3-manifolds with(out) metrics of nonpositive curvature *

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October 3, 1994

Abstract. In the context of Thurstons geometrisation program we address the question which compact aspherical 3-manifolds admit Riemannian metrics of nonpositive curvature. We prove that a Haken manifold with, possibly empty, boundary of zero Euler characteristic admits metrics of nonpositive curvature if the boundary is non-empty or if at least one atoroidal component occurs in its canonical topological decomposition. Our arguments are based on Thurstons Hyperbolisation Theorem. We give examples of closed graph-manifolds with linear gluing graph and arbitrarily many Seifert components which do not admit metrics of nonpositive curvature.

Contents

1 Introduction

It is known since the last century that a closed surface admits particularly nice Riemannian metrics, namely metrics of constant curvature. All aspherical surfaces, i.e. all surfaces besides the sphere and the projective plane can be given a metric of nonpositive curvature.

*appeared in Inventiones Mathematicae vol. 122, no. 2 (1995), pp. 277–289.

Thurston suggested a geometrisation procedure in dimension 3. Here, constant curvature metrics are a too restricted class of model geometries, and one allows more generally complete locally homogeneous metrics. There are 8 types of 3-dimensional geometries which can occur, namely the model spaces are $S^3, S^2 \times R^1, R^3, Nil, Sol$, $\widetilde{SL(2,\mathbb{R})}$, $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{H}^3$. The classification is due to Thurston [Th1] and a detailed exposition can be found in Scott's article [S]. Despite the large supply of model geometries it is far from being true that any closed 3-dimensional manifold M is *geometric*, that is, admits a geometric structure. One must cut M into suitable pieces. The *canonical topological decomposition* is obtained as follows: According to Kneser, there is a maximal decomposition of M as a finite connected sum. The summands are called prime manifolds and their homeomorphism type is determined by M if M is orientable, whereas the decomposition itself is in general not unique (Milnor). The aspherical prime manifolds can be further decomposed by cutting along incompressible embedded tori and Klein bottles, see [J-S, Jo]. The components which one obtains in this second step are Seifert fibered or atoroidal. Thurstons *Geometrisation Conjecture* asserts that one can put a geometric structure of unique type on each piece of the canonical topological decomposition of M. It is well-known that a closed 3-manifold is Seifert if and only if it can be given one of the geometries different from Sol and $\mathbb{H}^3$, see [S]. The atoroidal components conjecturally admit a hyperbolic structure and this has been proven by Thurston [Th2] for components which are Haken, i.e. contain a closed incompressible surface. In particular, all Haken 3-manifolds are geometrisable.

We are interested in those 3-manifolds M which have a chance to admit metrics of nonpositive curvature. In the context of Thurstons geometrisation program, we address in this paper the

Question: *Which compact aspherical 3-manifolds M admit Riemannian metrics of nonpositive sectional curvature?*

The situation is well-understood for geometric 3-manifolds. If M is Haken and non-geometric, then each component in the canonical decomposition admits a geometric structure which is modelled on one of the nonpositively curved geometries $\mathbb{R}^3, \mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{H}^3$. The known obstructions to the existence of a nonpositively curved metric on M vanish, i.e. all solvable subgroups of $\pi_1(M)$ are virtually abelian and centralisers virtually split. It is therefore only natural to ask whether one can put compatible nonpositively curved metrics on all components of M simultaneously. We show that, as suggested by the geometrisation program, non-geometric Haken manifolds indeed generically admit metrics of nonpositive curvature. More precisely, we prove the following existence results:

Theorem 3.2 *Suppose that M is a graph-manifold, i.e. contains only Seifert components, and has non-empty boundary. Then there exists a Riemannian metric of nonpositive curvature on M.*

And, relying on Thurstons Hyperbolisation Theorem [Th2]:

Theorem 3.3 *Let M be a Haken manifold with, possibly empty, boundary of zero Euler characteristic. Suppose that at least one atoroidal component occurs in its canonical decomposition. Then M admits a Riemannian metric of nonpositive curvature.*

The Riemannian metrics can be chosen smooth and flat near the boundary. Our existence results yield new examples of closed nonpositively curved manifolds of geometric rank one with non-hyperbolic fundamental group.

