The Center Conjecture for spherical buildings of types F_4 and E_6

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Abstract

We prove that a convex subcomplex in a spherical building of type F_4 or E_6 is a subbuilding or the building automorphisms preserving the subcomplex fix a point on it. Our approach is differential-geometric and based on the theory of metric spaces with curvature bounded above. We use these techniques also to give another proof of the same result for the spherical buildings of classical type.

1 Introduction

A subset in a CAT(1) space, i.e. in a space with curvature ≤ 1 in the comparison sense, is called convex if it contains with any two points of distance $< \pi$ the unique minimizing geodesic segment connecting them.

In a Euclidean unit sphere there are no proper convex subsets beyond a certain threshold: A convex subset is either contained in a convex metric ball, that is, a ball of radius $\leq \frac{\pi}{2}$ or it fills out the entire sphere. To put it more intrinsically, the convex subset is either contained in a convex metric ball centered around one of its points or it is a geodesic subsphere. Thus its intrinsic circumradius is $\leq \frac{\pi}{2}$ or $= \pi$.

Spherical buildings are a very special kind of CAT(1) spaces. Their geometry is rigidified by the property that they contain "plenty of apartments", i.e. top-dimensional convex subsets isometric to a unit sphere. A metric ball of radius $< \pi$ in a spherical building is convex if and only if it has radius $\leq \frac{\pi}{2}$. It is natural to ask whether the "circumradius gap phenomenon" for convex subsets of spheres holds more generally in spherical buildings, compare [KL06, Question 1.5]:

Question 1.1. *Suppose that B is a spherical building and that $C \subseteq B$ is a convex subset. Is it true that C is either a subbuilding or it is contained in a convex metric ball centered in C?*

It is easy to see that the answer is yes, if $dim(C) \leq 1$. A one-dimensional convex subset is either a subbuilding or a tree. In the latter case, it contains a unique circumcenter.

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Regarding isometric group actions on spherical buildings, one can ask the following weaker version of Question 1.1, see [BL05]:

Question 1.2. *Suppose that B is a spherical building and that $C \subseteq B$ is a convex subset. Is it true that C is a subbuilding or the action $Isom(C) \curvearrowright C$ has a fixed point?*

A positive answer to Question 1.1 implies a positive answer to Question 1.2.

Question 1.2 has been answered positively in [BL05, Thm. 1.1] when $dim(C) \leq 2$.

In higher dimensions, both questions seem to become considerably more approachable when one restricts to convex subsets which are *subcomplexes* with respect to the natural polyhedral structure of the spherical building. Question 1.2 then becomes a geometric version of Tits' Center Conjecture, compare [MT06] and [Se05, Conjecture 2.8]:

Conjecture 1.3 (Center Conjecture). *Suppose that B is a spherical building and that $K \subseteq B$ is a convex subcomplex. Then K is a subbuilding or the action $Stab_{Aut(B)}(K) \curvearrowright K$ of the automorphisms of B preserving K has a fixed point.*

The automorphisms of a spherical building are the isometries which preserve its combinatorial (i.e. polyhedral) structure.

A positive answer to Question 1.2 implies a positive answer to Conjecture 1.3.

Conjecture 1.3 easily reduces to the irreducible case.

It has been proven for irreducible buildings of types A_n, B_n and D_n in [MT06]. The F_4-case has been announced in a talk at Oberwolfach by Parker and Tent [PT08]. These approaches are incidence-geometric.

Our main result is the proof of the Center Conjecture 1.3 for spherical buildings of types F_4 and E_6, see Theorems 3.1 and 3.18. Our methods are differential-geometric and based on the theory of metric spaces with curvature bounded above. The arguments rely on the specific features of F_4- and E_6-geometry. We use these techniques also to give another proof of the Center Conjecture for the spherical buildings of classical type.

The approach in this paper has been carried further by the second author in [Ra09a, Ra09b] where he proves Conjecture 1.3 for spherical buildings of types E_7 and E_8. It follows that the Center Conjecture holds for all spherical buildings without factors of type H_4, and in particular for all thick spherical buildings.

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2.1 CAT(1) spaces

A complete metric space X is called a *CAT(1) space*, if any two points with distance $< \pi$ are connected by a minimizing geodesic segment and if geodesic triangles with perimeter $< 2\pi$ are not thicker than the corresponding comparison triangles in the unit sphere $S^2(1)$ with Gauß curvature $\equiv 1$. We refer to [KL98, ch. 2.1-2], [BH99, ch. 2.1-3] and [BBI01, 9.1] for basic information about CAT(1) spaces.

We denote by $B_r(x)$ the open metric ball of radius r centered at x, and by xy a minimizing geodesic segment with endpoints x and y.

The *link* or *space of directions* $\Sigma_x X$ of X at a point x equipped with the angle metric is again a CAT(1) space. It can be thought of as an analogue of the unit tangent sphere of a Riemannian manifold. If $x \neq y$, we denote by $\overrightarrow{xy} \in \Sigma_x X$ the direction of the segment xy at x.

2.1.1 Convexity

One calls a subset $C \subseteq X$ *π-convex* or simply *convex*, if with any two points $x, x' \in C$ of distance $< \pi$ the unique minimizing geodesic segment xx' is contained in C. Closed convex subsets of CAT(1) spaces are CAT(1) spaces themselves. Metric balls with radius $\leq \frac{\pi}{2}$ in CAT(1) spaces are convex. The *closed convex hull* $CH(A)$ of a subset $A \subseteq X$ is the smallest closed convex subset of X containing A. We will denote the closed convex hull of finitely many points $a_1, \ldots, a_m$ by $CH(a_1, \ldots, a_m)$.

2.1.2 Circumradius and circumcenters

For a subset $A \subseteq X$ and a point $x \in X$ we denote by $\mathrm{rad}(A, x)$ the radius of the smallest closed metric ball around x which contains A. We define the *circumradius* $\mathrm{rad}(A) = rad_X(A)$ of A as the infimum of the function $\mathrm{rad}(A, \cdot)$ on X. A point where the infimum is attained is called a *circumcenter* of A. If A is convex, we call the infimum of $\mathrm{rad}(A, \cdot)$ *on* A the *intrinsic* circumradius of A.

If $\mathrm{rad}(A) < \frac{\pi}{2}$, then by standard comparison arguments A has a unique circumcenter which must be contained in the closed convex hull $CH(A)$ of A. (Indeed, suppose that (x_n) is a sequence of points in X with $\mathrm{rad}(A, x_n) \searrow \mathrm{rad}(A)$, and let m_{ij} be the midpoint of $x_i x_j$. Then $\mathrm{rad}(A, m_i) \geq \mathrm{rad}(A)$ and the CAT(1) inequality imply that (x_n) is a Cauchy sequence. Its limit is a circumcenter of A, and it must be unique. The circumcenter must belong to $CH(A)$ because due to the CAT(1) inequality its nearest point projection to $CH(A)$ is also a circumcenter)

If $\mathrm{rad}(A) = \frac{\pi}{2}$, then the set $Cent(A)$ of circumcenters of A is closed and convex. (Its convexity follows from the CAT(1) inequality.) Since $\mathrm{rad}(CH(A)) = \frac{\pi}{2}$ and $Cent(A) = Cent(CH(A))$, the closed convex set $CH(A) \cap Cent(A)$ has diameter $\leq \frac{\pi}{2}$.

2.2 Spherical Coxeter complexes

We refer to [GB71, ch. 4-5], [Bou81, ch. V, VI.4] and [KL98, ch. 3.1, 3.3] for more information.

2.2.1 General definitions and facts

Let S be the unit sphere in a finite dimensional Euclidean vector space V. The reflection at a hyperplane in V through the origin induces an involutive isometry of S. One refers to such isometries briefly as *reflections*. If $W \subset Isom(S)$ is a finite subgroup generated by reflections, one calls the pair (S, W) a *(spherical) Coxeter complex* and W its *Weyl group*. (Note that we allow W to have fixed points.)

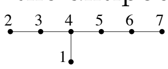
The Weyl group W induces a *polyhedral structure* on S. The fixed point sets of the reflections in W are great spheres of codimension one, the *walls*. There are finitely many walls and they divide S into open convex subsets whose closures are called *chambers*. If W is nontrivial, then the chambers are convex spherical polyhedra because they are finite intersections of closed hemispheres. A *half-apartment* or *root* is a hemisphere bounded by a wall, a *singular sphere* is an intersection of walls, a *face* of S is the intersection of a chamber with a singular sphere, a *panel* is a codimension one face, a *vertex* is a zero-dimensional face. Two faces are called *opposite* or *antipodal* if they are exchanged by the antipodal involution of S. The face *spanned* by a point is the face containing it as an interior point, equivalently, the smallest face containing it. A point is called *regular* if it spans a chamber, and *singular* otherwise. A minimizing geodesic segment connecting two vertices is called *singular* if it is contained in a singular 1-sphere. A vertex is called of *root type* if the hemisphere centered at it is a root.

Each chamber Δ is a fundamental domain for the action $W \curvearrowright S$ and W is generated by the reflections at the codimension one faces of Δ, that is, at the walls containing them. We call $\Delta_{mod} = \Delta_{mod}^{(S,W)} := S/W$ the *model Weyl chamber*. Its isometry type determines W up to conjugacy. The quotient map $\theta_S : S \to \Delta_{mod}$ is 1-Lipschitz and restricts to isometries on chambers. We call the image $\theta_S(x)$ of a point $x \in S$ its *θ_S-type* or just its *type*.

The *link* $\Sigma_x S$ of a point $x \in S$ is the unit tangent sphere of S at x in the sense of Riemannian geometry. It inherits from S a natural structure as the spherical Coxeter complex $(\Sigma_x S, Stab_W(x))$ with Weyl group $Stab_W(x)$ and with model Weyl chamber $\Delta_{mod}^{(\Sigma_x S, Stab_W(x))} \cong \Sigma_x \Delta_{mod}^{(S,W)}$.

More generally, let $\sigma \subset S$ be a face of codimension ≥ 1. Then for an interior point $x \in \sigma$, the link $\Sigma_x S$ splits as the spherical join $\Sigma_x S \cong \Sigma_x \sigma \circ \nu_x \sigma$ of the unit tangent sphere $\Sigma_x \sigma$ of σ and the unit normal sphere $\nu_x \sigma$ of σ in S. The unit normal sphere has dimension $\dim(S) - \dim(\sigma) - 1$, and there is a natural isometric identification $\nu_x \sigma \cong Poles(\sigma)$ of $\nu_x \sigma$ with the sphere $Poles(\sigma) := \{p \in S : d(p, \cdot)|_\sigma \equiv \frac{\pi}{2}\}$ of *poles* of σ in S. This provides one way to see, that one can consistently identify with each other the normal spheres $\nu_x \sigma$ for all interior points $x \in \sigma$ to obtain the *link $\Sigma_\sigma S$ of the face σ*. It inherits a natural structure as the spherical Coxeter complex $(\Sigma_\sigma S, Stab_W(\sigma))$ with Weyl group the stabilizer (fixator) of σ in W and with model Weyl chamber $\Delta_{mod}^{(\Sigma_\sigma S, Stab_W(\sigma))} \cong \Sigma_\sigma \Delta_{mod}^{(S,W)}$.

