Rigidity of quasi-isometries for symmetric spaces and Euclidean buildings

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Abstract. We study quasi-isometries between products of symmetric spaces and Euclidean buildings. The main results are that quasi-isometries preserve product structure, and in the irreducible higher rank case quasi-isometries are at finite distance from homotheties.

Rigidité des quasiisométries pour les espaces symétriques et les immeubles euclidéens

Résumé. On étudie les quasiisométries entre des produits d'éspaces symétriques et d'immeubles euclidéens. Les résultats principaux sont que les quasiisométries préservent la structures de produit et, dans le cas irréductible de rang ≥ 2, les quasiisométries sont à distance finie de homothéties.

VERSION FRANCAISE ABRÉGÉE

Une (L, C) quasiisométrie est une application $\Phi : X \to X'$ d'espaces métriques telle que

$$L^{-1}d(x_1, x_2) - C \leq d(\Phi(x_1), \Phi(x_2)) \leq Ld(x_1, x_2) + C$$

pour tous $x_1, x_2 \in X$ et

$$d(x', Im(\Phi)) < C$$

pour tous $x' \in X'$, voir [1]. Dans cette note on annonce des résultats concernant les quasiisométries entre des produits d'espaces symétriques, d'immeubles euclidiens et de groupes de Lie nilpotents. Tous les espaces symétriques considérés ici sont de type non compact et sans facteurs euclidiens. Tous les immeubles sont épais, mais pas nécessairement discrets ou localement compacts (voir [2] pour les définitions précises).

Théorème 1 *Pour $1 \leq i \leq k$, $1 \leq j \leq k'$ soit chaque X_i, X'_j un espace symétrique irréductible ou un immeuble euclidien de groupe de Weyl affine cocompact; soient Nil et Nil' des groupes de Lie nilpotents et simplement connexes munis de métriques Riemanniennes invariantes à gauche. Soit $X = Nil \times \prod_{i=1}^{k} X_i$, $X' = Nil' \times \prod_{j=1}^{k'} X'_j$ les produits métriques. Alors pour chaque constantes L, C il existe des constantes $\bar{L}, \bar{C}$ et $\bar{D}$ telles que: Si $\Phi : X \to X'$ est une (L, C) quasiisométrie, alors $k = k'$, Nil et Nil' sont quasiisométriques et on peut réindexer les facteurs de X', tel qu'il existe des $(\bar{L}, \bar{C})$ quasiisométries $\Phi_i : X_i \to X'_i$ telles que $d(p' \circ \Phi, \prod \Phi_i \circ p) < \bar{D}$, où $p : X \to \prod_{i=1}^{k} X_i$ et $p' : X' \to \prod_{i=1}^{k} X'_i$ sont les projections.*

Théorème 2 *Soient X et X' comme dans le théorème 1 et supposons de plus que X soit irréductible et de rang ≥ 2. Alors chaque (L, C) quasiisométrie $\Phi : X \to X'$ est à distance finie $< D$ d'une homothétie $\Phi_0 : X \to X'$, où D ne dépend que de (L, C).*

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Les théorèmes 1 et 2 entraînent le théorème de rigidité de Mostow pour les espaces symétriques sans facteurs de rang un. On obtient de plus une version feuilletée:

Théorème 7 [cf. [3]] *Pour $i = 1, 2$ soit X_i un espace symétrique sans facteurs hyperboliques réels ou complexes. Soit $(M_i, \mathcal{F}_i)$ une variété compacte feuilletée, et supposons que les feuilles des feuilletages $\mathcal{F}_i$ soient munies de métriques riemanniennes g_i localement isométriques à X_i qui varient continûment dans la direction transverse. Alors chaque homéomorphisme $\Phi : M_1 \to M_2$ qui porte $\mathcal{F}_1$ à $\mathcal{F}_2$ est homotope à un homéomorphisme $\Phi_0 : M_1 \to M_2$ affine sur les feuilles.*

Les théorèmes 1, 2 et [4] impliquent la classification des espaces symétriques à quasiisometrie près:

Théorème 8 *Soient X et X' des espaces symétriques de type non compact. Si X et X' sont quasiisométriques alors ils sont isométriques après renormalisation appropriée des métriques sur les facteurs.*

Les mêmes théorèmes fournissent une contribution importante à la classification des groupes de type fini quasiisométriques aux espaces symétriques. On y utilise des résultats de Tukia, Gabai, Casson-Jungreis, Hinkkanen, R. Chow and Pansu pour les facteurs de rang un.