Differently from the situation in dimension 2, not all closed aspherical 3-manifolds admit metrics of nonpositive curvature. Examples can already be found among geometric manifolds, namely quotients of Nil, Sol or $\widetilde{SL(2,\mathbb{R})}$. We show that also non-geometric Haken manifolds cannot always be equipped with a metric of nonpositive curvature. We give the following

Example 4.2 *There are closed graph-manifolds glued from arbitrarily many Seifert components which do* **not** *admit metrics of nonpositive curvature.*

Note that nevertheless the fundamental groups of such manifolds are nonpositively curved *in the large*: We prove in [K-L2] that the fundamental group of a closed Haken 3-manifold is quasi-isometric to Nil, Sol or the fundamental group of a closed nonpositively curved 3-manifold.

The paper is organised as follows: In section 2, we explain that any nonpositively curved metric on a Haken manifold arises from nonpositively curved metrics on its geometric components (section 2.1) and is *rigid* on Seifert components. The atoroidal components, on the other hand, are *flexible* (section 2.2). The extent of rigidity for the Seifert pieces is described in section 2.3. The rigidity is responsible for the non-existence examples (section 4.1) but leaves sufficient degrees of freedom (Proposition 2.6) to produce nonpositively curved metrics on Haken manifolds with atoroidal components *or* non-empty boundary (section 3).

We refer to [Ch-E] for an introduction to the geometry of nonpositive curvature and to [Ja] for concepts from 3-dimensional topology.

Acknowledgements. The results in this paper are part of my thesis [L]. I benefitted from the inspiring environment provided by the geometry group at the University of Maryland. I am particularly grateful to Bill Goldman, Karsten Grove and John Millson for their advice and encouragement during my time as a graduate student and to Misha Kapovich for many helpful discussions. I thank the Alfred P. Sloan foundation for supporting me with a dissertation fellowship.

2 Nonpositively curved metrics on the geometric components

2.1 Reduction to geometric components

In this section M will always denote a compact Haken 3-manifold with boundary of zero Euler characteristic. We recall that a compact smooth irreducible 3-manifold is *Haken* if it contains a closed incompressible surface, that is, a closed smooth embedded 2-sided surface whose fundamental group is infinite and injects via the canonical inclusion homomorphism, cf. [Ja].

According to [J-S, Jo], there is a canonical topological decomposition of M into compact 3-manifolds which are Seifert fibered or atoroidal. It is obtained by cutting M along finitely many disjoint closed embedded incompressible tori and Klein bottles Σ_i which we call the *splitting* or *decomposing* surfaces. We refer to the pieces of the decomposition as *geometric components* because they can be equipped with canonical geometric structures. If the topological decomposition of M is non-trivial, the geometric components have non-empty incompressible boundary. The Seifert pieces then admit a $\mathbb{R}^3$- or $\mathbb{H}^2 \times \mathbb{R}$-structure (compare [S]) whereas the atoroidal pieces admit $\mathbb{H}^3$-structures [Th2]. A minimal topological decomposition of M is unique up to isotopy.

Assume now that M carries a Riemannian metric g of nonpositive sectional curvature. We only allow metrics where the boundary is totally-geodesic and (hence) flat. In this geometric situation, there is an analogous geometric decomposition of (M, g) along totally-geodesically embedded flat surfaces, cf. [L, L-S]. Since a minimal topological decomposition of M is unique, it can therefore be geometrised in the presence of g:

Lemma 2.1 ([L, L-S]) *There is an isotopy of M which moves the splitting surface $\Sigma := \cup_i \Sigma_i$ to a totally-geodesically embedded flat surface in M.*

It follows that each metric of nonpositive curvature on M arises modulo isotopy in the following way: Put suitable flat metrics on the decomposing surfaces Σ_i and extend them to nonpositively curved metrics on the geometric components of the topological decomposition. It is therefore crucial to understand the following question:

Extension Problem 2.2 *Let X be a geometric component. Which flat metrics on the boundary ∂X can be extended to metrics of nonpositive curvature on X?*

Atoroidal and Seifert components behave differently with respect to the extendability of metrics. As discussed below, atoroidal components are so *flexible* that the extension problem can be solved for all flat metrics prescribed on the boundary, whereas the rigidity of nonpositively curved Seifert manifolds puts restrictions on the solvability of the extension problem for Seifert components. The interdependence of flat metrics on the boundary tori of a nonpositively curved Seifert manifold is the source of obstructions to the existence of nonpositively curved metrics on graph-manifolds.

2.2 Nonpositively curved metrics on atoroidal manifolds

Let X be an atoroidal component of M. We already mentioned a deep result of Thurston [Th2], his Hyperbolisation Theorem, that the interior $intX$ of X admits a complete metric g_0 of constant negative curvature, say -1, and finite volume. The hyperbolic structure g_0 is in fact unique according to Mostows rigidity theorem. By modifying g_0 we can solve the extension problem:

Proposition 2.3 (Flexibility of atoroidal components) *Any flat metric on ∂X can be extended to a smooth nonpositively curved metric on X which is flat near the boundary.*

Proof: Let h be a prescribed flat metric on ∂X. We will change the hyperbolic structure g_0 in order to extend h to all of X. The ends of $(intX, g_0)$ are hyperbolic cusps: Outside a suitable compact subset, the metric g_0 is isometric to a warped product metric

$$e^{-2t} g_{\partial X} + dt^2$$

on $\partial X \times \mathbb{R}^+$ where $g_{\partial X}$ is a flat metric on ∂X. The conformal type of $g_{\partial X}$ is determined by g_0. There is no relation between the metrics $g_{\partial X}$ and h. As a first step, we adjust the conformal type of the cusps of g_0 to the prescribed metric h. To do so it suffices to allow more general warped product metrics: after isotoping h if necessary we can diagonalise h with respect to $g_{\partial X}$, that is, we write $g_{\partial X}$ and h as

$$g_{\partial X} = dx^2 + dy^2 \qquad \text{and} \qquad h = a^2 \cdot dx^2 + b^2 \cdot dy^2$$

where dx^2 and dy^2 are positive-semidefinite sub-Riemannian metrics parallel with respect to $g_{\partial X}$ with orthogonal one-dimensional kernels and a, b are positive functions constant on each boundary component Σ_i. To interpolate between the conformal types of $g_{\partial X}$ and h, we put a metric of the form

$$e^{-2t}[\phi + (1-\phi) \cdot a]^2 \cdot dx^2 + e^{-2t}[\phi + (1-\phi) \cdot b]^2 \cdot dy^2 + dt^2$$

on $\partial X \times \mathbb{R}^+$ where the smooth function $\phi : \mathbb{R}^+ \to [0,1]$ is required to be equal to 1 in a neighborhood of 0 and equal to 0 in a neighborhood of ∞. A curvature calculation shows that the sectional curvature of a metric of this type is negative if the component functions $e^{-t}[\phi + (1-\phi) \cdot a]$ and $e^{-t}[\phi + (1-\phi) \cdot b]$ are strictly monotonically decreasing and convex. This holds for all functions ϕ whose first and second derivatives are bounded by sufficiently small constants depending on a and b. Hence we can find a complete negatively curved metric g_1 on $intX$ which is outside a suitable compact subset hyperbolic and isometric to the warped product metric

$$e^{-2t} h + dt^2$$

on $\partial X \times (T_0, \infty)$ for some $T_0 \in \mathbb{R}$.