Let $s \subset S$ be a singular sphere. Then s inherits a natural structure as a Coxeter complex as follows. By a *reflection on s* we mean an involutive isometry of s whose fixed point set is a codimension one subsphere. We define the *induced Weyl group $W_s \subset Isom(s)$* on s as the subgroup generated by those reflections on s which are induced by isometries in W. The pair (s, W_s) is a Coxeter complex and we refer to it as a *Coxeter subcomplex* of (S, W). The Coxeter tesselation of s is in general coarser than its polyhedral structure inherited from S. Let us call the fixed point set of a reflection on s and in W_s an *s-wall*. Every face of codimension ≥ 1 in s with respect to the (coarser) intrinsic polyhedral structure is contained in an s-wall. A codimension one face in s with respect to the (finer) polyhedral structure induced from S is contained in an s-wall if and only if both top-dimensional faces in s adjacent to it (again with respect to the finer polyhedral structure) have the same type, i.e. the same θ_S-image.

Remark 2.1. Note that W_s can be strictly smaller than the image of the natural homomorphism $Stab_W(s) \to Isom(s)$. An example for this phenomenon can be seen in the E_7-Coxeter complex: It contains a singular 1-sphere s of type 13756137561 (the first and last 1 to be identified). The induced Weyl group W_s is trivial, but the antipodal involution on s is induced by isometries in $Stab_W(s)$. Here we use the labelling 

for the Dynkin diagram.

We call a Coxeter complex *trivial* or a *sphere* if its Weyl group is trivial. The Coxeter complex (S, W) splits off a sphere factor if and only if W has fixed points, equivalently, if and only if Δ_{mod} has diameter π. In this case the sphere $Fix(W) \subseteq S$ is canonically identified with the unique maximal sphere factor of (S, W), its *spherical de Rham factor*.

We call a Coxeter complex *reducible* if it decomposes as the spherical join of Coxeter complexes. Join decompositions of a Coxeter complex correspond to join decompositions of its model Weyl chamber. For a Coxeter complex without spherical factor holds $\text{diam}(\Delta_{mod}) \leq \frac{\pi}{2}$. (If two W-orbits in S have distance $> \frac{\pi}{2}$ then each of them is contained in an open hemisphere and has a center fixed by W.) It is irreducible if and only if $\text{diam}(\Delta_{mod}) < \frac{\pi}{2}$. The *de Rham decomposition* of (S, W) is the unique maximal decomposition as the join of the spherical de Rham factor and some irreducible nontrivial Coxeter complexes.

Suppose now that W has no fixed points on S. Then (S, W) has no spherical factor and Δ_{mod} is a spherical simplex. As remarked above, Δ_{mod} has diameter $\leq \frac{\pi}{2}$ with equality if and only if (S, W) is reducible. The dihedral angle between any two panels of Δ_{mod} equals $\frac{\pi}{p}$ for some integer $p \geq 2$. If Δ_{mod} has no one-dimensional join factor, then the only possible values for p are 2,3,4 and 5. The geometry of Δ_{mod} can be encoded in a marked (or weighted) graph Γ, the *Coxeter graph*, as follows. The vertices of Γ correspond to the panels of Δ_{mod}. Two vertices are not connected if the corresponding dihedral angle equals $\frac{\pi}{2}$; they are connected by an edge if the angle equals $\frac{\pi}{3}$, and by an edge with label p if the angle equals $\frac{\pi}{p}$ with $p \geq 4$. If no edge labels $\neq 4, 6$ occur, as it is the case for the Coxeter complexes coming from a root system, then one often replaces the edges with label 4 by double edges and the edges with label 6 by triple edges. The resulting graph with multiple edges is called the *Dynkin diagram.* The Coxeter graph determines Δ_{mod} up to isometry. Note that Γ is disconnected if and only if (S, W) is reducible, equivalently, if Δ_{mod} decomposes as a spherical join. The classification of irreducible spherical Coxeter complexes can be found in [GB71, Thm. 5.3.1]. The irreducible Coxeter complexes of dimension ≥ 2 which occur for thick spherical buildings have the Dynkin diagrams $A_{n \geq 3}$, $B_{n \geq 3}$, $D_{n \geq 4}$, F_4, E_6, E_7 and E_8, cf. [Ti77] and [Ti74, pp. 274]. For a face $\sigma \subset S$ of codimension ≥ 1, the Coxeter graph of its link $(\Sigma_\sigma S, Stab_W(\sigma))$ is obtained from the Coxeter graph of (S, W) by deleting those vertices which correspond to the vertices of σ.

When discussing a concrete Coxeter complex we will label the vertices of its Coxeter graph by some index set I. This induces also a labelling of the vertices of the Weyl chamber Δ_{mod} by assigning to a vertex v of Δ_{mod} the label of the vertex of Γ corresponding to the panel opposite to v.

An *automorphism* of the Coxeter complex (S, W) is an isometry α of S which preserves the tesselation into chambers, equivalently, which normalizes W, i.e. $Aut((S, W)) = N_{Isom(S)}(W)$. The isometries in W are the *inner* automorphisms and the automorphisms outside W are the *outer* automorphisms. The *outer automorphism group* $Out((S, W)) := Aut((S, W))/Inn((S, W))$ $= N_{Isom(S)}(W)/W$ is canonically identified with $Isom(\Delta_{mod})$ and with the automorphism group $Aut(\Gamma)$ of the Coxeter graph.

The antipodal involution of S is always an automorphism of (S, W). It induces the *canonical involution* ι of Δ_{mod}. For a chamber $\Delta \subset S$ there is a unique Weyl isometry $w \in W$ with $w\Delta = -\Delta$. The composition $-w$ of w with the antipodal involution of S is an isometric involution of Δ. It coincides with ι modulo the natural identification $\Delta \stackrel{\theta_S}{\rightarrow} \Delta_{mod}$. Note that ι is trivial if and only if any two opposite vertices in the Coxeter complex have equal type. Regarding the irreducible Coxeter complexes one has that $\iota = id_{\Delta_{mod}}$ for the Coxeter complexes of types A_1, B_n, D_{2n}, F_4, E_7 and E_8, and $\iota \neq id_{\Delta_{mod}}$ for the Coxeter complexes of types $A_{n\geq 2}$, D_{2n+1} and E_6.

2.2.2 The Coxeter complex of type F_4

Let (S^3, W_{F_4}) be the Coxeter complex of type F_4. We use the labelling $\overset{1}{\bullet}\!-\!\overset{2}{\bullet}\!=\!\overset{3}{\bullet}\!-\!\overset{4}{\bullet}$ for its Dynkin diagram Γ_{F_4}. We collect here some geometric properties of (S^3, W_{F_4}) which will be needed in the proof of Theorem 3.1 and which can be deduced from the information in [GB71, ch. 5.3].

We have $Out((S^3, W_{F_4})) \cong Isom(\Delta^{F_4}_{mod}) \cong Aut(\Gamma_{F_4}) \cong \mathbb{Z}_2$. The nontrivial involutive isometry of $\Delta^{F_4}_{mod}$ exchanges the vertices $1 \leftrightarrow 4$ and $2 \leftrightarrow 3$. Hence the properties of i- and $(5-i)$-vertices are dual to each other.

If we consider (S^3, W_{F_4}) embedded in $\mathbb{R}^4$ as the unit sphere, we can describe the Weyl group as a group of isometries of $\mathbb{R}^4$ as follows. The Weyl group W_{F_4} is the finite group generated by the reflections at the hyperplanes orthogonal to the *fundamental root vectors*:

$$r_1 = -\frac{1}{2}(1,1,1,1), \quad r_2 = e_1, \quad r_3 = e_2 - e_1, \quad r_2 = e_3 - e_2.$$

The *fundamental Weyl chamber* Δ is given by the inequalities:

$$x_1 + \cdots + x_4 \overset{(1)}{\leq} 0\ ; \qquad 0 \overset{(2)}{\leq} x_1 \overset{(3)}{\leq} x_2 \overset{(4)}{\leq} x_3.$$

We list vectors representing the vertices of Δ:

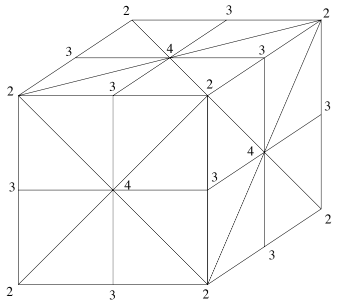
1-vertex:	v_1	$(\ 0,\ 0,\ 0,-1)$
2-vertex:	v_2	$(\ 1,\ 1,\ 1,-3)$
3-vertex:	v_3	$(\ 0,\ 1,\ 1,-2)$
4-vertex:	v_4	$(\ 0,\ 0,\ 1,-1)$

All half-apartments of (S^3, W_{F_4}) are centered at a vertex. The vertices of types 1 and 4 are the vertices of *root type*. We list vectors representing these vertices:

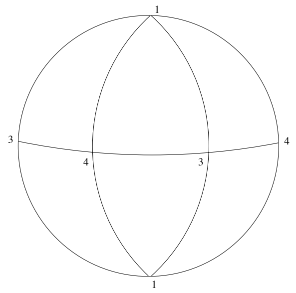
1-vertices: $\pm e_i$ for $i = 1, \ldots, 4$; $\frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4)$

4-vertices: $\pm e_i \pm e_j$ for $1 \leq i < j \leq 4$

The vertices of root type are better separated from each other than the other types of vertices. The possible mutual distances between 1-vertices (4-vertices) are $0, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}$ and π. Any two pairs of 1-vertices with the same distance are equivalent modulo the action of the Weyl group. Two 1-vertices with distance $\frac{\pi}{3}$ are connected by a singular segment of type 121, two 1-vertices with distance $\frac{\pi}{2}$ by a singular segment of type 141, and two 1-vertices with distance $\frac{2\pi}{3}$ by a singular segment of type 12121.

