptotically embedded by [?, Lemma 5.38].

7.3 More general relative settings

Let $\Gamma < G$ be a discrete subgroup. We equip Γ as an abstract discrete group with an additional *intrinsic* geometric structure in the form of a properly discontinuous isometric action $\Gamma \curvearrowright Y$ on a Gromov hyperbolic proper geodesic space Y . This action is *not* required to be cocompact. (If it is cocompact or, more generally, undistorted, then Γ is word hyperbolic and the additional intrinsic structure amounts to the choice of a word metric. This is the context in which we mostly worked in our earlier papers.) We are interested in geometric and dynamical properties of the action $\Gamma \curvearrowright X$ *relative* to the action $\Gamma \curvearrowright Y$.

To relate the actions $\Gamma \curvearrowright X$ and $\Gamma \curvearrowright Y$, we fix base points $x \in X$ and $y \in Y$ so that $\Gamma_y \leq \Gamma_x$ and consider the Γ -equivariant map of orbits

$$o_{x,y} : \Gamma y \rightarrow \Gamma x, \quad \gamma y \mapsto \gamma x.$$

Note that for any point $y \in Y$ there exists a point $x \in X$ fixed by Γ_y , because Γ_y is finite and finite groups acting isometrically on symmetric spaces have fixed points.

We further extend the relative versions of asymptotic and boundary embeddedness (see Definitions ?? and ??) to our present more general setting.

Definition 7.9 (Relatively boundary and asymptotically embedded II). A discrete subgroup $\Gamma < G$ is called

(i) τ_{mod} -boundary embedded relative $\Gamma \curvearrowright Y$ if there exists a Γ -equivariant antipodal embedding, called a *boundary embedding*,

$$\beta : \Lambda_Y \rightarrow \text{Flag}_{\tau_{mod}}.$$

(ii) τ_{mod} -asymptotically embedded relative $\Gamma \curvearrowright Y$ if it is τ_{mod} -regular, antipodal and there exists a Γ -equivariant homeomorphism, called an *asymptotic embedding*,

$$\alpha : \Lambda_Y \xrightarrow{\cong} \Lambda_{X, \tau_{mod}} \subset \text{Flag}_{\tau_{mod}}.$$

As before, in the non-degenerate case when $|\Lambda_Y| \geq 3$, an asymptotic embedding continuously extends the maps of orbits to infinity. Lemma ?? and its proof directly generalize:

Lemma 7.10. *If $\Gamma < G$ is τ_{mod} -asymptotically embedded relative $\Gamma \curvearrowright Y$ and if $|\Lambda_Y| \geq 3$, then the combined map*

$$\bar{o}_{x,y} = o_{x,y} \sqcup \alpha : \overline{\Gamma y} = \Gamma y \sqcup \Lambda_Y \rightarrow \overline{\Gamma x}^{\tau_{mod}} = \Gamma x \sqcup \Lambda_{X, \tau_{mod}} \subset X \sqcup \text{Flag}_{\tau_{mod}}$$

is continuous.

The continuity on the orbit Γy is trivial by discreteness.

8 Coarse geometric conditions for subgroups

We introduce two coarse geometric conditions which are a priori stronger than the asymptotic conditions discussed above. The advantage of these coarse geometric properties is that they allow for a *local-to-global principle* similar to the one for Morse subgroups (cf. [?, §7]) and hence define classes of discrete subgroups which are *structurally stable*. These conditions are also sometimes easier to verify in concrete situations. These aspects will be discussed elsewhere. The main results in this section compare the coarse geometric conditions to asymptotic embeddedness (see Theorems ??, ??, ?? and ??).

8.1 Relatively Morse subgroups

In our earlier paper [?] we defined Morse subgroups as finitely generated word-hyperbolic subgroups whose orbit maps are Morse quasiisometric embeddings. We relativize this as follows:

Definition 8.1 (Relatively Morse). A subgroup $\Gamma < G$ is called *relatively τ_{mod} -Morse* if there exists a relatively hyperbolic structure \mathcal{P} on Γ with a Gromov model Y and a Γ -equivariant τ_{mod} -Morse quasiisometric embedding $f : Y \rightarrow X$.

The peripheral subgroups have to be virtually nilpotent, as follows from Corollary ?. We will see that the peripheral structure \mathcal{P} is uniquely determined because relatively Morse implies relatively asymptotically embedded (see Corollary ??).

Relatively Morse subgroups are uniformly regular, since Morse quasiisometric embeddings are. The latter continuously extend to infinity (see Corollary ??).

In the equivariant situation (for general discrete subgroups which need not be relatively Morse) one can relate the limit sets:

Lemma 8.2. *Let $\Gamma < G$ be a discrete subgroup. Suppose that $\Gamma \curvearrowright Y$ is a properly discontinuous isometric action on a proper geodesic hyperbolic space Y and that $f : Y \rightarrow X$ is a Γ -equivariant τ_{mod} -Morse quasiisometric embedding.*

Then Γ is τ_{mod} -uniformly regular and $\Lambda_{X, \tau_{mod}} = \partial_\infty f(\Lambda_Y)$ is antipodal.

Proof. This follows from the uniform regularity of Morse quasiisometric embeddings, the continuity of $f \sqcup \partial_\infty f$ at $\partial_\infty Y$ and the antipodality of $\partial_\infty f$. \square

In the relatively Morse setting, we obtain:

Theorem 8.3. *Every relatively τ_{mod} -Morse subgroup $\Gamma < G$ is relatively τ_{mod} -asymptotically embedded. If $f : Y \rightarrow X$ is an equivariant τ_{mod} -Morse quasiisometric embedding as in the definition of relatively Morse subgroups, then $\partial_\infty f$ is an asymptotic embedding.*

Proof. In this situation $\Lambda_Y = \partial_\infty Y$ and the lemma yields the assertion. \square

Corollary 8.4. *The relatively hyperbolic structure on a relatively Morse subgroup is unique.*

Proof. This follows from the uniqueness of the relatively hyperbolic structure on relatively asymptotically embedded subgroups. \square

Theorem 8.5. *If X has rank one, then relatively Morse is equivalent to geometrically finite.*

Proof. According to Theorem ??, a relatively Morse subgroup is relatively asymptotically embedded and hence its action on its limit set satisfies the Beardon-Maskit condition. In rank one, this is classically known to be equivalent to geometric finiteness (see [?]).