In the second step, we replace e^{-2t} by a convex and monotonically decreasing function $\psi : (T_0, \infty) \to (0, \infty)$ which coincides with e^{-2t} in a neighborhood of T_0 and is constant in a neighborhood of ∞. The curvature of the resulting complete metric g_2 is nonpositive because ψ is convex. After rescaling, $(intX, g_2)$ is outside some compact subset isometric to a Euclidean cylinder with base $(\partial X, h)$. We cut off the ends along cross sections of the cylinders and obtain a smooth nonpositively curved metric on X which is flat near the boundary and extends h. $\square$

2.3 Nonpositively curved metrics on Seifert manifolds

We address the Extension Problem 2.2 for Seifert manifolds. Let X be a Seifert fibered manifold with non-empty incompressible boundary and denote by O its base orbifold. We have the exact sequence

$$1 \to \langle f \rangle \to \pi_1(X) \to \pi_1(O) \to 1$$

where the element f is represented by a generic Seifert fiber. We recall that due to the cyclic normal subgroup of the fundamental group, any nonpositively curved metric

g on X must have a special form: The universal cover decomposes as a Riemannian product

$$(\tilde{X}, g) \cong \mathbb{R} \times Y \tag{1}$$

where Y is nonpositively curved, cf. [E]. The lines $\mathbb{R} \times \{y\}$ are the axes of the deck-transformation f, and they project down to a Seifert fibration of X by closed geodesics of, apart from finitely many singular fibers, equal length. The deck-action of $\pi_1(X)$ preserves the splitting (1) and hence decomposes as the product of a representation

$$\phi : \pi_1(X) \to Isom(\mathbb{R}) \cong \mathbb{R} \rtimes \mathbb{Z}/2\mathbb{Z} \tag{2}$$

and a cocompact discrete action of $\pi_1(O)$ on Y which induces an orbifold-metric of nonpositive curvature on O. Since f acts on its axes by translations, the representation ϕ satisfies the condition

$$\phi(f) \in \mathbb{R} \setminus \{0\}. \tag{3}$$

We see that a nonpositively curved metric on X correponds to the choice of a nonpositively curved metric on O and a representation (2) satisfying (3). It is well-known that the Seifert manifold X does admit metrics of nonpositive curvature. This follows, for instance, from Theorem 5.3 in [S]; namely, the closed Seifert manifold obtained from doubling X admits a metric of nonpositive curvature which yields a representation (2) for $\pi_1(X)$ so that (3) holds. We call a Seifert manifold of *Euclidean* respectively *hyperbolic* type according to whether its base orbifold O admits flat or hyperbolic metrics. All nonpositively curved metrics on a Seifert manifold of Euclidean type are flat.

Suppose in the remainder of this section that X has hyperbolic type. We want to determine which flat metrics h on ∂X can be extended to a nonpositively curved metric on X. One necessary condition is immediate from the above discussion: The closed geodesics in $(\partial X, h)$ which are (homotopic to) Seifert fibers must have equal lenghts. This is sufficient in the non-orientable case:

Lemma 2.4 *If X is non-orientable, then any flat metric on ∂X so that Seifert fibers have equal length can be extended to a nonpositively curved metric on X.*

In the orientable case, there is a second necessary condition because the values of the representation ϕ on the boundary tori T_i are interrelated. To describe it we need some notation: The flat metric h on ∂X induces scalar products σ_i on the abelian groups $\pi_1(T_i) \cong H_1(T_i, \mathbb{Z})$. The values of ϕ on $H_1(T_i, \mathbb{Z})$ lie in $\mathbb{R}$ and they are related to σ_i by the formula

$$\sigma_i(f, \cdot) = \phi(f) \cdot \phi \qquad \text{on } H_1(T_i, \mathbb{Z}).$$

We choose in each group $H_1(T_i, \mathbb{Z})$ an element b_i which forms together with f a basis compatible with the orientation of T_i. The compatibility condition for the flat metrics on the boundary tori is given by

Lemma 2.5 (Rigidity of Seifert components) *If X is oriented, there is a rational number c which depends on the types of singular fibers and the choice of the basis elements b_i, so that the following is true: A flat metric h on ∂X can be extended to* *a nonpositively curved metric on X if and only if the Seifert fibers of $(\partial X, h)$ have equal length and the relation*

$$\sum_i \sigma_i(f, b_i) = c \cdot \|f\|^2 \tag{4}$$

holds.