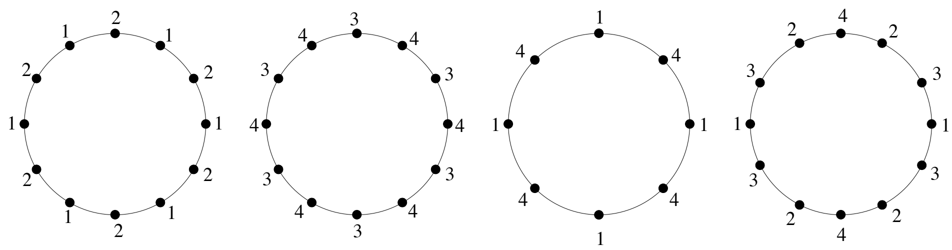


The link of a 1-vertex is a Coxeter complex Σ_1 of type B_3 with induced labelling 2 3 4 for its Dynkin diagram. Any two different 2-vertices in Σ_1 with distance $< \frac{\pi}{2}$ are connected by a singular segment of type 232, and any two non-antipodal 2-vertices with distance $> \frac{\pi}{2}$ are connected by a singular segment of type 242.



The link of a 2-vertex is a Coxeter complex Σ_2 of type $A_1 \circ A_2$ with induced labelling 1 3 4 for its Dynkin diagram.

The following types of singular 1-spheres occur in (S^3, W_{F_4}):



The last type can be verified e.g. using the Dynkin diagrams of the links Σ_i, compare section 2.2.1, and the edge lengths of Δ.

The canonical involution $\iota : \Delta_{mod}^{F_4} \to \Delta_{mod}^{F_4}$ is trivial. Accordingly, the antipodes of i-vertices in the Coxeter complex are i-vertices.

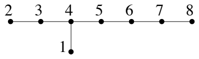
Remark 2.2. Since $Out((S^3, W_{F_4})) \cong \mathbb{Z}_2$, the normalizer $Aut((S^3, W_{F_4}))$ of W_{F_4} in $Isom(S^3)$ is an index two extension of W_{F_4}. However, it is not a reflection group, because the nontrivial isometry of Δ fixes no vertex and therefore is not induced by a hyperplane reflection.

2.2.3 The Coxeter complex of type E_6

Let (S^5, W_{E_6}) be the Coxeter complex of type E_6. We use the labelling 2 3 4 5 6 1 for its Dynkin diagram Γ_{E_6}. We collect here some geometric properties of (S^5, W_{E_6}) which will be needed in the proof of Theorem 3.18 and which can be deduced from the information in [GB71, ch. 5.3].

We have $Out((S^5, W_{E_6})) \cong Isom(\Delta_{mod}^{E_6}) \cong Aut(\Gamma_{E_6}) \cong \mathbb{Z}_2$. The nontrivial involutive isometry of $\Delta_{mod}^{E_6}$ fixes the vertices 1 and 4 and exchanges the vertices $2 \leftrightarrow 6$ and $3 \leftrightarrow 5$.

Our model for (S^5, W_{E_6}) is based on a model for the Coxeter complex (S^7, W_{E_8}) of type E_8. We consider (S^7, W_{E_8}) embedded in $\mathbb{R}^8$ as the unit sphere and will use the labelling

 for its Dynkin diagram. The root system of E_8 consists of the vectors

$$\begin{array}{ll} \pm e_i \pm e_j & \text{for } 1 \leq i < j \leq 8, \\ \frac{1}{2}\sum_{i=1}^{8} \pm\epsilon_i e_i & \text{with } \epsilon_i = \pm 1 \text{ such that } \prod_{i=1}^{8} \epsilon_i = -1. \end{array}$$

(The walls are the intersections of S^7 with the hyperplanes perpendicular to a vector in the root system.)

The link $\Sigma_\sigma S^7$ of a type 78 edge σ is a Coxeter complex of type E_6. The 8-vertices in (S^7, W_{E_8}) are the vertices of root type. The 78-edges have length $\frac{\pi}{6}$ and a pair of 8-vertices with distance $\frac{\pi}{3}$ or $\frac{2\pi}{3}$ lies on a singular circle of type $\dots 8787\dots$. Thus a model for (S^5, W_{E_6}) can be obtained by choosing two E_8-root vectors r, r' with angle $\frac{2\pi}{3}$, taking $S^5 := S^7 \cap \langle r, r'\rangle^\perp$ and $W_{E_6} = Fix_{W_{E_8}}(\{r, r'\})$. The model in [GB71] uses $r = e_8 - e_7$ and $r' = \frac{1}{2}(1,1,1,1,1,1,1,-1)$. For us the choice $r = e_8 - e_7$ and $r' = e_7 - e_6$ is more convenient, i.e. we realize (S^5, W_{E_6}) as the unit sphere in $\mathbb{R}^6 \cong \{(x_1,\dots,x_8) \in \mathbb{R}^8 \mid x_6 = x_7 = x_8\}$.

The E_6-root system then consists of the E_8-root vectors perpendicular to r and r', i.e. of

$$\begin{array}{ll} \pm e_i \pm e_j & \text{for } 1 \leq i < j \leq 5, \\ \frac{1}{2}\sum_{i=1}^{8} \pm\epsilon_i e_i & \text{with } \epsilon_i = \pm 1 \text{ such that } \prod_{i=1}^{8} \epsilon_i = -1 \text{ and } \epsilon_6 = \epsilon_7 = \epsilon_8. \end{array} \tag{2.3}$$

The Weyl group W_{E_6} is the finite group of isometries of generated by the reflections at the hyperplanes orthogonal to the *fundamental root vectors*:

$$r_1 = \tfrac{1}{2}(1,1,1,-1,-1,-1,-1,-1), \quad r_i = e_i - e_{i-1} \text{ for } 2 \leq i \leq 5;$$
$$\text{and} \quad r_6 = \tfrac{1}{2}(1,1,1,1,-1,1,1,1).$$

It contains as a proper subgroup the group W' which permutes the first five coordinates and changes an even number of their signs.

The *fundamental Weyl chamber* Δ is given by the inequalities:

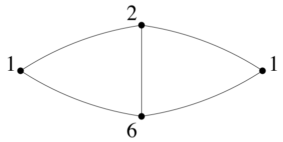
$$x_4 + x_5 + \dots + x_8 \overset{(1)}{\leq} x_1 + x_2 + x_3\ ; \quad x_1 \overset{(2)}{\leq} x_2 \overset{(3)}{\leq} \dots \overset{(5)}{\leq} x_5\ ; \quad x_5 \overset{(6)}{\leq} x_1 + \dots + x_4 + x_6 + x_7 + x_8.$$

We list vectors representing the vertices of Δ:

1-vertex: v_1	$(\ 1,\ 1,\ 1,\ 1,\ 1,-1,-1,-1)$
2-vertex: v_2	$(-3,\ 3,\ 3,\ 3,\ 3,-1,-1,-1)$
3-vertex: v_3	$(\ 0,\ 0,\ 3,\ 3,\ 3,-1,-1,-1)$
4-vertex: v_4	$(\ 1,\ 1,\ 1,\ 3,\ 3,-1,-1,-1)$
5-vertex: v_5	$(\ 3,\ 3,\ 3,\ 3,\ 9,-1,-1,-1)$
6-vertex: v_6	$(\ 3,\ 3,\ 3,\ 3,\ 3,\ 1,\ 1,\ 1)$

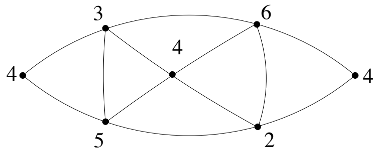
All half-apartments in (S^5, W_{E_6}) are centered at a vertex. The *1-vertices* are the vertices of *root type*, and they are represented by the root vectors (2.3).

The possible mutual distances between 1-vertices are 0, $\frac{\pi}{3}$, $\frac{\pi}{2}$, $\frac{2\pi}{3}$ and π. The pairs of 1-vertices with the same distance are equivalent modulo the action of the Weyl group. (This can be verified e.g. by considering the pairs containing v_1 which has a large stabilizer in W'.) A pair of 1-vertices with distance $\frac{\pi}{3}$ ($\frac{2\pi}{3}$) is connected by a type 141 (14141) singular segment. (Note that v_4 is the midpoint of v_1 and the 1-vertex represented by $e_4 + e_5$.)



The segment connecting two 1-vertices with distance $\frac{\pi}{2}$ is not singular. Its simplicial convex hull is a rhombus whose other diagonal is a 26-edge. (The midpoint of v_2v_6 equals the midpoint of v_1 and the 1-vertex represented by $\frac{1}{2}(-1, 1, \ldots, 1)$.)

An equilateral triangle with 141-sides is not a simplicial subcomplex. Its center is the midpoint of a 35-edge perpendicular to the triangle. A square with 141-sides is not a simplicial subcomplex either. Its center is the midpoint of a 26-edge perpendicular to the square.



To verify these last facts, note that the link of a 1-vertex is a 4-dimensional Coxeter complex Σ_1 of type A_5 with induced labelling for its Dynkin diagram. Any two distinct non-antipodal 4-vertices in Σ_1 have distance $\arccos(\pm\frac{1}{3})$, cf. section 2.2.4. The segment connecting them is not singular. Its simplicial convex hull is a rhombus whose other diagonal is a 35-edge if the 4-vertices have distance $\arccos\frac{1}{3}$, and a 26-edge if they have distance $\arccos(-\frac{1}{3})$. Furthermore, the link of an edge of type 35 (26) in (S^5, W_{E_6}) is a Coxeter complex Σ_{35} (Σ_{26}) of type $A_2 \circ A_1 \circ A_1$ (D_4) with Dynkin diagram ( ) and contains a singular 1-sphere of type $\ldots 1414 \ldots$ with edge length $\frac{\pi}{3}$ ($\frac{\pi}{4}$).

Next we list the *2-vertices* modulo the action of $W' \subset W_{E_6}$, more precisely, vectors representing them:

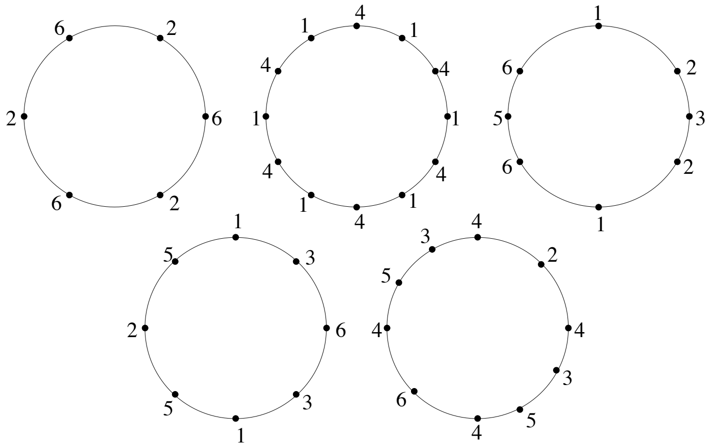
$$(-3, \ 3, \ 3, \ 3, \ 3, -1, -1, -1), \quad (\ 0, \ 0, \ 0, \ 0, \ 6, \ 2, \ 2, \ 2)$$
$$(\ 0, \ 0, \ 0, \ 0, \ 0, -4, -4, -4)$$

Since $-v_6$ is a 2-vertex, the *6-vertices* are just the antipodes of the 2-vertices. We see that the canonical involution $\iota : \Delta^{E_6}_{mod} \to \Delta^{E_6}_{mod}$ is the nontrivial isometry. Accordingly, the antipodes of i-vertices in the Coxeter complex are i-vertices for $i = 1, 4$ and $(8 - i)$-vertices for $i = 2, 3, 5, 6$.