Théorème 9 *Soit X un espace symétrique et soit Nil un groupe de Lie nilpotent simplement connexe muni d'une métrique riemannienne invariant à gauche. Soit Γ un groupe de type fini muni d'une métrique de mots (voir [1]). Si Γ est quasiisométrique à $Nil \times X$, alors il'y a une suite exacte $1 \to N \to \Gamma \to L \to 1$ où N est un groupe nilpotent de type fini quasiisométrique à Nil, et $L \subset Isom(X)$ est un réseau cocompact.*

En utilisant le théorème de rigidité topologique de [7] on en déduit:

Théorème 10 *Soit X un espace symétrique et soit M une variété asphérique compact de dimension ≥ 5 dont le revêtement universel $\tilde{M}$ est quasiisométrique à X. Alors M est homéomorphe à un quotient compact X/Γ où $\Gamma \subset Isom(X)$ est un réseau cocompact.*

ENGLISH VERSION

1. THE MAIN RESULTS. An (L, C) quasi-isometry is a map $\Phi : X \to X'$ between metric spaces such that for all $x_1, x_2 \in X$ we have

$$L^{-1} d(x_1, x_2) - C \leq d(\Phi(x_1), \Phi(x_2)) \leq L d(x_1, x_2) + C$$

and

$$d(x', Im(\Phi)) < C$$

for all $x' \in X'$ (see [1] for background). In this note we announce results concerning quasi-isometries between products of symmetric spaces, Euclidean buildings and nilpotent Lie groups. All symmetric spaces considered here are assumed to be of noncompact type and to have no Euclidean de Rham factors. All Euclidean buildings will be thick, but they need not be discrete or locally compact (see [2] for precise definitions).

Theorem 1 (Splitting) *For $1 \leq i \leq k$, $1 \leq j \leq k'$ let each X_i, X'_j be either an irreducible symmetric space or an irreducible Euclidean building with cocompact affine Weyl group; let Nil and Nil' be simply connected nilpotent Lie groups with left invariant Riemannian metrics. Let $X = Nil \times \prod_{i=1}^{k} X_i$, $X' = Nil' \times \prod_{j=1}^{k'} X'_j$ be metric products. Then for every L, C there are constants $\bar{L}$, $\bar{C}$, and $\bar{D}$ such that the following holds. If $\Phi : X \to X'$ is an (L, C) quasi-isometry, then $k = k'$, Nil and Nil' are quasiisometric and, after reindexing the factors of X', there are $(\bar{L}, \bar{C})$ quasi-isometries $\Phi_i : X_i \to X'_i$ so that $d(p' \circ \Phi, \prod \Phi_i \circ p) < \bar{D}$, where $p : X \to \prod_{i=1}^{k} X_i$ and $p' : X' \to \prod_{i=1}^{k} X'_i$ are the projections.*

Theorem 2 (Rigidity) *Let X and X' be as in theorem 1, but assume in addition that X is either an irreducible symmetric space of rank at least 2, or an irreducible Euclidean building of rank at least 2 with cocompact affine Weyl group. Then any (L, C) quasi-isometry $\Phi : X \to X'$ lies at distance $< D$ from a homothety $\Phi_0 : X \to X'$, where D depends only on (L, C).*

Theorem 2 settles a conjecture made by Margulis in the late 1970's [1, 5]. The analogous statement is false for real and complex hyperbolic spaces and true for the quaternionic hyperbolic spaces $H_{\mathbb{H}}^{n\geq 2}$ and the Cayley hyperbolic plane H_{Cay}^2 [6].