Conversely, let $\Gamma < G$ be geometrically finite.

If $|\Lambda| \geq 2$, the closed convex hull of the limit set serves as a Gromov model, $Y := \text{CH}(\Lambda)$. The subgroups Π_i are the stabilizers of the bounded parabolic fixed points of Γ . There exists a Γ -invariant family of pairwise disjoint horoballs B_i in X such that Γ acts cocompactly on

$$Y^{th} = Y - \bigcup_i B_i,$$

and we use this as a Gromov model of (Γ, \mathcal{P}) . The Morse quasiisometric embedding is the inclusion $Y \hookrightarrow X$, and we see that $\Gamma < G$ is a relatively Morse subgroup.

If Λ consists of a single ideal point λ , we equip it with the trivial relatively hyperbolic structure $\mathcal{P} = \{\Gamma\}$. Let $B \subset X$ be a horoball centered at λ . It is preserved by Γ , and according to [?, sect. 4] there exists a Γ -invariant closed convex subset $C \subseteq X$ such that the action $\Gamma \curvearrowright \partial B \cap C$ is cocompact. (C can be obtained as the closed convex hull of a Γ -orbit in $\partial_\infty X - \{\lambda\}$.) One can then take $B \cap C$ as a Gromov model. \square

Corollary 8.6. *In rank one, relatively Morse is equivalent to relatively asymptotically embedded.*

Example 8.7. Let $\text{rank}(X) \geq 2$ and let $X_1 \subset X$ be a totally geodesic subspace of rank 1. It is τ_{mod} -regular for ι -invariant face-types $\tau_{mod} \subset \tau_{mod}(X_1)$, see Example ???. Let $\Gamma < G$ be a subgroup which preserves X_1 and acts on it as a geometrically finite group. Then Γ is τ_{mod} -Morse, the Gromov model being a convex subset of X_1 as described in the proof of Theorem ???.

8.2 Relatively Finsler-straight subgroups

This section is the heart of the paper. We define the notion of relatively Finsler-straight groups of isometries of symmetric spaces. For actions of relatively hyperbolic groups, we prove its equivalence to relative asymptotic embeddedness.

8.2.1 From straightness to boundary maps

We take up the discussion of Finsler-straight maps in an equivariant situation. We deduce from the results in section ?? that Finsler-straightness of maps of orbits implies the existence of partial and, under suitable assumptions, of full boundary maps.

We work in the general relative setting of section ???. The notion of Finsler-straightness for maps (see Definition ??) carries over to subgroups:

Definition 8.8 (Finsler-straight subgroup). A discrete subgroup $\Gamma < G$ is said to be

- (i) τ_{mod} -straight rel $\Gamma \curvearrowright Y$ if the map $o_{x,y}$ is τ_{mod} -straight.
- (ii) *uniformly* τ_{mod} -straight rel $\Gamma \curvearrowright Y$ if $o_{x,y}$ is uniformly τ_{mod} -straight.

Note that a τ_{mod} -straight subgroup $\Gamma < G$ is uniformly τ_{mod} -straight if and only if it is uniformly τ_{mod} -regular.

We will consider the notion of relative straightness only in the context of regular subgroups.

Now we use the results from section ?? in order to obtain boundary maps for Finsler-straight subgroups. Proposition ??, applied to the maps of orbits $o_{x,y}$, yields a *partial* asymptotic embedding:

Corollary 8.9. *If $\Gamma < G$ is τ_{mod} -straight relative $\Gamma \curvearrowright Y$, then there exists an antipodal map $\partial_\infty^{con} o_{x,y} : \Lambda_Y^{con} \rightarrow \Lambda_{X,\tau_{mod}} \subset \text{Flag}_{\tau_{mod}}$ such that the extended map*

$$o_{x,y} \sqcup \partial_\infty^{con} o_{x,y} : \Gamma y \sqcup \Lambda_Y^{con} \rightarrow \Gamma x \sqcup \Lambda_{X,\tau_{mod}} \subset X \sqcup \text{Flag}_{\tau_{mod}}$$

is continuous (at infinity).

If $\Gamma < G$ is uniformly τ_{mod} -straight relative $\Gamma \curvearrowright Y$, then $\partial_\infty^{con} o_{x,y}(\Lambda_Y^{con}) \subseteq \Lambda_{X,\tau_{mod}}^{con}$.

Note that the boundary map $\partial_\infty^{con} o_{x,y}$ is independent of the choice of the base points y, x .

Note also that, if $|\Lambda_Y| \geq 2$, then Λ_Y^{con} is nonempty [?, Thm. 2R] and hence dense in Λ_Y .

In the relatively hyperbolic setting, i.e. when $\Gamma \curvearrowright Y$ is the action on a Gromov model, we obtain a *full* asymptotic embedding under an additional assumption on the actions of the peripheral subgroups. Namely, we consider the following condition:

Definition 8.10 (Tied-up horospheres). A τ_{mod} -regular subgroup $\Gamma < G$ is said to have *tied-up horospheres* with respect to a relatively hyperbolic structure \mathcal{P} if the limit set $\Lambda_{X, \tau_{mod}}(\Pi_i) \subset \text{Flag}_{\tau_{mod}}$ of each peripheral subgroup $\Pi_i < \Gamma$ is a singleton.

We adapt Finsler-straightness as follows to relatively hyperbolic subgroups:

Definition 8.11 (Relatively Finsler-straight). A τ_{mod} -regular subgroup $\Gamma < G$ is called

(i) *relatively τ_{mod} -straight* if there exists a relatively hyperbolic structure \mathcal{P} on Γ with a Gromov model Y such that Γ is *τ_{mod} -straight* relative $\Gamma \curvearrowright Y$ and has tied-up horospheres.

(ii) *relatively uniformly τ_{mod} -straight* if in addition Γ is uniformly τ_{mod} -regular.