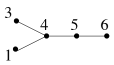
The possible mutual distances between 2-vertices (6-vertices) are 0, $\arccos\frac{1}{4}$ and $\frac{2\pi}{3}$. Any two pairs of 2-vertices with the same distance are equivalent modulo the action of the Weyl group. (This is obvious from considering the W'-orbits.) Two 2-vertices (6-vertices) with distance $\arccos\frac{1}{4}$ are connected by a singular segment of type 232 (656), and two 2-vertices (6-vertices) with distance $\frac{2\pi}{3}$ by a singular segment of type 262 (626). (One sees this by extending the edges v_2v_3 and v_2v_6 of Δ.)

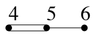
The possible distances between a 2-vertex and a 6-vertex are $\frac{\pi}{3}$, $\arccos(-\frac{1}{4})$ and π. If their distance is $\frac{\pi}{3}$, they are connected by an edge of type 26; if their distance is $\arccos(-\frac{1}{4})$, they are connected by a singular segment of type 216. (This and the next list can be verified using the Dynkin diagrams of the links Σ_i and the edge lengths of Δ.)

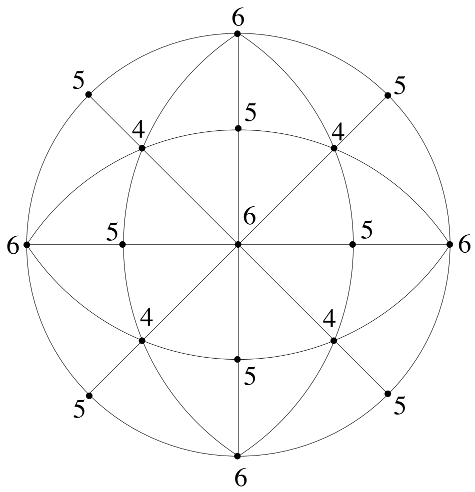
The following types of singular 1-spheres occur in (S^5, W_{E_6}):



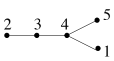
We will also need some information about the geometry of the links of (S^5, W_{E_6}), and in one case about the links of the links. These Coxeter complexes are of classical type, see section 2.2.4. The links of 1-vertices have already been mentioned above.

The link of a 2-vertex in (S^5, W_{E_6}) is a 4-dimensional Coxeter complex Σ_2 of type D_5 with induced labelling  for its Dynkin diagram. In view of the symmetry of the Dynkin diagram, the roles of the 1- and 3-vertices are equivalent.

Any two distinct 6-vertices in Σ_2 have distance $\frac{\pi}{2}$ or π. In the first case, their midpoint is a 5-vertex and they are connected by a singular segment of type 656. The convex hull of a triple of 6-vertices in Σ_2 with pairwise distances $\frac{\pi}{2}$ is a right-angled equilateral triangle centered at a 4-vertex. The 2-sphere containing it is a singular 2-sphere isomorphic to the B_3-Coxeter complex with Dynkin diagram . It is tesselated by forty eight 654-triangles.



The convex hull of a quadruple of 6-vertices in Σ_2 with pairwise distances $\frac{\pi}{2}$ is an equilateral tetrahedron with edge lengths $\frac{\pi}{2}$. Its codimension-one faces are simplicial subcomplexes composed of six 654-triangles. However, the tetrahedron itself is *not* a simplicial subcomplex; its center is the midpoint of a 13-edge. Accordingly, the geodesic 3-sphere containing the tetrahedron is not a subcomplex either, and its simplicial convex hull is the entire Coxeter complex.

The link of a 6-vertex in (S^5, W_{E_6}) is a 4-dimensional Coxeter complex Σ_6 of type D_5 with induced labelling  for its Dynkin diagram. Its geometric properties are dual to the ones of Σ_2. For instance, the singular 2-sphere containing a 234-triangle is isomorphic to the

B_3-Coxeter complex with Dynkin diagram $\overset{4}{\bullet}\!=\!\overset{3}{\bullet}\!-\!\overset{2}{\bullet}$. It is tesselated by forty eight 234-triangles, compare the figure for Lemma 3.9 below.

We will use the fact that the possible types of singular segments of length π in Σ_6 connecting antipodal 2-vertices are 23232, 24342 and 2512.

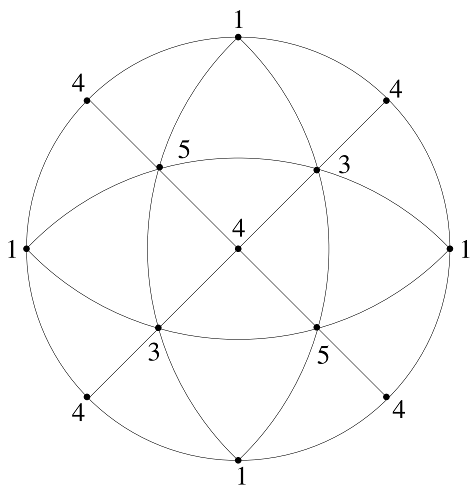
As already mentioned above, the link of a 26-edge in (S^5, W_{E_6}) is a 3-dimensional Coxeter complex Σ_{26} of type D_4 with induced labelling $\overset{3}{\bullet}\,\overset{4}{\curlywedge}\,\overset{5}{\bullet}\ {}_{1}$ for its Dynkin diagram. In view of the symmetries of the Dynkin diagram, the roles of the i-vertices for $i \neq 4$ are equivalent.

Any vertex adjacent to a 4-vertex in Σ_{26} has distance $\frac{\pi}{4}$ from it.

Any two non-adjacent vertices in Σ_{26} of different types $\neq 4$ are connected by a singular segment through a vertex of the third type $\neq 4$, and they lie on a singular 1-sphere of type 1351351 (the first and last 1 to be identified). For instance, two non-adjacent vertices of types 1 and 3 are connected by a segment of type 153.

Any two distinct vertices in Σ_{26} of the same type $i \neq 4$ have distance $\frac{\pi}{2}$ or π. In the first case, their midpoint is a 4-vertex and they are connected by a singular segment of type $i4i$. The convex hull of a triple of i-vertices with pairwise distances $\frac{\pi}{2}$ is a right-angled equilateral triangle. We observe that it is *not* a simplicial subcomplex of the Coxeter complex; its center is the midpoint of an edge perpendicular to it and with endpoints of the two types $\neq i, 4$. For instance, if $i = 3$ then it is the midpoint of a 15-edge. Accordingly, the geodesic 2-sphere containing such a triple of i-points is not a subcomplex, and its simplicial convex hull is the whole Coxeter complex.

There is only one type of singular 2-spheres in Σ_{26}, equivalently, the singular 2-spheres are composed of all four types of 2-dimensional faces.



Remark 2.4. We have $Out((S^5, W_{E_6})) \cong \mathbb{Z}_2$. However, the index two extension $Aut((S^5, W_{E_6}))$ of W_{E_6} is not a reflection group, because the nontrivial isometry of Δ fixes only two vertices and therefore is not induced by a hyperplane reflection, compare Remark 2.2.

2.2.4 The Coxeter complexes of classical types

We consider a spherical Coxeter complex (S^{n-1}, W) embedded in $\mathbb{R}^n$ as the unit sphere and we describe the Weyl group as a group of isometries of $\mathbb{R}^n$.

Type A_n. Let $n \geq 1$. We use the labelling $\overset{1}{\bullet}\!-\!\overset{2}{\bullet}\cdots\overset{n-1}{\bullet}\!-\!\overset{n}{\bullet}$ for the Dynkin diagram of type A_n. The Weyl group W_{A_n} is the finite group of isometries of $\mathbb{R}^n \cong \{x_0 + \cdots + x_n = 0\} \subset \mathbb{R}^{n+1}$ generated by the reflections at the hyperplanes orthogonal to the *fundamental root vectors*

$$r_i = e_i - e_{i-1} \text{ for } 1 \leq i \leq n.$$

The *fundamental Weyl chamber* Δ is given by the inequalities:

$$x_0 \overset{(1)}{\leq} x_1 \overset{(2)}{\leq} \ldots \overset{(n)}{\leq} x_n \tag{2.5}$$

We list vectors representing the vertices of Δ:

$$\begin{array}{rcllllllll}
\text{1-vertex:} & v_1 & (& -n, & 1, & 1, & \ldots, & 1, & 1, & 1\;) \\
\text{2-vertex:} & v_2 & (& -(n-1), & -(n-1), & 2, & \ldots, & 2, & 2, & 2\;) \\
\vdots & \vdots & & & & & \vdots & & & \\
(n-1)\text{-vertex:} & v_{n-1} & (& -2, & -2, & -2, & \ldots, & -2, & n-1, & n-1\;) \\
n\text{-vertex:} & v_n & (& -1, & -1, & -1, & \ldots, & -1, & -1, & n\;)
\end{array}$$

The *root system* consists of the vectors

$$\pm(e_i - e_j) \qquad \text{for } 0 \leq i < j \leq n.$$

(The walls are the intersections of S with the hyperplanes perpendicular to a vector in the root system.) The Weyl group W_{A_n} acts on $\mathbb{R}^{n+1}$ by permutations of the coordinates.

The canonical involution $\iota : \Delta_{mod}^{A_n} \to \Delta_{mod}^{A_n}$ is the nontrivial isometry. Accordingly, the antipodes of i-vertices in the Coxeter complex are $(n+1-i)$-vertices.

Remark 2.6. We have $Out((S^{n-1}, W_{A_n})) \cong \mathbb{Z}_2$ for $n \geq 2$. However, the index two extension $Aut((S^{n-1}, W_{A_n}))$ of W_{A_n} is not a reflection group if $n \geq 4$, because the nontrivial isometry of Δ moves more than two vertices and therefore is not induced by a hyperplane reflection. If $n \leq 3$, we have $Aut((S^2, W_{A_3})) \cong W_{B_3}$ and $Aut((S^1, W_{A_2})) \cong W_{G_2}$.