Note that the nonpositively curved metrics on X can always be chosen flat near the boundary by adjusting the metric on the base-orbifold O.

The proofs are straight-forward. By varying the hyperbolic structure on O one can realize arbitrary lengths for the boundary curves. The possible values for the representation ϕ on the boundary ∂X can be analysed as in [S] by using a canonical presentation for $\pi_1(X)$. A proof for the non-orientable case is given in [K-L1].

Observe that by varying the metric on the base orbifold O, exactly those modifications of the induced flat metric on ∂X are possible which preserve the scalar products $\sigma_i(f, \cdot)$ with the fiber direction and, consequently, the length $\|f\|$ of the fiber. We will refer to this way of changing the flat metric on ∂X as *rescaling orthogonally to the Seifert fiber*.

In our construction of nonpositively curved metrics on graph-manifolds in section 3 we need to extend flat metrics which are prescribed only on part of the boundary. Lemmata 2.4 and 2.5 imply that the Extension Problem 2.2 with loose ends is always solvable:

Proposition 2.6 (Restricted flexibility of Seifert components) *Suppose that X is a Seifert manifold with non-empty incompressible boundary and hyperbolic base orbifold. Let h be a flat metric which is prescribed on some but not all of the boundary components of X so that Seifert fibers have equal lengths. Then h can be extended to a nonpositively curved metric on X which is flat near the boundary.*

For the sake of completeness, we show how one can deduce the proposition from the existence of a single nonpositively curved metric g on X. This is done by modifying the representation $\phi : \pi_1(X) \to Isom(\mathbb{R})$. There is a simple closed loop in O which separates the boundary from the singularities and orientation reversing loops. Accordingly X can be obtained from gluing a Seifert manifold Y and a circle bundle Z over a punctured sphere along one boundary component. Denote by $a_0, \ldots, a_k$ the compatibly oriented boundary curves of a section of Z so that a_0 corresponds to the boundary surface along which Z is glued to Y. The fundamental group of Z is presented by

$$\left\langle f, a_0, \ldots, a_k \middle| a_i f a_i^{-1} = f^{\epsilon_i}, \prod_i a_i = 1 \right\rangle$$

with $\epsilon_i \in \{\pm 1\}$ and $\Pi_i \epsilon_i = 1$. A new presentation ϕ' is obtained from ϕ as follows. We leave the presentation ϕ unchanged on $\pi_1(Y)$ and choose ϕ' on Z so that $\phi'(a_i)$ is in the same component of $Isom(\mathbb{R})$ as $\phi(a_i)$ and so that the conditions

$$\phi'(f) = \phi(f), \qquad \phi'(a_0) = \phi(a_0) \qquad \text{and} \qquad \prod_i \phi'(a_i) = \prod_i \phi(a_i)$$

hold. This shows that we have the freedom to prescribe flat metrics on ∂X as claimed in Proposition 2.6.

3 Existence results

3.1 Graph-manifolds with boundary

There is no unanimous notion of graph-manifold available in the literature. We choose the

Definition 3.1 *A* **graph-manifold** *is a Haken manifold which contains only Seifert components in its topological decomposition.*

Any metric of nonpositive curvature on a graph-manifold M is rigid in the sense that it splits almost everywhere locally as a product, see section 2.3. Nevertheless, the restricted flexibility of Seifert manifolds as stated in Proposition 2.6 suffices to construct nonpositively curved metrics on M in the presence of boundary:

Theorem 3.2 *Suppose that M is a graph-manifold with non-empty boundary. Then there exists a Riemannian metric of nonpositive curvature on M.*
Addendum. *Moreover, let $\{\gamma_i\}$ be a collection of homotopically non-trivial simple loops in the boundary, one on each component, so that none of them represents the fiber of the adjacent Seifert component. Then, given positive numbers l_i, the nonpositively curved metric on M can be chosen so that the loops γ_i are geodesics of length l_i.*

We will give examples of closed graph-manifolds without metrics of nonpositive curvature in section 4.