In view of Lemma ??, relative Finsler-straightness does not depend on the choice of the Gromov model Y .

Applying Proposition ?? now yields:

Theorem 8.12. *If $\Gamma < G$ is relatively τ_{mod} -straight, then it is relatively τ_{mod} -asymptotically embedded.*

Proof. We apply Proposition ?? with $A = \Gamma y$ and $f = o_{x,y}$. Hypothesis (i) of the proposition is satisfied, because Γ acts cocompactly on the thick part $Y^{th} \subset Y$ of the Gromov model, which equals the complement of the family of peripheral horoballs B_i . In hypothesis (ii) we can take $S_i = \Pi_i y$, because Π_i acts cocompactly on horospheres at ζ_i (cf. Lemma ??), and the condition is satisfied because Γ has tied-up horospheres. Since $\partial_\infty A = \partial_\infty(\Gamma y) = \partial_\infty Y$ and $\partial_\infty^{\tau_{mod}}(f(A)) = \partial_\infty^{\tau_{mod}}(\Gamma x) = \Lambda_{X, \tau_{mod}}$, the proposition yields an antipodal continuous map $\partial_\infty o_{x,y} : \partial_\infty Y \rightarrow \Lambda_{X, \tau_{mod}}$ sending $\zeta_i \mapsto \tau_i$ so that the extension $\bar{o}_{x,y} = o_{x,y} \sqcup \partial_\infty o_{x,y}$ is continuous at infinity. The latter implies that the image of $\partial_\infty o_{x,y}$ equals $\Lambda_{X, \tau_{mod}}$. \square

Note that, as a consequence of the theorem, the relatively hyperbolic structure in the definition of relative Finsler-straightness is *unique*.

8.2.2 From boundary maps to straightness

We now, conversely, explore what the existence of boundary maps implies for the orbit geometry of actions $\Gamma \curvearrowright X$. We show that asymptotic embeddedness implies Finsler-straightness. Our discussion follows [?, §5.3], generalizing it.

Suppose first that $\beta : \Lambda_Y \rightarrow \text{Flag}_{\tau_{mod}}$ is a *boundary embedding* relative $\Gamma \curvearrowright Y$.

Our preliminary step concerns the position of the Γ -orbits in X relative to the parallel sets spanned by pairs of simplices in the image of β (cf. [?, Lemma 5.3]):

Lemma 8.13. *For every D there exists d (also depending on the Γ -actions and -orbits) so that:*

If a triple $(\eta_-, \gamma y, \eta_+)$ with $\eta_{\pm} \in \Lambda_Y$ is D -straight, then the triple $(\beta(\eta_-), \gamma x, \beta(\eta_+))$ is (τ_{mod}, d) -straight.

Proof. By equivariance, we may assume that $\gamma = e$.

The set of pairs $(\eta_-, \eta_+) \in (\partial_{\infty} Y \times \partial_{\infty} Y) - \Delta_{\partial_{\infty} Y}$, for which the triple (η_-, y, η_+) is D -straight, is compact. It follows that the set C of their images $(\beta(\eta_-), \beta(\eta_+))$ in the space $(\text{Flag}_{\tau_{mod}} \times \text{Flag}_{\tau_{mod}})^{opp} \subset \text{Flag}_{\tau_{mod}} \times \text{Flag}_{\tau_{mod}}$ is also compact. Since $(\text{Flag}_{\tau_{mod}} \times \text{Flag}_{\tau_{mod}})^{opp}$ is a homogeneous G -space, it is of the form $C = C' \cdot (\tau_0^-, \tau_0^+)$ with a compact subset $C' \subset G$ and some antipodal pair (τ_0^-, τ_0^+) . The set of triples $(\beta(\eta_-), x, \beta(\eta_+)) = (g\tau_0^-, x, g\tau_0^+) = g(\tau_0^-, g^{-1}x, \tau_0^+)$ for $g \in C'$ is (τ_{mod}, d) -straight for some $d = d(D)$, because the set $C'^{-1}x \subset X$ is compact and depends on D . \square

The conclusion can be rephrased as follows: If γy lies within distance D of a line $\eta_- \eta_+ \subset Y$, $\eta_{\pm} \in \Lambda_Y$, then γx lies within distance $d(D)$ of the parallel set $P(\beta(\eta_-), \beta(\eta_+)) \subset X$.

In order to get more control, we strengthen our assumptions for the rest of this section:

Assumption 8.14. $\Gamma < G$ is τ_{mod} -asymptotically embedded relative $\Gamma \curvearrowright Y$ with asymptotic embedding $\alpha : \Lambda_Y \xrightarrow{\cong} \Lambda_{X, \tau_{mod}}$ and $|\Lambda_Y| \geq 3$.

Then $\Gamma < G$ is τ_{mod} -regular and α continuously extends the map of orbits $o_{x,y}$ (Lemma ??).

Now we can relate the position of the Γ -orbits in X to Weyl cones:

Lemma 8.15. *For every D there exists d such that:*

If a triple $(\gamma_- y, \gamma y, \eta_+)$, $\eta_+ \in \Lambda_Y$, is D -straight, then $(\gamma_- x, \gamma x, \alpha(\eta_+))$ is (τ_{mod}, d) -straight.

Proof. Let

$$R := d(y, \text{QCH}(\Lambda_Y)).$$

We may assume that $\gamma_- = e$, and denote $\eta_+ =: \eta$.

Suppose that the triple $(y, \gamma y, \eta)$ with $\eta \in \Lambda_Y$ is D -straight. Due to the quasiconvexity of $\text{QCH}(\Lambda_Y)$, the ray $y\eta$ lies within distance $R + C\delta$ of a geodesic line $\hat{\eta}\eta \subset Y$ with $\hat{\eta} \in \Lambda_Y$, and it follows that the triple $(\hat{\eta}, \gamma y, \eta)$ is $(D + R + C\delta)$ -straight.