Type B_n. Let $n \geq 2$. We use the labelling $\overset{1}{\bullet}\!=\!\overset{2}{\bullet}\!-\!\overset{3}{\bullet}\cdots\overset{n-1}{\bullet}\!-\!\overset{n}{\bullet}$ for the Dynkin diagram. The Weyl group W_{B_n} is the finite group of isometries of $\mathbb{R}^n$ generated by the reflections at the hyperplanes orthogonal to the *fundamental root vectors*

$$r_1 = e_1, \quad r_i = e_i - e_{i-1} \text{ for } 2 \leq i \leq n.$$

The *fundamental Weyl chamber* Δ is given by the inequalities:

$$0 \overset{(1)}{\leq} x_1 \overset{(2)}{\leq} x_2 \overset{(3)}{\leq} \ldots \overset{(n)}{\leq} x_n \tag{2.7}$$

We list vectors representing the vertices of Δ:

$$\begin{array}{rcl}
\text{1-vertex:} & v_1 & (1, 1, 1, \ldots, 1) \\
\text{2-vertex:} & v_2 & (0, 1, 1, \ldots, 1) \\
\text{3-vertex:} & v_3 & (0, 0, 1, \ldots, 1) \\
\vdots & \vdots & \vdots \\
(n-1)\text{-vertex:} & v_{n-1} & (0, \ldots, 0, 1, 1) \\
n\text{-vertex:} & v_n & (0, \ldots, 0, 0, 1)
\end{array}$$

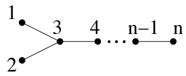
All half-apartments in (S^{n-1}, W_{B_n}) are centered at a vertex. The vertices of types n and $n-1$ are the vertices of *root type*. They are represented by the vectors

$$\begin{array}{ll} \pm e_i & \text{for } 1 \le i \le n, \\ \pm e_i \pm e_j & \text{for } 1 \le i < j \le n. \end{array}$$

The Weyl group W_{B_n} acts on $\mathbb{R}^n$ by permutations of the coordinates and change of signs.

The canonical involution $\iota : \Delta^{B_n}_{mod} \to \Delta^{B_n}_{mod}$ is trivial. Accordingly, the antipodes of i-vertices in the Coxeter complex are i-vertices.

Remark 2.8. $Out((S^{n-1}, W_{B_n}))$ is trivial for $n \ge 3$. For $n = 2$, we have $Aut(S^1, W_{B_2}) \cong W_{I_2(8)}$.

Type D_n. Let $n \ge 4$. We use the labelling  for the Dynkin diagram. The Weyl group W_{D_n} is the finite group of isometries of $\mathbb{R}^n$ generated by the reflections at the hyperplanes orthogonal to the *fundamental root vectors*

$$r_1 = e_1 + e_2, \quad r_i = e_i - e_{i-1} \text{ for } 2 \le i \le n.$$

The *fundamental Weyl chamber* Δ is given by the inequalities:

$$-x_2 \overset{(1)}{\le} x_1 \overset{(2)}{\le} x_2 \overset{(3)}{\le} \ldots \overset{(n)}{\le} x_n \tag{2.9}$$

We list vectors representing the vertices of Δ:

$$\begin{array}{rcl} \text{1-vertex:} & v_1 & (\ \ 1, 1, 1, \ldots, 1) \\ \text{2-vertex:} & v_2 & (-1, 1, 1, \ldots, 1) \\ \text{3-vertex:} & v_3 & (\ \ 0, 0, 1, \ldots, 1) \\ \vdots & \vdots & \vdots \\ (n-1)\text{-vertex:} & v_{n-1} & (\ \ 0, \ldots, 0, 1, 1) \\ n\text{-vertex:} & v_n & (\ \ 0, \ldots, 0, 0, 1) \end{array}$$

All half-apartments in (S^{n-1}, W_{D_n}) are centered at a vertex. The vertices of type $n-1$ are the vertices of *root type*. They are represented by the vectors

$$\pm e_i \pm e_j \qquad \text{for } 1 \le i < j \le n.$$

The Weyl group W_{D_n} acts on $\mathbb{R}^n$ by permutations of the coordinates and change of signs in an even number of places.

The canonical involution $\iota : \Delta^{D_n}_{mod} \to \Delta^{D_n}_{mod}$ is trivial for n even. Accordingly, in this case the antipodes of i-vertices in the Coxeter complex are i-vertices. For n odd, the canonical involution is the nontrivial isometry. The antipodes of 1-vertices are then 2-vertices and for $i = 3, \ldots, n$ the antipodes of i-vertices are i-vertices.

In the models chosen here, the root system of D_n is contained in the root system of B_n, and thus the B_n-Coxeter complex is a subdivision of the D_n-Coxeter complex. The vertices of root type $(n-1)$ in D_n are the vertices of a fixed type (also $n-1$) in B_n. Hence the D_n-root system is preserved by W_{B_n} and $W_{D_n} \subset W_{B_n} \subseteq Aut((S^{n-1}, W_{D_n}))$, cf. section 2.2.1. If $n \geq 5$, then $Out((S^{n-1}, W_{D_n})) \cong \mathbb{Z}_2$ and it follows that

$$Aut((S^{n-1}, W_{D_n})) = W_{B_n} \qquad \text{for } n \geq 5 \tag{2.10}$$

and $W_{B_n} \cong W_{D_n} \rtimes \mathbb{Z}_2$. The subdivision of Δ^{D_n} into two B_n-Weyl chambers is obtained by cutting along the hyperplane perpendicular to $\pm e_1$.

The Dynkin diagram of Γ_{D_4} has more than two symmetries, $Aut(\Gamma_{D_4}) \cong Out(S^3, W_{D_4}) \cong S_3$, and indeed the root system of D_4 is contained in a larger root system than the one of B_4, namely in the root system of F_4, Thus the D_4-Coxeter complex can be subdivided into the F_4-Coxeter complex. More precisely, the D_4-root system is preserved by W_{F_4}, because the vertices of root type (3) in D_4 are the vertices of a fixed type (4) in F_4. Accordingly, W_{D_4} is a normal subgroup of W_{F_4}, $W_{D_4} \subset W_{F_4} \subseteq Aut((S^3, W_{D_4}))$. Since also $W_{D_4} \subset W_{B_4} \subset W_{F_4}$, it is clear that $[W_{F_4} : W_{D_4}] = 6$. (This can also easily be checked directly, for instance, by determining the F_4-walls intersecting the fundamental Weyl chamber Δ^{D_4}, respectively, containing its center.) It follows that

$$Aut((S^3, W_{D_4})) = W_{F_4}, \tag{2.11}$$

and $W_{F_4} \cong W_{D_4} \rtimes S_3$. The subdivision of Δ^{D_4} into six F_4-Weyl chambers is obtained by taking the barycentric subdivision of the equilateral type 124 face and coning off at the 3-vertex.

2.3 Spherical buildings

We refer to [KL98, ch. 3] for a treatment of spherical buildings from the perspective of comparison geometry.

2.3.1 Some basic definitions

We recall the geometric definition of spherical buildings as given in [KL98, ch. 3.2]. A *spherical building* modelled on a Coxeter complex (S, W) is a CAT(1) space B equipped with an *atlas* consisting of isometric embeddings $\iota : S \hookrightarrow B$, the *charts*, satisfying certain properties. The images of the charts are called the *apartments*. Any two points must lie in an apartment. The atlas must be closed under precomposition with isometries in W, and the charts must be compatible in the sense that the coordinate changes are restrictions of isometries in W.

The underlying set may be empty, in which case the building is called a spherical *ruin*.

One defines walls, roots, singular spheres, faces, chambers, panels, regular and singular points as the images of the corresponding objects in the model Coxeter complex. In particular, a spherical building carries a natural structure as a piecewise spherical *polyhedral* complex, in fact, as a *simplicial* complex if W has no fixed points on S. The building B is called *thick*, if every panel is adjacent to at least three chambers.

Two points in B are called *antipodal* if they have maximal distance π.

Let $\sigma \subset B$ be a face of codimension ≥ 1. Then for an interior point $x \in \sigma$, the link $\Sigma_x B$ splits as the spherical join $\Sigma_x B \cong \Sigma_x \sigma \circ \nu_x \sigma$ of the unit sphere $\Sigma_x \sigma$ and the unit normal space $\nu_x \sigma$ of σ in B. One can consistently identify with each other the unit normal spaces $\nu_x \sigma$ for all interior points $x \in \sigma$. This identification can be described as follows: For interior points $x_1, x_2 \in \sigma$, let $c_i : [0, \epsilon) \to B$ be unit speed geodesic segments emanating from x_i orthogonal to σ. Then the directions $\dot{c}_i(0) \in \nu_{x_i}\sigma$ are identified if and only if for small $t > 0$ the convex hulls $CH(\sigma \cup \{c_i(t)\})$ locally coincide near x_1 and x_2. We call the resulting identification space the *link* $\Sigma_\sigma B$ *of the face* σ. It inherits a natural structure as a spherical building modelled on the Coxeter complex $\left(\Sigma_{\iota^{-1}(\sigma)} S, Stab_W(\iota^{-1}(\sigma))\right)$ where $\iota : S \hookrightarrow B$ is a chart with $\sigma \subset \iota(S)$. For faces $\sigma \subset \tau$ there is a canonical identification $\Sigma_\tau B \cong \Sigma_{\Sigma_\sigma \tau}(\Sigma_\sigma B)$.

There is a natural "accordion" *anisotropy map* $\theta_B : B \to \Delta_{mod}$ folding the building onto the model Weyl chamber. It is determined by the property that for any chart $\iota : S \hookrightarrow B$ holds $\theta_B \circ \iota = \theta_S$. The anisotropy map is 1-Lipschitz and restricts to isometries on chambers. Furthermore, for any apartment $a \subset B$ and any chamber $\sigma \subset a$ there is a natural 1-Lipschitz retraction $\rho_{a,\sigma} : B \to a$, $\rho_{a,\sigma}|_a = id_a$, which can be described as follows: For a regular point $x \in \sigma$ and any point $y \in B$ not antipodal to x, the segment connecting x to $\rho_{a,\sigma}(y)$ coincides with the segment xy near x and has the same length. The retraction is type preserving, $\theta_B \circ \rho_{a,\sigma} = \theta_B$, and it restricts to an isometry on every apartment containing σ.

The following result in the spirit of [KL98, Prop. 3.5.1] allows to recover the building structure from the anisotropy map.