Proof: The Seifert components of M have non-empty incompressible boundary and hence admit nonpositively curved metrics. We can assume that each Seifert component of Euclidean type has one end, since otherwise it would be homeomorphic to a torus or Klein bottle cross the unit interval and hence be obsolete in the topological decomposition. Furthermore, we may assume that no gluing of ends of Seifert components identifies the Seifert fibers.

Let n be the number of Seifert components of M and suppose that the claim has been proven for graph-manifolds with less than n components. Choose a Seifert component X which contributes to the boundary of M. It has hyperbolic type. The complement C of X in M consists of some Seifert manifolds X_i^{eu} of Euclidean type with one end and a graph-manifold M_0 whose Seifert components adjacent to X have hyperbolic base. We pick flat metrics on the Euclidean pieces X_i^{eu} and, using the induction assumption, a nonpositively curved metric on M_0. We can arrange this metric g_0 on C so that all closed geodesics in the boundary which are identified with fibers in ∂X via the gluing, have the same length l. This is achieved on M_0 by rescaling orthogonally to the Seifert fiber, as described in section 2.3. The metric g_0 determines via the gluing map a flat metric on $\partial X \cap C$ with the property that Seifert fibers have equal length l. It may also happen that boundary surfaces of X are glued to each other. We pick flat metrics on those which are compatible with the identification and give length l to the Seifert fibers as well. Our choices prescribe a flat metric on a proper part of ∂X so that Seifert fibers have equal length. According to Proposition 2.6, this flat metric can be extended to a nonpositively curved metric on X. Combined with g_0, this yields a smooth nonpositively curved metric on M. The Addendum follows by rescaling orthogonally to the fiber and ordinary rescaling. □

3.2 Haken manifolds with atoroidal components

By now we have collected all ingredients for our main existence result. We saw in Proposition 2.3 that atoroidal manifolds are completely flexible, and the existence theorem 3.2 for graph-manifolds with boundary allows to put nonpositively curved metrics on the complement of the atoroidal components of a Haken manifold. Combining these facts we get:

Theorem 3.3 (Existence in presence of an atoroidal piece) *Let M be a Haken manifold with, possibly empty, boundary of zero Euler characteristic. Suppose that at least one atoroidal component occurs in its canonical decomposition. Then M admits a Riemannian metric of nonpositive curvature.*

We obtain new examples of closed nonpositively curved manifolds which are not locally symmetric and do not admit metrics of strictly negative curvature. These manifolds have geometric rank one in the sense of Ballmann, Brin and Eberlein. The first 3-manifold examples of this kind were given by Heintze [H] and Gromov [G].

4 Closed graph-manifolds

In section 3, we proved existence of nonpositively curved metrics on Haken manifolds which have non-empty boundary or contain an atoroidal component in their topological decomposition. This leaves open the case of closed graph-manifolds which we address now.

Consider a closed non-geometric graph-manifold M. As explained in section 2, a metric g of nonpositive curvature on M induces flat metrics g_i on the decomposing surfaces Σ_i and g locally splits as a product on each Seifert component. This rigidity interrelates the metrics g_i and serves as the source of obstructions to the existence of nonpositively curved metrics. A nonpositively curved metric exists on M if and only if we can put flat metrics on the Euclidean Seifert components and the surfaces Σ_i so that all orientable Seifert components with hyperbolic base satisfy the linear compatibility condition stated in Lemma 2.5. This reduces the existence question to a purely algebraic problem which is, however, in general fairly complicated. In section 4.1 below we discuss a class of graph-manifolds with linear gluing graph where the compatibility conditions can be translated into a simple combinatorial criterion for finite point configurations in hyperbolic plane. Although a majority of these manifolds admit metrics of nonpositive curvature, we will also obtain examples of non-existence.