2. ON THE PROOFS OF THEOREMS 1 AND 2. In our approach, we use the concept of asymptotic cone introduced by van den Dries, Wilkie and Gromov, see [1]. A quasi-isometry $\Phi : X \to X'$ induces a bilipschitz homeomorphism $\Phi_\omega : X_\omega \to X'_\omega$ of asymptotic cones. Our first step is to prove that for any symmetric space or Euclidean building X, the asymptotic cone X_ω is a Euclidean building; this building is non-discrete and homogeneous as a metric space. In the second step we prove topological analogs of our main results; these are most interesting if the buildings are non-discrete:

Theorem 3 *Any homeomorphism between Eulidean buildings carries apartments to apartments.*

Theorem 4 *Let Y_i, Y'_j be irreducible Euclidean buildings whose affine Weyl group has dense orbits, Let $Y = \mathbb{E}^n \times \prod_{i=1}^k Y_i$, $Y' = \mathbb{E}^{n'} \times \prod_{j=1}^{k'} Y'_j$. If $\Psi : Y \to Y'$ is a homeomorphism, then $n = n'$, $k = k'$, and after reindexing factors there are homeomorphisms $\Psi_i : Y_i \to Y'_i$ so that $p' \circ \Psi = \prod \Phi_i \circ p$, where $p : X \to \prod_{i=1}^k X_i$ and $p' : X' \to \prod_{i=1}^k X'_i$ are the projections.*

Theorem 5 *Let Y be an irreducible thick Euclidean building of rank ≥ 2 whose affine Weyl group has dense orbits. Then any homeomorphism from Y to a Euclidean building is a homothety.*

The final step in the proof of theorem 2 is a characterization of Euclidean buildings by their Tits boundary and cone topology:

Theorem 6 *Let X, X' be irreducible Euclidean buildings of rank ≥ 2, and let $\partial_{Tits} X$, $\partial_{Tits} X'$ denote their Tits boundaries. If $\psi : \partial_{Tits} X \to \partial_{Tits} X'$ is Tits isometry which is a cone topology homeomorphism, then there exists a unique homothety $\Psi : X \to X'$ inducing ψ, i.e. $\partial_\infty \Psi = \psi$.*

We remark that theorems 1,2 and 6 apply to Euclidean buildings whether they arise from algebraic groups or not. In particular, they apply to the plethora of rank 2 examples constructed by Tits, Ballmann-Brin and Swiatkowski, many of which have trivial isometry group.

3. APPLICATIONS. Theorems 1 and 2 immediately imply Mostow's rigidity theorem for symmetric spaces with no rank 1 de Rham factors. In fact, they yield a foliated version as well:

Theorem 7 (cf. [3]) *For $i = 1, 2$ let X_i be a symmetric space with no Euclidean, real hyperbolic or complex hyperbolic de Rham factors. Let $(M_i, \mathcal{F}_i)$ be a compact foliated manifold, and suppose the leaves of the foliation $\mathcal{F}_i$ are equipped with Riemannian metrics g_i which are locally isometric to X_i and which vary continuously in the transverse direction. Then any homeomorphism $\Phi : M_1 \to M_2$ which carries $\mathcal{F}_1$ to $\mathcal{F}_2$ can be homotoped to a homeomorphism $\Phi_0 : M_1 \to M_2$ which is affine on leaves.*

Another consequence of theorems 1 and 2 and [4] is the classification of symmetric spaces up to quasi-isometry:

Theorem 8 *Let X, X' be symmetric spaces of noncompact type. If X and X' are quasi-isometric, then they become isometric after the metrics on their irreducible factors are suitably renormalized.*

The same theorems provide an important ingredient in the classification of finitely generated groups which are quasi-isometric to symmetric spaces:

Theorem 9 *Let X be a symmetric space and let Nil be a simply connected nilpotent Lie group with a left invariant metric. Let Γ be a finitely generated group endowed with a word metric (see [1]). If Γ is quasi-isometric to $Nil \times X$, then there is an exact sequence $1 \to N \to \Gamma \to L \to 1$ where N is a finitely generated nilpotent group which is quasi-isometric to Nil, and $L \subset Isom(X)$ is a cocompact lattice. In the case that $Nil \simeq \mathbb{R}^n$, N will be virtually free abelian of rank n.*

This theorem also uses results of Tukia, Gabai, Casson-Jungreis, Hinkkanen, R. Chow and Pansu for the rank one cases.

Theorem 9 together with the topological rigidity theorem of [7] implies:

Theorem 10 *Let X be a symmetric space and let M be a closed aspherical manifold of dimension ≥ 5 whose universal cover $\tilde{M}$ is quasi-isometric to X. Then M is homeomorphic to a compact quotient X/Γ where $\Gamma \subset Isom(X)$ is a cocompact lattice.*

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