Proposition 2.12 (Recognizing a building structure). *Suppose that X is a CAT(1) space which is equipped with a structure as a piecewise spherical polyhedral complex of dimension equal to* $\dim(S)$. *Let $\theta_X : X \to \Delta_{mod}$ be a 1-Lipschitz map which restricts on every top-dimensional face to an isometry onto Δ_{mod}. Suppose furthermore that any two points in X lie in an isometrically embedded copy of S. Then X carries a natural structure as a spherical building modelled on the Coxeter complex (S, W) and with anisotropy map θ_X.*

Proof. We call a top-dimensional face of X a *chamber* and an isometrically embedded copy of S an *apartment*. Due to our assumption, the apartments are tesselated by chambers. For any two adjacent chambers σ_1 and σ_2 in an apartment a there is the isometry $(\theta_X|_{\sigma_2})^{-1} \circ (\theta_X|_{\sigma_1}) : \sigma_1 \to \sigma_2$. It must coincide with the reflection at the common codimension-one face $\sigma_1 \cap \sigma_2$. Hence there is an isometric identification $\iota : S \to a$ satisfying $\theta_X \circ \iota = \theta_S$ which is unique up to precomposition with Weyl isometries. The compatibility of all these *charts* for all apartments is automatic and they form an atlas for a spherical building structure modelled on (S, W) with anisotropy map θ_X. $\square$

An isometry $\alpha : B \to B$ is called an *automorphism* of the spherical building B if it preserves the polyhedral structure. If B is a *thick* building, then all its isometries are automorphisms. An automorphism is called *inner* if it is type preserving, $\theta_B \circ \alpha = \theta_B$. We denote the automorphism group of B by $Aut(B)$, and the subgroup of inner automorphisms by $Inn(B)$. Then $Inn(B)$ is a finite index normal subgroup of $Aut(B)$ and $Out(B) := Aut(B)/Inn(B)$ embeds as a subgroup of $Isom(\Delta_{mod}) \cong Out((S, W))$.

2.3.2 Convex subcomplexes and subbuildings

Let B be a spherical building. By a *convex subcomplex* of B we mean a closed convex subset which is also a subcomplex with respect to the natural polyhedral structure on B. The *simplicial* *convex hull* of a subset $A \subseteq B$ is the smallest convex subcomplex of B containing A.

We call a convex subcomplex $K \subseteq B$ a *subbuilding* if any two of its points are contained in a singular sphere $s \subseteq K$ with $\dim(s) = \dim(K)$. The next result tells that a subbuilding inherits a natural structure as a spherical building. To describe the associated Coxeter complex, let $a \subset B$ be an apartment containing a singular sphere $s \subset K$ with $\dim(s) = \dim(K)$ and let $\iota : S \xrightarrow{\cong} a \subset B$ be a chart. As explained in section 2.2.1, the singular sphere $\iota^{-1}(s) \subseteq S$ inherits from S a natural structure as a Coxeter complex with a possibly coarser polyhedral structure. Its W*-type*, that is, its equivalence class modulo the action of the Weyl group does not depend on the choice of s.

Proposition 2.13 (Building structure on subbuildings). *The subbuilding $K \subseteq B$ carries a natural structure as a spherical building modelled on the Coxeter complex $(\iota^{-1}(s), W_{\iota^{-1}(s)})$.*

Proof. We fix a singular sphere $s \subset K$ with $\dim(s) = \dim(K)$. Let a be an apartment containing s and let $\sigma \subset a$ be a chamber such that $\sigma \cap s$ is a top-dimensional face of s. Then the retraction $\rho_{a,\sigma} : B \to a$ restricts to a retraction $\rho_{s,\sigma\cap s} : K \to s$ of K.

We note that $\rho_{s,\sigma\cap s}$ restricts to an isometry on every singular sphere $s' \subset K$ containing $\sigma \cap s$. Using a chart $\iota : S \xrightarrow{\cong} a \subset B$, we can pull back the intrinsic polyhedral structure (as a Coxeter complex) on $\iota^{-1}(s)$ to s' via $\iota^{-1} \circ \rho_{s,\sigma\cap s}|_{s'}$. We will refer to the pulled back structure as the *intrinsic* polyhedral structure on s'. The main point to verify is that the intrinsic polyhedral structures on all such singular spheres s' match and yield a polyhedral structure on K.

At this point, we have on K only the polyhedral structure which it inherits from S. We say that K *branches* along a codimension one face ϕ, if K has at least three top-dimensional faces τ_1, τ_2 and τ_3 adjacent to ϕ, i.e. with ϕ as a common codimension one face. It then follows that the τ_i (and all top-dimensional faces of K adjacent to ϕ) must have the same θ_B-type. Indeed, the unions $\tau_i \cup \tau_j$ for $i \neq j$ are convex, because K is convex and the τ_i are top-dimensional in K. Let ι_{ij} be charts whose images contain $\tau_i \cup \tau_j$. We may choose, say, ι_{12} and ι_{13} so that $\iota_{12}^{-1}(\tau_1) = \iota_{13}^{-1}(\tau_1)$. Then necessarily $\iota_{12}^{-1}(\tau_2) = \iota_{13}^{-1}(\tau_3)$ and hence $\theta_B(\tau_2) = \theta_B(\tau_3)$. Similarly, $\theta_B(\tau_1) = \theta_B(\tau_2)$.

Let $s' \subset K$ be a singular sphere containing ϕ and $\sigma \cap s$. Since the two top-dimensional faces in s' adjacent to ϕ have the same θ_B-type, the codimension one singular subsphere $t' \subset s'$ containing ϕ is an s'*-wall* in the sense that $(\iota^{-1} \circ \rho_{s,\sigma\cap s})(t')$ is an $\iota^{-1}(s)$-wall as defined in section 2.2.1.

The intersection $s' \cap s''$ of any two singular spheres $s', s'' \subset K$ containing $\sigma \cap s$ is obviously top-dimensional in K. Our discussion implies that it is a subcomplex with respect to the *intrinsic* polyhedral structures of these singular spheres, because its boundary consists of codimension one faces of K along which K branches and thus is contained in a union of s'-walls (s''-walls). It follows that the intersection of any two top-dimensional faces $\tau' \subset s'$ and $\tau'' \subset s''$ with respect to the intrinsic polyhedral structures is either empty or a face of τ' (and τ''). This means that the intrinsic polyhedral structures on the singular spheres in K containing $\sigma \cap s$ match and form together a polyhedral structure on K.

To conclude the argument, we observe that the 1-Lipschitz map

$$\theta_{\iota^{-1}(s)} \circ \iota^{-1} \circ \rho_{s,\sigma\cap s} : K \to \iota^{-1}(s)/W_{\iota^{-1}(s)} = \Delta_{mod}^{(\iota^{-1}(s), W_{\iota^{-1}(s)})}$$

restricts on top-dimensional faces (for the new polyhedral structure just defined) to surjective isometries, because for singular spheres $s' \subset K$ containing $\sigma \cap s$ the map $\iota^{-1}(s) \circ \rho_{s,\sigma\cap s}|_{s'} : s' \to \iota^{-1}(s)$ is an isometry preserving the polyhedral structure. The assertion follows now from Proposition 2.12. $\square$

We discuss now some conditions implying that a convex subcomplex is a subbuilding. To begin with, the existence of a top-dimensional subsphere is sufficient.

Proposition 2.14. *If a convex subcomplex $K \subseteq B$ contains a singular sphere $s \subseteq K$ with $\dim(s) = \dim(K)$, then it is a subbuilding.*

Proof. For top-dimensional subcomplexes this is [KL98, Prop. 3.10.3]. The proof in the general case is similar. One observes first that every point $x \in K$ has an antipode $\hat{x}$ in s. More precisely, for an arbitrary point $y \in s$ the antipode $\hat{x}$ can be chosen so that y lies on a geodesic segment $x\hat{x}$ of length π. To see this, let $\sigma \subset s$ be a face containing y with $\dim(\sigma) = \dim(s)$ and let y_1 be an interior point of σ. Then the geodesic segment xy_1 is contained in s near y_1 and therefore can be extended beyond y_1 inside s to a geodesic segment $xy_1\hat{x}$ of length π. The convex hull $CH(\{x, \hat{x}\} \cup \sigma)$ is a bigon and contains a geodesic segment $xy\hat{x}$ of length π.

The convex hull of x and a small disk in s around $\hat{x}$ is a singular sphere s' with $y \in s' \subset K$. We see that any point and also any pair of points in K lies in a singular sphere of dimension $\dim(K)$ contained in K. $\square$

If every point of K has an antipode *in* K, then clearly plenty of spheres $s \subset K$ as in Proposition 2.14 exist. Just take the simplicial convex hull of a pair of maximally regular antipodes in K. (A point in K is said to be *maximally regular* if it is an interior point of a face $\sigma \subset K$ which is top-dimensional in K, $\dim(\sigma) = \dim(K)$.) The next result says that it is enough to assume the existence of antipodes only for vertices.

Proposition 2.15 ([Se05, Thm. 2.2]). *If every vertex of a convex subcomplex $K \subseteq B$ has an antipode in K, then K is a subbuilding.*

Proof. Let $\sigma \subset K$ be a simplex with vertices $p_0, \ldots, p_k$. In view of Proposition 2.14, it suffices to show that there exists a singular k-sphere s_k with $\sigma \subset s_k \subset K$.

We proceed by induction over k. By assumption, the assertion holds for $k = 0$. To do the induction step, we consider a singular $(k-1)$-sphere s_{k-1} with $p_0, \ldots, p_{k-1} \in s_{k-1} \subset K$. The convex hull of s_{k-1} and p_k is a k-hemisphere $h_k \subset K$ with boundary $\partial h_k = s_{k-1}$. By assumption, p_k has an antipode $\hat{p}_k$ in K. The convex hull of h_k and $\hat{p}_k$ contains a singular k-sphere s_k with $\sigma \subset s_k \subset K$. $\square$

The following simple observation will be useful when we search for antipodes in convex subcomplexes.

Lemma 2.16. *Let $x_1x_2 \subset K$ be a segment. Suppose that z is an interior point of x_1x_2 which has an antipode $\hat{z} \in K$. Then the x_i also have antipodes in K.*

Proof. Let $\gamma_i \subset K$ for $i = 1, 2$ be the geodesic connecting z and $\hat{z}$ with initial direction $\overrightarrow{zx_{3-i}}$ at z. Then $x_i z \cup \gamma_i$ is a geodesic of length $> \pi$, and γ_i contains an antipode of x_i. $\square$

We will call a point x in a convex subcomplex $K \subset B$ an *interior* point of K if $\Sigma_x K$ is a subbuilding of $\Sigma_x B$, and a *boundary* point otherwise.

2.3.3 Circumcenters

Let B be a spherical building and let $K \subset B$ be a non-empty convex subcomplex.

We recall that if $\mathrm{rad}(K) < \frac{\pi}{2}$ then K has a unique circumcenter which must be contained in K, cf. section 2.1.2.

If $\mathrm{rad}(K) = \frac{\pi}{2}$, then the arguments in [BL05, ch. 3] for general CAT(1) spaces of finite dimension yield that $Cent(K) \neq \emptyset$. Moreover, if $K \cap Cent(K) \neq \emptyset$, then $K \cap Cent(K)$ has a unique circumcenter. In our special situation, the proofs simplify and we include them for the sake of completeness.