The method in section 4.1 can be extended to the more general case when the dual graph to the canonical decomposition is a tree (joint work with Misha Kapovich). Examples of graph-manifolds with no metrics of nonpositive curvature when the dual graph contains cycles are given in [K-L1]; they arise as mapping tori of reducible surface diffeomorphisms. Recently, Buyalo and Kobelskii [B-K] announced the construction of a numerical invariant for graph-manifolds which detects the existence of a nonpositively curved metric.

4.1 An example

We consider closed graph-manifolds M built from two Seifert components X_0 and X_{n+1}, each with one boundary torus (denoted by $\partial_+ X_0$ respectively $\partial_- X_{n+1}$), and $n \geq 0$ Seifert components $X_1, \dots, X_n$ with two boundary tori (denoted by $\partial_\pm X_i$). The X_i are assumed to be trivial circle bundles over compact orientable surfaces Σ_i of genus ≥ 1 and hence they are of hyperbolic type. The graph-manifold M is obtained by performing gluing homeomorphisms

$$\alpha_i : \partial_+ X_i \to \partial_- X_{i+1} \qquad (i = 0, \dots, n).$$

Denote by $T_i := \partial_+ X_i = \partial_- X_{i+1}$ the decomposing tori of M.

We define an abstract free abelian group A of rank 2 by identifying the homology groups $H_1(T_i, \mathbb{Z})$ with each other in a canonical way: For $i = 1, \dots, n$, we define the elements $x_{i-1} \in H_1(T_{i-1}, \mathbb{Z})$ and $x_i \in H_1(T_i, \mathbb{Z})$ to be equivalent iff their images under the inclusion monomorphisms

$$H_1(\partial_\pm X_i, \mathbb{Z}) \hookrightarrow H_1(X_i, \mathbb{Z})$$

coincide. The Seifert fibers of the components X_i yield distinguished elements $f_0, \dots, f_{n+1} \in A$. In addition there are elements $b_0, b_{n+1} \in A$ which correspond to the well-defined horizontal directions in the boundary tori of the one-ended components; they can be defined as generators of the kernels of the natural homomorphisms

$$H_1(\partial_+ X_0, \mathbb{Z}) \to H_1(X_0, \mathbb{Z})$$

and

$$H_1(\partial_- X_{n+1}, \mathbb{Z}) \to H_1(X_{n+1}, \mathbb{Z}).$$

Note that there are no well-defined horizontal directions in the boundary tori of components with more than one end. The collection of vectors $b_0, b_{n+1}, f_0, \dots, f_{n+1}$ encodes the gluing maps and hence the topology of the closed graph-manifold M.

To put flat metrics on the tori T_i is equivalent to choosing scalar products, i.e. positive-definite symmetric bilinear forms σ_i on the group A. We recall that the space $\mathcal{H}$ of projective equivalence classes of scalar products on A carries a natural metric which is isometric to hyperbolic plane. A geodesic c in $\mathcal{H}$ corresponds to a splitting $V = L_1 \oplus L_2$ of the 2-dimensional vector space $V := A \otimes_{\mathbb{Z}} \mathbb{R}$ into 1-dimensional subspaces; namely c consists of all (classes of) scalar products with respect to which the splitting is orthogonal. The ideal boundary points in the geometric compactification of $\mathcal{H}$ correspond to projective equivalence classes of positive-semidefinite forms or, in other words, to 1-dimensional subspaces of V. The geometric boundary $\partial_{geo}\mathcal{H}$ can hence be canonically identified with the projective line $\mathbb{P}V$. In particular, each element $a \in A$ can be interpreted as a point $[a] \in \partial_{geo}\mathcal{H}$.