By Lemma ??, the triple $(\alpha(\hat{\eta}), \gamma x, \alpha(\eta))$ is d -straight for some $d = d(D + R + C\delta)$. This means that γx lies within distance d of the parallel set $P(\alpha(\hat{\eta}), \alpha(\eta))$. It applies in particular to $x = ex$. It follows (compare [?, Dichotomy Lemma 5.5 and Proposition 5.16]) that γx lies within distance d of a Weyl cone $V(\bar{x}, \text{st}(\tau'))$ for a point $\bar{x} \in P(\alpha(\hat{\eta}), \alpha(\eta))$ with $d(x, \bar{x}) \leq d$ and a type τ_{mod} simplex $\tau' \subset \partial_{\infty} P(\alpha(\hat{\eta}), \alpha(\eta))$, that is, the triple $(x, \gamma x, \tau')$ is (τ_{mod}, d) -straight. It is important to note that $\alpha(\eta)$ is the only simplex contained in $\partial_{\infty} P(\alpha(\hat{\eta}), \alpha(\eta))$ which is antipodal to $\alpha(\hat{\eta})$. Hence either $\tau' = \alpha(\eta)$ or τ' is *not* antipodal to $\alpha(\hat{\eta})$.

Consider a sequence of D -straight triples $(y, \gamma_n y, \eta_n)$ with $\eta_n \in \Lambda_Y$ and $\gamma_n \rightarrow \infty$ in Γ , and corresponding sequences of ideal points $\hat{\eta}_n \in \Lambda_Y$, points $\bar{x}_n \in P(\alpha(\hat{\eta}_n), \alpha(\eta_n))$ and simplices $\tau'_n \subset \partial_{\infty} P(\alpha(\hat{\eta}_n), \alpha(\eta_n))$ as above. Then $\gamma_n x$ lies within distance d of the Weyl cone $V(\bar{x}_n, \text{st}(\tau'_n))$.

Suppose that the simplices τ'_n are not antipodal to the simplices $\alpha(\hat{\eta}_n)$ for all n . After extraction, we may assume that $\eta_n \rightarrow \eta$, $\hat{\eta}_n \rightarrow \hat{\eta}$, $\bar{x}_n \rightarrow \bar{x}$ and $\tau'_n \rightarrow \tau'$. Then $\alpha(\eta_n) \rightarrow \alpha(\eta)$, $\alpha(\hat{\eta}_n) \rightarrow \alpha(\hat{\eta})$, $\gamma_n x \rightarrow \tau'$ and $\tau' \subset \partial_\infty P(\alpha(\hat{\eta}), \alpha(\eta))$. Moreover, since the relation of being non-antipodal is closed with respect to the visual topology, τ' is not antipodal to $\alpha(\hat{\eta})$, and hence $\tau' \neq \alpha(\eta)$. On the other hand, $\gamma_n y \rightarrow \eta$ and hence $\gamma_n x \rightarrow \alpha(\eta)$ due to the continuity of $\bar{o}_{x,y}$, a contradiction.

It follows that, for all $\gamma \in \Gamma$ outside a finite subset of Γ depending on D , a triple $(x, \gamma x, \alpha(\eta))$ is (τ_{mod}, d) -straight whenever the triple $(y, \gamma y, \eta)$ with $\eta \in \Lambda_Y$ is D -straight. After suitably enlarging d , the implication holds for all $\gamma \in \Gamma$. \square

We rephrase the conclusion: If γy lies within controlled distance of a ray $(\gamma_- y)\eta_+ \subset Y$, $\eta_+ \in \Lambda_Y$, then γx lies within controlled distance of the Weyl cone $V(\gamma_- x, \alpha(\eta_+)) \subset X$.

In the next step, we control the straightness of triples in Γ -orbits:

Lemma 8.16. *For every D there exists d such that:*

If a triple $(\gamma_- y, \gamma y, \gamma_+ y)$ is D -straight, then $(\gamma_- x, \gamma x, \gamma_+ x)$ is (τ_{mod}, d) -straight.

Proof. Suppose that the triple $(\gamma_- y, \gamma y, \gamma_+ y)$ is D -straight. By the quasiconvexity of $\text{QCH}(\Lambda_Y)$, it is $C \cdot (D + R + \delta)$ -Hausdorff close to a triple of points lying in the same order on a geodesic line $\eta_- \eta_+ \subset Y$ with $\eta_\pm \in \Lambda_Y$. Hence its middle point γy lies within distance $C' \cdot (D + R + \delta)$ of geodesic rays $(\gamma_\mp y)\eta_\pm$.

By Lemmas ?? and ??, the points $\gamma_\pm x, \gamma x$ lie within distance d from the parallel set $P = P(\alpha(\eta_-), \alpha(\eta_+))$, and γx lies within distance d from the two Weyl cones $V_\pm = V(\gamma_\mp x, \text{st}(\alpha(\eta_\pm)))$ for some $d = d(D)$. (We suppress the dependence on the actions and orbits.)

Let $\bar{x}_\pm, \bar{x} \in P$ denote the nearest-point projections of $\gamma_\pm x, \gamma x$. Then the Weyl cones V_\pm and $\bar{V}_\pm = V(\bar{x}_\mp, \text{st}(\alpha(\eta_\pm))) \subset P$ have Hausdorff distance $\leq d(\gamma_\mp x, \bar{x}_\mp) \leq d$. Hence γx lies within distance $2d$ from both Weyl cones \bar{V}_\pm , and \bar{x} lies within distance $3d$ from them.

Now we invoke again the τ_{mod} -regularity of Γ . It implies the existence of a finite subset $F \subset \Gamma$ depending on D such that: If $\gamma^{-1}\gamma_\mp \notin F$, then $\bar{x} \in \bar{V}_\pm$. If both conditions are satisfied, then \bar{x} lies on a τ_{mod} -Finsler geodesic $\bar{x}_- \bar{x}_+$, and hence the triple $(\gamma_- x, \gamma x, \gamma_+ x)$ is (τ_{mod}, d) -straight.