Lemma 2.17. *If B is a spherical building and if $K \subset B$ is a convex subcomplex with $rad(K) = \frac{\pi}{2}$, then $Cent(K) \neq \emptyset$.*

Proof. Consider a sequence of points $x_i \in B$ with $\mathrm{rad}(K, x_i) \searrow \frac{\pi}{2}$. We may assume that $\theta_B(x_i) \to t \in \Delta_{mod}$ by compactness. Let σ_i denote the face of B containing x_i as an interior point. For sufficiently large i, σ_i contains a point x_i' with $\theta_B(x_i') = t$. (Namely, when $d(\theta_B(x_i), t) < \delta$ where δ is the distance between t and the union of those faces of Δ_{mod} which do not contain t in their closure.) We have $d(x_i', x_i) \to 0$ and hence also $\mathrm{rad}(K, x_i') \to \frac{\pi}{2}$. Since the x_i' have fixed θ_B-type t and since K is a subcomplex, it follows that $\mathrm{rad}(K, x_i') = \frac{\pi}{2}$ for large i. This is due to the fact that the radius of a face of B with respect to a point of fixed type t can take only finitely many values (depending on t and the Coxeter complex). □

The following observations apply to the situation when the closed convex subset $K \cap Cent(K)$ is non-empty. Clearly, it has diameter $\leq \frac{\pi}{2}$. We begin with a consequence of [BL05, Prop. 1.2]:

Lemma 2.18. *Suppose that $A \subset B$ is a non-empty subset with $diam(A) \leq \frac{\pi}{2}$ and such that $\theta_B(A) \subset \Delta_{mod}$ is finite. Then $rad(A) < \frac{\pi}{2}$.*

Proof. By the finiteness assumption on types, the distances between points in A take only finitely many values. Therefore, if for some point $x \in A$ holds $d(x, y) < \frac{\pi}{2}$ for all $y \in A$, then $\mathrm{rad}(A, x) < \frac{\pi}{2}$ and we are done.

Otherwise, we pick some point $x \in A$ and consider the set $A' \subset \Sigma_x B$ of directions $\overrightarrow{xy}$ to the points $y \in A$ with $d(x, y) = \frac{\pi}{2}$. The directions in A' have only finitely many $\theta_{\Sigma_x B}$-types and, by triangle comparison, $\mathrm{diam}(A') \leq \frac{\pi}{2}$. Thus A' satisfies the same assumptions as A. Moreover, $\mathrm{rad}(A') < \frac{\pi}{2}$ implies $\mathrm{rad}(A) < \frac{\pi}{2}$. Indeed, let $\gamma : [0, \epsilon) \to B$ be a geodesic segment with initial point $\gamma(0) = x$ and initial direction $\dot\gamma(0)$ satisfying $\mathrm{rad}(A', \dot\gamma(0)) < \frac{\pi}{2}$. Then $\mathrm{rad}(A, \gamma(t)) < \frac{\pi}{2}$ for small $t > 0$. Here we use again the finiteness of $\theta_B(A)$; it yields a constant $\epsilon > 0$ with the property that $d(x, y) = \frac{\pi}{2}$ or $d(x, y) \leq \frac{\pi}{2} - \epsilon$ for all $y \in A$.

We can therefore proceed by induction on the dimension of B. The assertion holds trivially for buildings of dimension zero. □

Corollary 2.19. *Let $C \subset B$ be a non-empty closed convex subset with diameter $\leq \frac{\pi}{2}$. Then the action $Stab_{Aut(B)}(C) \curvearrowright C$ has a fixed point.*

Proof. Pick any point in C. By Lemma 2.18, its $Stab_{Aut(B)}(C)$-orbit has circumradius $< \frac{\pi}{2}$ and therefore a unique circumcenter which is contained in C. It is fixed by $Stab_{Aut(B)}(C)$. $\square$

Corollary 2.20. *If K is as in Lemma 2.17 and if $K \cap Cent(K) \neq \emptyset$, then $Stab_{Aut(B)}(K) \curvearrowright K$ has a fixed point.*

Proof. By Corollary 2.19, $Stab_{Aut(B)}(K)$ fixes a point in $K \cap Cent(K)$. $\square$

Remark 2.21. $Cent(K)$ is the intersection of the closed $\frac{\pi}{2}$-balls centered at the vertices of K. Hence $Cent(K)$ is a subcomplex of B with respect to a refinement of the polyhedral structure of B which corresponds to a refinement of the polyhedral structure of the Coxeter complex (S, W). This refinement can be described as follows: Consider the boundaries of the hemispheres in the Coxeter complex centered at its vertices. In general, not all of these codimension one great spheres are walls. However there are only finitely many of them and they yield a refinement of the polyhedral structure of (S, W) which projects to a subdivision of Δ_{mod}. If $K \cap Cent(K) \neq \emptyset$, we may therefore apply Lemma 2.18 to the set of vertices of $K \cap Cent(K)$ with respect to the refined polyhedral structure and conclude that $\mathrm{rad}(K \cap Cent(K)) < \frac{\pi}{2}$. It follows that $K \cap Cent(K)$ contains a unique circumcenter which must be fixed by $Isom(K) \supseteq Stab_{Aut(B)}(K)$.

Combining these results with Proposition 2.14, we obtain:

Corollary 2.22. *If a convex subcomplex $K \subseteq B$ contains a singular sphere $s \subseteq K$ with $\dim(s) = \dim(K) - 1$, then K is a subbuilding or it is contained in a closed $\frac{\pi}{2}$-ball centered in K and the action $Stab_{Aut(B)}(K) \curvearrowright K$ has a fixed point.*

Proof. K contains a (polyhedral) hemisphere $h \subseteq K$ with boundary s. One can obtain h as the convex hull of s and a top-dimensional face σ of K such that $\sigma \cap s$ is a top-dimensional face of s. Let z be the center of h.

If $K \subseteq \overline{B}_{\frac{\pi}{2}}(z)$, then $\mathrm{rad}(K) = \frac{\pi}{2}$ and $z \in K \cap Cent(K)$. Corollary 2.20 yields that $Stab_{Aut(B)}(K) \curvearrowright K$ has a fixed point.

If $K \not\subseteq \overline{B}_{\frac{\pi}{2}}(z)$, let $x \in K$ be a point with $d(x, z) > \frac{\pi}{2}$. There exists an antipode $\hat{x} \in h - s$ of x. (Just pick a maximally regular point $y \in h$ close to z and extend the geodesic segment xy beyond y inside h up to length π.) It follows that the convex hull of x and h contains a singular sphere s' with $\dim(s') = \dim(K)$. Proposition 2.14 then implies that K is a subbuilding. $\square$

The proof shows that if K is not a subbuilding and if $h \subset K$ is a hemisphere with $\dim(h) = \dim(K)$ and center z, then $K \subseteq \overline{B}_{\frac{\pi}{2}}(z)$.

3 On the Center Conjecture

3.1 General properties of potential counterexamples

Let B be a spherical building and let $K \subseteq B$ be a convex subcomplex. We call K a *counterexample* to the Center Conjecture 1.3, if K is not a subbuilding and if the action $Stab_{Aut(B)}(K) \curvearrowright K$ has no fixed point.

It is easy to see that a one-dimensional convex subcomplex $G \subset B$ is either a subbuilding or a metric tree with intrinsic circumradius $\leq \frac{\pi}{2}$. In the latter case, $G \cap Cent(G)$ consists of precisely one point which then must be fixed by the action $Isom(G) \curvearrowright G$.

Hence a counterexample K must have dimension ≥ 2. According to [BL05], it must even have dimension ≥ 3. However, we will not use this fact in order to keep our arguments self-contained.

Our considerations in chapter 2 imply that K cannot contain a singular sphere of codimension one (in K), cf. Corollary 2.22. Furthermore, K can neither contain a $Stab_{Aut(B)}(K)$-invariant subset with circumradius $< \frac{\pi}{2}$ nor one with diameter $\leq \frac{\pi}{2}$, cf. Corollary 2.19.

3.2 The F_4-case

We now prove our first main result.

Theorem 3.1. *The Center Conjecture 1.3 holds for spherical buildings of type F_4.*

Proof. We will use the information on the geometry of the F_4-Coxeter complex collected in section 2.2.2.

Let B be a spherical building of type F_4 and let $K \subseteq B$ be a convex subcomplex which is a counterexample in the sense of section 3.1. Then K must have dimension 2 or 3.

We start by checking that also the action of the potentially smaller group of *inner* automorphisms of B preserving K has no fixed point on K.

Lemma 3.2. *The action $Stab_{Inn(B)}(K) \curvearrowright K$ has no fixed point.*

Proof. We assume that there exists an automorphism $\alpha \in Aut(B) - Inn(B)$ preserving K, because otherwise there is nothing to prove. In view of the natural embedding $Out(B) \hookrightarrow Isom(\Delta^{F_4}_{mod}) \cong \mathbb{Z}_2$, α induces on $\Delta^{F_4}_{mod}$ the nontrivial isometric involution. Hence it switches the vertex types $1 \leftrightarrow 4$ and $2 \leftrightarrow 3$. Moreover, $Stab_{Aut(B)}(K)$ is generated by α and the index two normal subgroup $Stab_{Inn(B)}(K)$.

The fixed point set of any inner automorphism of B is a convex subcomplex, therefore also the $Stab_{Aut(B)}(K)$-invariant subset $F := K \cap Fix(Stab_{Inn(B)}(K))$. Note that α acts on F as an isometric involution without fixed point (since K is a counterexample). Hence α must map any point $x \in F$ to an antipode, because otherwise x and $\alpha(x)$ would have a unique midpoint which would be a fixed point of α in F. On the other hand, for any vertex $v \in F$, v and $\alpha(v)$ have different θ_B-types and therefore cannot be antipodal. This shows that $F = \emptyset$, as claimed. ☐

Proof of Theorem 3.1 continued. Due to the symmetry of the Dynkin diagram, the roles of i- and $(5-i)$-vertices are equivalent (dual). Our strategy will be to investigate the pattern of 1- and 4-vertices in K, because vertices of these types are better separated from each other than 2- and 3-vertices. We recall that the possible mutual distances between 1-vertices (4-vertices) in B are $0, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}$ and π.

Case 1: All vertices of types 1 and 4 in K have antipodes in K. Let $q \in K$ be a 2-vertex. Since $dim(K) \geq 2$, there is a vertex $p \in K$ of type 1 or 4 adjacent to q. Let $\hat{p} \in K$ be an antipode of p. The geodesic segment $pq\hat{p}$ of length π contains an interior 1-vertex p' (between q and $\hat{p}$). Since also p' has an antipode in K, it follows from Lemma 2.16 that q has an antipode in K. Hence all 2-vertices in K have antipodes in K, and analogously all 3-vertices in K do. This contradicts Proposition 2.15.