As in Lemma 2.5 we obtain the following necessary and sufficient compatibility conditions for the scalar products $\sigma_0, \dots, \sigma_n$ on A so that the corresponding flat metrics on the decomposing tori T_i can be extended to a nonpositively curved metric on M: For the one-ended components we have

$$\sigma_0(f_0, b_0) = 0 \qquad \text{and} \qquad \sigma_n(f_{n+1}, b_{n+1}) = 0,$$

because the horizontal boundary curves representing b_0, b_{n+1} are homologically trivial in X_0 respectively X_{n+1}. For the components with two ends we get

$$\sigma_{i-1}(f_i, \cdot) = \sigma_i(f_i, \cdot) \qquad \text{for } i = 1, \ldots, n.$$

This translates into geometric conditions relating the ideal points $[b_0], [b_{n+1}], [f_0], \ldots, [f_{n+1}] \in \partial_{geo}\mathcal{H}$ and the points $[\sigma_0], \ldots, [\sigma_n] \in \mathcal{H}$:

1. $[\sigma_0]$ lies on the geodesic with ideal endpoints $[b_0]$ and $[f_0]$. $[\sigma_n]$ lies on the geodesic with endpoints $[b_{n+1}]$ and $[f_{n+1}]$.
2. $[\sigma_{i-1}]$ and $[\sigma_i]$ lie on a geodesic asymptotic to $[f_i]$ $(i = 1, \ldots, n)$.

Note that this is a condition on the conformal types of the metrics on the tori T_i. But since the gluing graph does not contain cycles, a collection of compatible conformal structures gives rise to a collection of compatible flat metrics. Our discussion yields the following

Criterion. *There exists a metric of nonpositive curvature on M if and only if there is a configuration of points $[\sigma_0], \ldots, [\sigma_n]$ in $\mathcal{H}$ which satisfies the above conditions 1 and 2.*

Let us first consider the simplest case when M is obtained from gluing two Seifert pieces with one end:

Example 4.1 *If $n = 0$, then M admits a metric of nonpositive curvature if and only if the gluing map $\alpha_0 : \partial_+ X_0 \to \partial_- X_1$ preserves the canonical bases of the first homology groups, i.e. if $\alpha_{0,*} : H_1(\partial_+ X_0, \mathbb{Z}) \to H_1(\partial_- X_1, \mathbb{Z})$ satisfies:*

$$\alpha_{0,*}\{\pm b_0, \pm f_0\} = \{\pm b_1, \pm f_1\}$$

Proof: If a nonpositively curved metric exists on M, then the splittings $A = \langle b_0 \rangle \oplus \langle f_0 \rangle$ and $A = \langle b_1 \rangle \oplus \langle f_1 \rangle$ must be orthogonal with respect to σ_0. The sets $\{\pm b_0, \pm f_0\}$ and $\{\pm b_1, \pm f_1\}$ contain the shortest lattice vectors with respect to σ_0 and therefore coincide. $\square$

One can as well produce examples with arbitrary number of Seifert components: Let C_0 be the geodesic in $\mathcal{H}$ with endpoints $[f_0]$ and $[b_0]$. For $i = 1, \ldots, n$ define C_i to be the union of all geodesics in $\mathcal{H}$ which are asymptotic to $[f_i]$ and intersect C_{i-1}. This is an increasing sequence of convex subsets of $\mathcal{H}$, and C_i consists of all conformal structures on T_i which can be induced by a nonpositively curved metric on $X_0 \cup \ldots \cup X_i$. There exists a nonpositively curved metric on M if and only if the geodesic connecting $[f_{n+1}]$ and $[b_{n+1}]$ intersects C_n. The contrary can easily be arranged, choose for instance:

$$\begin{aligned} f_i &:= f_0 + i \cdot b_0 \qquad \text{for } i = 1, \ldots, n+1 \\ b_{n+1} &:= f_0 + (n+2) \cdot b_0 \end{aligned}$$

Then $b_0, f_0, \ldots, f_{n+1}, b_{n+1}$ lie in this order on the circle $\partial_{geo}\mathcal{H}$ and no metric of nonpositive curvature exists on the corresponding graph-manifold M. This yields:

Example 4.2 *There exist closed graph-manifolds with arbitrarily many Seifert components which do* **not** *admit metrics of nonpositive curvature.*

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