If one of the elements $\gamma^{-1}\gamma_\mp$ lies in F , then the corresponding distance $d(\gamma_\mp x, \gamma x)$ is bounded and the conclusion holds trivially after increasing d sufficiently. \square

Lemmas ??, ?? and ?? yield together:

Proposition 8.17. *The extension $\bar{o}_{x,y} = o_{x,y} \sqcup \alpha$ of the map of orbits $o_{x,y}$ is τ_{mod} -straight.*

Specializing to the relatively hyperbolic setting, we obtain the converse to Theorem ??:

Theorem 8.18. *If $\Gamma < G$ is relatively τ_{mod} -asymptotically embedded, then it is relatively τ_{mod} -straight.*

Proof. In the case when $|\partial_\infty Y| \geq 3$, i.e. when Assumption ?? is satisfied, the straightness of

Γ relative to the action on the Gromov model is the content of the proposition. That Γ has tied-up horospheres is immediate from asymptotic embeddedness.

If $|\partial_\infty Y| = 2$, then $\mathcal{P} = \emptyset$ and we are in the absolute case. There, asymptotic embeddedness implies Morse, and this in turn straightness [?].

If $|\partial_\infty Y| = 1$, then $\mathcal{P} = \{\Gamma\}$ and relative straightness amounts to the τ_{mod} -regularity of Γ and $|\Lambda_{X,\tau_{mod}}| = 1$. Both properties follow from asymptotic embeddedness. \square

8.2.3 Further to uniform straightness

We return to the more general setting of a subgroup $\Gamma < G$ which is asymptotically embedded relative to an action $\Gamma \curvearrowright Y$ as in Assumption ???. We show that under suitable assumptions the subgroup Γ is *uniformly* regular.

We continue the discussion of the previous section and further promote the control on the position of triples in orbits of $\Gamma \curvearrowright X$ to control on the position of *holey lines*. By Proposition ??, we know that straight holey lines $q : H \rightarrow \Gamma y$ go to Finsler-straight holey lines $o_{x,y} \circ q : H \rightarrow \Gamma x$. We establish next that the latter lie near Finsler geodesics in X . To achieve this, we use that the holey lines $o_{x,y} \circ q$ are asymptotically embedded and satisfy, as parts of orbits of the regular action $\Gamma \curvearrowright X$, a weak form of *uniform* regularity. This allows us to apply Addendum ??, and we obtain:

Proposition 8.19. *For D there exists d such that:*

If $q : H \rightarrow \Gamma y$ is a D -straight holey line, then there exists a τ_{mod} -Finsler line $\tau_- \tau_+$, $\tau_\pm \in \Lambda_{X,\tau_{mod}}$, and a monotonic map $q' : H \rightarrow \tau_- \tau_+$ which is d -close to the holey line $o_{x,y} \circ q : H \rightarrow \Gamma x$.

Proof. By straightness, $q(H)$ lies within distance $D' = D'(D)$ of a line in Y , and within distance $D'' = D''(D, R)$ of a line $\eta_- \eta_+ \subset Y$ asymptotic to Λ_Y , $\eta_\pm \in \Lambda_Y$. The extended holey line $\bar{q} : \bar{H} = H \sqcup \{\pm\infty\} \rightarrow \bar{Y} = \Gamma y \sqcup \Lambda_Y$ with $\bar{q}(\pm\infty) = \eta_\pm$ is D'' -straight.

Since $\bar{o}_{x,y} = o_{x,y} \sqcup \alpha$ is τ_{mod} -straight by Proposition ??, it follows that the extended holey line $\overline{o_{x,y} \circ q} = \bar{o}_{x,y} \circ \bar{q} : \bar{H} \rightarrow \Gamma x \sqcup \Lambda_{X,\tau_{mod}}$ mapping $\pm\infty \mapsto \alpha(\eta_\pm) =: \tau_\pm$ is (τ_{mod}, d) -straight for some $d = d(D)$. Furthermore, the holey lines $o_{x,y} \circ q$ are weakly uniformly τ_{mod} -regular as a consequence of the τ_{mod} -regularity of the action $\Gamma \curvearrowright X$. Applying Addendum ?? yields the assertion. \square

Since the image holey lines $o_{x,y} \circ q$ in X follow Finsler geodesics, their weak uniform regularity turns into (strong) uniform regularity when there are no arbitrarily large holes, i.e. when the holey lines $o_{x,y} \circ q$, equivalently, the straight holey lines q are coarsely connected:

Claim 8.20. *For D, L there exist Θ, d such that:*

If $q : H \rightarrow \Gamma y$ is a D -straight holey line which is coarsely L -connected, then the τ_{mod} -Finsler line $\tau_- \tau_+$ in Proposition ?? can be chosen to be Θ -regular.

Proof. The holey line $o_{x,y} \circ q : H \rightarrow \Gamma x$ is then coarsely l -connected with $l = l(L)$ and the assertion is a consequence of Claim ???. \square

Straight holey lines in Γy with holes of bounded size are, up to reparameterization, *quasi-geodesics*. We conclude that $o_{x,y}$ sends uniform quasigeodesics in Γy to uniformly τ_{mod} -regular uniform quasigeodesics in Γx :

Theorem 8.21. *Suppose that $\Gamma < G$ is τ_{mod} -asymptotically embedded rel $\Gamma \curvearrowright Y$. Then for L, A there exist l, a, Θ, d such that:*

If $q : I \rightarrow \Gamma y \subset Y$ is an (L, A) -quasigeodesic, then $o_{x,y} \circ q : I \rightarrow \Gamma x \subset X$ is an (l, a) -quasigeodesic which is contained in the d -neighborhood of a Θ -Finsler geodesic.

Remark 8.22. In the ‘‘absolute’’ case, that is, when $\Gamma \curvearrowright Y$ is cocompact (or undistorted) and hence Γ is Gromov hyperbolic, this recovers our earlier result that τ_{mod} -asymptotically embedded subgroups $\Gamma < G$ are τ_{mod} -Morse [?].

The theorem yields some *partial* uniform regularity for the map of orbits $o_{x,y}$. In order to obtain full uniform regularity, we impose an additional assumption, cf. Definition ???. We then can extend Claim ??? as follows:

Claim 8.23. *For $D, \hat{\Theta}, l$ there exist Θ, d such that:*

If $q : H \rightarrow \Gamma y$ is a D -straight holey line and $o_{x,y} \circ q : H \rightarrow \Gamma x$ has $(\hat{\Theta}, l)$ -regular large holes, then the τ_{mod} -Finsler line $\tau_- \tau_+$ in Proposition ??? can be chosen to be Θ -regular.

Proof. The assertion follows from Claim ???.

□

If any two orbit points can be connected by such a holey line, we obtain uniform regularity:

Corollary 8.24. *Let $\Gamma < G$ be τ_{mod} -asymptotically embedded rel $\Gamma \curvearrowright Y$. Suppose that there exist data $D, \hat{\Theta}, l$ and $y \in Y$ such that for each $\gamma \in \Gamma$ the points y and γy can be connected by a D -straight holey line $q : H \rightarrow \Gamma y$ so that $o_{x,y} \circ q : H \rightarrow \Gamma x$ has $(\hat{\Theta}, l)$ -regular large holes.*

Then Γ is uniformly τ_{mod} -regular and hence uniformly τ_{mod} -straight rel $\Gamma \curvearrowright Y$.

Proof. The uniform straightness follows from uniform regularity together with the straightness proven earlier in Proposition ???.

□

In the next section, we will apply this result in the relatively hyperbolic setting.

8.2.4 Relatively hyperbolic subgroups

We now restrict to relatively hyperbolic subgroups $\Gamma < G$ and the case when $\Gamma \curvearrowright Y$ is the action on a Gromov model. In this setting we obtain the following criterion for uniform regularity:

Theorem 8.25. *Suppose that $\Gamma < G$ is relatively τ_{mod} -asymptotically embedded and that each peripheral subgroup $\Pi_i < \Gamma$ is uniformly τ_{mod} -regular.*

Then $\Gamma < G$ is uniformly τ_{mod} -regular and hence relatively uniformly τ_{mod} -straight.

Proof. First consider the case when $|\partial_\infty Y| \geq 3$, i.e. when Assumption ??? is satisfied. In order

to apply Corollary ??, we need to check the connectability condition there for the orbits of the action $\Gamma \curvearrowright Y$ on a Gromov model (Y, \mathcal{B}) of (Γ, \mathcal{P}) .

We may assume that $y \in Y^{th}$. Given $\gamma \in \Gamma$, we connect y and γy by a geodesic in Y . Along this geodesic, we choose a monotonic sequence of points $y_0 = y, y_1, \dots, y_n = \gamma y$ in Y^{th} so that any two successive points y_{k-1} and y_k have distance $\leq d_0$ for some fixed constant $d_0 > 0$ or lie on the same peripheral horosphere ∂B , $B \in \mathcal{B}$.

Since the action $\Gamma \curvearrowright Y^{th}$ is cocompact, we may choose orbit points $\gamma_k y$ at uniformly bounded distance from the points y_k . The sequence $(\gamma_k y)$, viewed as a holey line $q : H = \{0, \dots, n\} \rightarrow \Gamma y$, is D -straight with a constant D independent of γ .

Since also the actions $\Pi_i \curvearrowright \partial B_i$ of the peripheral subgroups on the corresponding horospheres are cocompact, and since there are finitely many conjugacy classes of peripheral subgroups, there exists a finite subset $\Phi \subset \Gamma$ independent of γ such that

$$\gamma_{k-1}^{-1} \gamma_k \in \Phi \left(\bigcup_i \Pi_i \right) \Phi$$

for all k .

Due to our assumption that the subgroups $\Pi_i < G$ are uniformly τ_{mod} -regular, there exist $\hat{\Theta}$ and another finite subset $\Phi' \subset \Gamma$, both independent of γ , such that the pair of points $(\gamma' x, \gamma'' x)$ in Γx is $\hat{\Theta}$ -regular whenever $\gamma'^{-1} \gamma'' \in \Phi \left(\bigcup_i \Pi_i \right) \Phi - \Phi'$. This means that the holey line $o_{x,y} \circ q$, which corresponds to the sequence $(\gamma_k x)$ in Γx , has $(\hat{\Theta}, l)$ -large holes for a sufficiently large constant l independent of γ . Hence Corollary ?? implies the assertion.

If $|\partial_\infty Y| = 2$, then $\mathcal{P} = \emptyset$ and we are in the absolute case. There, asymptotic embeddedness implies Morse, and this in turn uniform regularity [?].

If $|\partial_\infty Y| = 1$, then $\mathcal{P} = \{\Gamma\}$ and the hypothesis of the theorem implies that Γ is uniformly τ_{mod} -regular. □

9 Comparing conditions for subgroups

The following is the main theorem. It summarizes the relation between different conditions on relatively hyperbolic subgroups established in the paper:

Theorem 9.1. *For subgroups $\Gamma < G$, the following implications hold:*

- (i) *relatively Morse \Rightarrow relatively uniformly Finsler-straight*
- (ii) *relatively Finsler-straight \Leftrightarrow relatively asymptotically embedded \Leftrightarrow relatively RCA*
- (iii) *relatively asymptotically embedded with uniformly regular peripheral subgroups \Rightarrow relatively uniformly Finsler-straight*
- (iv) *relatively boundary embedded and Zariski dense \Rightarrow relatively asymptotically embedded*

Proof. (i) The maps of orbits are uniformly straight because Morse quasiisometric embeddings are. Furthermore, since relatively Morse implies relatively asymptotically embedded, see Theorem ??, Γ has tied-up horospheres.

(ii) The first equivalence is the combination of Theorems ?? and ?. The second equivalence is Yaman's theorem, see Theorem ??.

(iii) is Theorem ??.

(iv) is Theorem ??. □

In rank one, all conditions become equivalent:

Corollary 9.2. *If the symmetric space X has rank one, then the following properties are equivalent for discrete subgroups $\Gamma < G$:*

(i) *relatively Morse*

(ii) *relatively straight*

(iii) *relatively asymptotically embedded*

(iv) *relatively RCA*

(v) *relatively boundary embedded*

(vi) *geometrically finite*

Proof. The implications (i) \Rightarrow (ii) \Leftrightarrow (iii) \Leftrightarrow (iv) \Rightarrow (v) hold in arbitrary rank. In rank one, relative RCA amounts to the usual Beardon-Maskit condition which is equivalent to geometric finiteness (see [?]), thus (iv) \Leftrightarrow (vi). Furthermore, (vi) \Leftrightarrow (i), see Theorem ??.

To get from (v) to the other conditions, we observe that for non-elementary subgroups (v) \Rightarrow (iii) holds because the limit set is the unique minimal nonempty Γ -invariant closed subset of $\partial_\infty X$ and hence must equal the image of the boundary embedding. In the elementary case, we have (v) \Rightarrow (vi) since in rank one all elementary discrete subgroups are geometrically finite. □

10 Appendix (by Grisha Soifer): Auslander's Theorem

Theorem 10.1. *Let G be a Lie group which splits as a semidirect product $G = N \rtimes K$ where N is connected nilpotent and K is compact. Then each discrete subgroup $\Gamma < G$ is finitely generated and virtually nilpotent.*

Proof. This theorem first appeared in Auslander's paper [?], but its proof was flawed. The theorem can be derived from a more general result [?] (in the torsion-free case). Notice that in the paper we are only interested in subgroups of finitely generated subgroups linear groups, which are virtually torsion-free by the Selberg Lemma. Hence, in this setting, Auslander's Theorem can be viewed as a corollary of [?]. If Γ is assumed to be finitely generated then Auslander's Theorem can be viewed as a corollary of Gromov's Polynomial Growth Theorem (which is much easier in this setting since Γ is already assumed to be a subgroup of a connected Lie group).

Nevertheless, for the sake of completeness, we present a direct and self-contained proof, which is well-known to experts, but which we could not find in the literature.

Step 1. Let us show that, after passing to a finite index subgroup in Γ , we can assume that Γ is a discrete subgroup of a closed connected solvable subgroup of G . Indeed, since K is compact, the quotient K/K^0 is compact, where K^0 is the identity component of K ; hence $\Gamma \cap NK^0$ is a finite index subgroup in Γ . Therefore we can assume that K is a connected compact Lie group. Let R be the solvable radical of G . Obviously, $R = NL$, where L is an abelian normal compact subgroup of K . Let $\pi : G \rightarrow G/R$ be the quotient homomorphism. By another Auslander theorem, [?, Theorem 8.24], the connected component of the closure $\overline{\pi(\Gamma)}$ in G/R is an abelian group. Since G/N is compact, G/R is compact as well. Let $\bar{\Gamma} = \pi^{-1}(\overline{\pi(\Gamma)}^0) \cap \Gamma$. Clearly, $\bar{\Gamma}$ is a finite index subgroup of Γ and $\bar{\Gamma}$ is a subgroup of the solvable group $\pi^{-1}(\overline{\pi(\Gamma)}^0)$.

Step 2. From now on, we will assume that G is a solvable connected group, $G = N \rtimes K$, where N is a connected nilpotent group and K is a compact abelian group.

Let us show that we can assume that N is simply connected.

Lemma 10.2. *For every connected nilpotent group Lie N there exists a maximal compact subgroup $T < N$ which is characteristic in N , such that the quotient N/T is a simply connected nilpotent Lie group.*

Proof. Consider a compact subgroup $C < N$ and the adjoint representation $Ad_N : N \rightarrow GL(\mathfrak{n})$, where \mathfrak{n} is the Lie algebra of N . Since C is a compact group, for every $c \in C$ we have that $Ad_N(c)$ is a semisimple linear transformation. On the other hand, $Ad_N(c)$ is nilpotent since the group N is nilpotent. Therefore $Ad_N(c) = \mathbf{1}$, hence, $C \subseteq Z(N)$. Thus, each compact subgroup of N is contained in the center $Z(N)$ of N and, by commutativity, the union of all compact subgroups of $Z(N)$ forms a compact subgroup $T < Z(N)$. Since the center is a characteristic subgroup and each automorphism of N sends compact subgroups to compact subgroups, T is a characteristic subgroup of N . \square

In our setting, the maximal compact subgroup $T < N$ will be a normal subgroup of G . By the compactness of T and discreteness of Γ , the intersection $\Gamma \cap T$ is a finite subgroup of Γ . Consider the quotient homomorphism $\pi : G \rightarrow G_1 = G/T$. The kernel of $\pi|_{\Gamma}$ is a finite normal subgroup of Γ . The quotient $\Gamma_2 = \pi(\Gamma)$ is a discrete subgroup of the group $G_1 = N_1K_1$, where $K_1 = \pi(K)$ and $N_1 = \pi(N)$ is a simply connected nilpotent group. We will work with the subgroup $\Gamma_1 := \pi(\Gamma)$ of G_1 . Once we know that Γ_1 is virtually a uniform lattice in a connected nilpotent Lie group, it is finitely generated and, hence, polycyclic. It then will follow that the group Γ itself is residually finite according to [?]; cf. Lemma 11.77 in [?]. Thus, Γ contains a finite index subgroup $\tilde{\Gamma}_1$ such that $\pi : \tilde{\Gamma}_1 \rightarrow \Gamma_1$ is injective. This reduces the problem to the case when N is simply connected and K is compact abelian.

Since G is a connected solvable group with simply connected nilpotent radical N , there exists a faithful linear representation $\rho : G \rightarrow GL(d, \mathbb{R})$ such that $\rho(n)$ is a unipotent matrix for every $n \in N$; see e.g. [?] and also [?]. We will identify G with $\rho(G)$. Then N is an algebraic subgroup of $GL(d, \mathbb{R})$ (since N is unipotent, both the exponential and logarithmic maps of N are polynomial).

Step 3. This is the key step in the proof. Let $N_2 < N$ be the Zariski closure of $\Gamma \cap N$. Since K is abelian, we have $[\Gamma, \Gamma] < N$; since N is normal in G , Γ normalizes N_2 .

Recall that a discrete subgroup of a simply connected (algebraic) nilpotent group H is Zariski dense in H if and only if it is a cocompact lattice in H , see [?, Theorem 2.3, page 30]. Therefore, in our case, $\Gamma \cap N = \Gamma \cap N_2$ is a cocompact lattice in N_2 . In particular, this intersection is finitely generated. As we noted above, the subgroup Γ normalizes N_2 . Our next goal is to prove that $N_2\Gamma = N_2 \rtimes \Gamma$ is a closed subgroup of G . This will be a corollary of a more general lemma about Lie groups:

Lemma 10.3. *Let H, N be closed subgroups of a Lie group G , such that the intersection $N \cap H$ is a cocompact subgroup of N and H normalizes N . Then the subgroup NH is closed in G .*

Proof. Let $K \subset H$ be a compact such that $HK = N$. Consider a convergent sequence (g_i) in NH . Then every g_i can be written as $g_i = k_i h_i$, $k_i \in K \subset N, h_i \in H$. After extraction, $k_i \rightarrow k \in K \subset N$, hence, $h_i \rightarrow h$ and, since H is closed, h is in H . Therefore,

$$g_i \rightarrow kh \in NH. \quad \square$$

Specializing to our situation, where the role of the closed subgroup H is played by Γ , and taking into account that N_2 is normalized by Γ , we obtain

Corollary 10.4. *$\Gamma N_2 < G$ is a closed Lie subgroup in G (in the classical topology).*

Corollary 10.5. *The identity component of ΓN_2 coincides with that of N_2 .*

Proof. This follows from countability of Γ . □

Let \mathfrak{n}_2 be the Lie algebra of N_2 . By the above observation about the identity component, the Lie algebra of ΓN_2 coincides with \mathfrak{n}_2 . Let $Ad : \Gamma N_2 \rightarrow GL(\mathfrak{n}_2)$ be the adjoint representation. We will use the following idea due to Margulis: There exists a full-rank lattice Δ in the vector space \mathfrak{n}_2 such that $\exp(\Delta)$ is a finite index subgroup of $\Gamma \cap N_2$; see e.g. [?, sect. 3.1].

The number of subgroups of the given index in a finitely generated group, such as $\Gamma \cap N$, is finite. Therefore, taking into account the fact that $\Gamma \cap N$ is normal in Γ , there exists a subgroup of finite index $\bar{\Gamma}$ of Γ such that $\exp \Delta$ is $\bar{\Gamma}$ -invariant. Since $|\Gamma : \bar{\Gamma}| < \infty$ we can and will assume that $\exp \Delta$ is Γ -invariant, i.e. is invariant under the action of Γ by conjugation on N . Thus Δ is $Ad(\Gamma)$ -invariant, and, by identifying Δ with \mathbb{Z}^m , $m = \dim \mathfrak{n}_2$, we have $Ad\gamma \in GL(m, \mathbb{Z})$ for each $\gamma \in \Gamma$. After passing to a further finite index subgroup of Γ , we can assume that $Ad(\Gamma) < SL(m, \mathbb{Z})$.

Consider the Jordan decomposition $Ad\gamma = \gamma_s \gamma_u$ of $Ad\gamma$, $\gamma \in \Gamma$, where γ_s is the semisimple and γ_u is the unipotent part of the decomposition. Since the maps $Ad\gamma \mapsto \gamma_u$ and $Ad\gamma \mapsto \gamma_s$ are restrictions of \mathbb{Q} -rational maps $GL(m, \mathbb{R}) \rightarrow GL(m, \mathbb{R})$ we have $\gamma_u \in SL(m, \mathbb{Q})$ and $\gamma_s \in SL(m, \mathbb{Q})$. (See [?, p. 158].)

Lemma 10.6. *For each unipotent element $u \in GL(m, \mathbb{Q})$, there exists $q = q_u \in \mathbb{N}$ such that $u^q \in GL(m, \mathbb{Z})$.*

Proof. There exists $q_1 = q_1(m)$ such that for each unipotent element $u \in GL(m, \mathbb{R})$,

$$(\mathbf{1} - u)^{q_1+1} = 0.$$

Assume now that $u \in GL(m, \mathbb{Q})$; let M denote the product of denominators of all matrix entries of $u_1, \dots, u_1^{m_1}$ and set $q_2 := M \cdot q_1!$, $u_1 := \mathbf{1} - u$. Then

$$u^{q_2} = \sum_{k=0}^{q_2} \binom{q_2}{k} u_1^k = \sum_{k=0}^{q_1} \binom{q_2}{k} u_1^k.$$

Clearly,

$$\binom{q_2}{k} u_1^k \in Mat(m, \mathbb{Z})$$

for all $k \leq n_1$. Hence, for $q = q_2$, $u^q \in GL(m, \mathbb{Z})$. □

In our case, given that $\gamma_u \in SL(m, \mathbb{Q})$, we conclude that there exists a positive integer q_γ such that $\gamma_u^{q_\gamma} \in SL(m, \mathbb{Z})$. Since $\gamma_s \gamma_u = \gamma_u \gamma_s$ and $Ad\gamma \in SL(m, \mathbb{Z})$, we have $\gamma_s^{q_\gamma} \in SL(m, \mathbb{Z})$. On the other hand, every $\gamma \in \Gamma$ is the product $\gamma = nk$ where $n \in N, k \in K$. As we noted above, $Ad(n)$ is unipotent; by compactness of K , for every eigenvalue λ of γ_s we have $|\lambda| = 1$. Hence $\gamma_s^{q_\gamma}$ is a finite order element of the discrete group $SL(m, \mathbb{Z})$. Thus there exists p_γ such that $\gamma_s^{p_\gamma} = \mathbf{1}$. From this it follows that $\gamma^{p_\gamma} \in N$.

Lemma 10.7. Γ is finitely generated.

Proof. The subgroup $\Gamma \cap N_2$ is finitely generated because it is a cocompact lattice. For the same reason, the projection of Γ to the connected abelian group G/N_2 is discrete. Therefore, this projection is finitely generated as well. It follows that Γ itself is finitely generated. □

Therefore, since the projection of Γ to G/N_2 is a finitely generated torsion group, this projection has to be finite. In particular, $\Gamma \cap N_2$ has finite index in Γ . Since $\Gamma \cap N_2$ is a cocompact lattice, Theorem ?? follows. □

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