Case 2: K contains vertices of type 1 or 4 without antipodes in K. We may assume without loss of generality that K contains 1-vertices without antipodes in K.

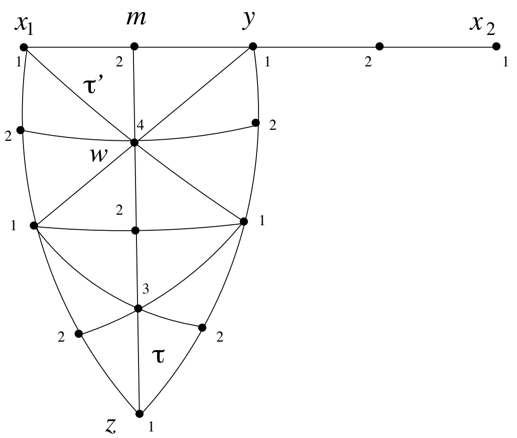
If P is any property defined for i-vertices in K and invariant under $Stab_{Inn(B)}(K)$, we call vertices with this property *iP-vertices*. If $1P$-vertices exist, then for any $1P$-vertex there is another $1P$-vertex at distance $\frac{2\pi}{3}$ or π. Namely, let $1P'$ be property $1P$ with the additional requirement that all other $1P$-vertices have distance $\leq \frac{\pi}{2}$. The set of $1P'$-vertices has diameter $\leq \frac{\pi}{2}$ and must therefore be empty in view of Corollary 2.19 and Lemma 3.2.

Let A be the property of *not having antipodes in* K. If P is a property implying A, $P \Rightarrow A$, then for any $1P$-vertex there exists another $1P$-vertex at distance *exactly* $\frac{2\pi}{3}$.

Let I be the property of *being an interior point of* K, compare the definition at the end of section 2.3.2. According to Proposition 2.14, a point $x \in K$ is an interior point of K, if and only if $\Sigma_x K$ contains a (singular) sphere of top dimension $\dim(K) - 1$. Clearly $I \Rightarrow A$, because K contains no top-dimensional (in K) singular spheres.

Let x_1 and x_2 be $1I$-vertices with distance $\frac{2\pi}{3}$. The segment x_1x_2 is of type 12121 and can be extended inside K beyond both endpoints to a segment y_1y_2 of length π and type 2121212. Note that the links of 2-vertices in B have the type $A_1 \circ A_2$ Dynkin diagram $\overset{1}{\bullet}\quad \overset{3}{\bullet}\!-\!\overset{4}{\bullet}$. Since x_i is an interior point of K, the link $\Sigma_{y_i} K$ contains a top-dimensional (in $\Sigma_{y_i} K$) hemisphere centered at $\overrightarrow{y_i x_i}$. Consequently, K contains a top-dimensional (in K) hemisphere (with y_1 and y_2 in its boundary). This is impossible and we see that K cannot contain $1I$-vertices. Dually, K cannot contain $4I$-vertices.

Below, we will consider yet another property. Note that a singular 1-sphere of type $\ldots 4343 \ldots$ in the B_3-Coxeter complex with Dynkin diagram $\overset{2}{\bullet}\!=\!\overset{3}{\bullet}\!-\!\overset{4}{\bullet}$ divides it into two hemispheres centered at 4-vertices. We say that a 1-vertex $x \in K$ has property H, if $\Sigma_x K$ *contains a 2-dimensional hemisphere centered at a 4-vertex*. Let $u \in K$ denote the 4-vertex adjacent to x such that $\overrightarrow{xu}$ is the center of this hemisphere. We note that $\Sigma_x K \subseteq \overline{B}_{\frac{\pi}{2}}(\overrightarrow{xu})$ by (the proof of) Corollary 2.22, because K contains no $1I$-vertices. One gives a dual definition of property H for 4-vertices of K. Clearly, $H \Rightarrow A$ for 1- and 4-vertices because K contains no 3-dimensional hemisphere.



Let us return to an arbitrary property P implying A. Let x_1 and x_2 be a pair of $1P$-vertices with distance $\frac{2\pi}{3}$. The midpoint y of x_1x_2 is a $1A$-vertex. There exists another 1-vertex $z \in K$ with $d(y, z) = \frac{2\pi}{3}$. Both angles $\angle_y(z, x_i)$ are $< \pi$, because $d(z, x_i) < \pi$. At least one of them, say $\angle_y(z, x_1)$, must be $\geq \frac{\pi}{2}$ and the arc in $\Sigma_y K$ connecting $\overrightarrow{yz}$ with $\overrightarrow{yx_1}$ is then of type 242. We observe that the convex hull $CH(x_1, y, z)$ is an isosceles spherical triangle (meaning the two-dimensional object). Indeed, let m denote the midpoint of the segment yx_1.

The convex hull of ym and z – which is contained in an apartment, as is the convex hull of any two faces – is a right-angled spherical triangle with the combinatorial structure as depicted in the figure. Let τ denote the type 123 face contained in it with vertex z. We denote the 4-vertex on mz adjacent to m by w. Then $\angle_m(w, x_1) = \frac{\pi}{2}$, and the arc connecting $\overrightarrow{mw}$ and $\overrightarrow{mx_1}$ in $\Sigma_m B$ consists only of a single 1-simplex of type 41. Accordingly, $\tau' := CH(m, w, x_1)$ is a face of type 124. Furthermore, $CH(\tau \cup \tau') = CH(x_1, y, z)$ is a spherical triangle with the combinatorial structure as shown in the figure. (That the geodesic triangle in K with vertices x_1, y and z is rigid, follows more directly from triangle comparison, because a comparison triangle in the unit sphere S^2 with vertices $\hat{x}_1, \hat{y}$ and $\hat{z}$ and with the same side lengths has angle $\angle_{\hat{y}}(\hat{x}_1, \hat{z}) = \angle_y(x_1, z)$.) We note that $\Sigma_w K$ contains a singular 1-sphere κ of type $\ldots 1212 \ldots$. Since K contains no $4I$-vertices, this implies in particular that $dim(K) = 3$. Then κ bounds a 2-dimensional hemisphere in $\Sigma_w K$ and w is a $4H$-vertex. In particular, we see that K contains $4H$-vertices if it contains $1P$-vertices.

Choosing $P := A$, we infer that K contains $4H$-vertices. The existence of 4-vertices in K implies, dually, that K also contains $1H$-vertices.

Now we choose $P := H$. It follows that there exists a configuration of $1H$-vertices $x_1, x_2 \in K$ and a 1-vertex $z \in K$ as considered above. Let $u \in K$ be a 4-vertex adjacent to x_1 such that $\Sigma_{x_1} K$ contains a two-dimensional hemisphere centered at $\overrightarrow{x_1 u}$. As noted before, $\Sigma_{x_1} K \subseteq \overline{B}_{\frac{\pi}{2}}(\overrightarrow{x_1 u})$. The spherical building $\Sigma_{x_1} B$ has Dynkin diagram $\overset{2}{\bullet}\!=\!\overset{3}{\bullet}\!-\!\overset{4}{\bullet}$ and, due to the geometry of the B_3-Coxeter complex, every 2-vertex in $\overline{B}_{\frac{\pi}{2}}(\overrightarrow{x_1 u})$ is adjacent to $\overrightarrow{x_1 u}$. Since $\angle_{x_1}(y, z) = \angle_{x_1}(y, u) + \angle_{x_1}(u, z)$, the direction $\overrightarrow{x_1 u}$ must bisect $\angle_{x_1}(y, z)$ and thus $u = w$. It follows that the segment $x_1 w = x_1 u$ can be extended beyond $w = u$ inside K. Since $\overrightarrow{x_1 u}$ is an interior vertex of $\Sigma_{x_1} K$, this in turn implies that w is a $4I$-vertex, a contradiction. $\square$

3.3 The E_6-case

We will use the information in section 2.2.3 regarding the geometry of the E_6-Coxeter complex.

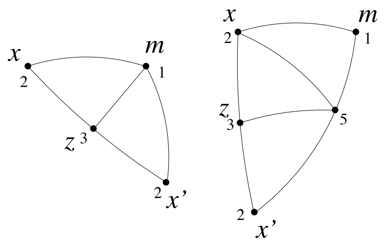
Let B be a spherical building of type E_6 and let $K \subset B$ be a convex subcomplex. We denote $G := Stab_{Aut(B)}(K)$ and $H := Stab_{Inn(B)}(K)$. Then H is a normal subgroup of G and, in view of $G/H \hookrightarrow Isom(\Delta^{E_6}_{mod}) \cong \mathbb{Z}_2$, it has index ≤ 2. The automorphisms in $G - H$ (if any) preserve the vertex types 1 and 4, and switch the types $2 \leftrightarrow 6$ and $3 \leftrightarrow 5$. We assume that K is a counterexample to the Center Conjecture in the sense of section 3.1, i.e. K is no subbuilding of B and the action $G \curvearrowright K$ has no fixed point.

If P is any property defined for i-vertices in K and invariant under H, we call vertices with this property *iP-vertices*. Let again A denote the property of *not having antipodes in K*.

Lemma 3.3. *Let P be an H-invariant property defined for 2- and 6-vertices in K and implying A, $P \Rightarrow A$. Then for any $2P$-vertex ($6P$-vertex) $x \in K$ exists another $2P$-vertex ($6P$-vertex) $x' \in Hx$ with $d(x, x') = \frac{2\pi}{3}$.*

Proof. Since the roles of 2- and 6-vertices in E_6-geometry are dual, it suffices to treat the case of $2P$-vertices.

Let us assume the contrary. Then the orbit Hx consists of $2P$-vertices with pairwise distances $\arccos \frac{1}{4}$, any two of which are connected by a type 232 singular segment. In particular, $\mathrm{diam}(Hx) < \frac{\pi}{2}$ and H has a fixed point in K, cf. section 2.1.2. Hence $G \supsetneq H$ and there exists $\alpha \in G - H$. The orbit $H\alpha x$ consists of $6P$-vertices. Since $P \Rightarrow A$, none of them is antipodal to x. On the other hand, they cannot all be adjacent to x, because then $\mathrm{diam}(Gx) \leq \arccos \frac{1}{4} < \frac{\pi}{2}$ and G would fix a point in K. Thus there exists $y \in H\alpha x$ such that xy is a type 216 singular segment of length $d(x,y) = \arccos(-\frac{1}{4})$.



We claim that for the 1-vertex m on xy holds $\mathrm{rad}(Gx, m) \leq \frac{\pi}{2}$. By duality, it suffices to check that $d(m, x') \leq \frac{\pi}{2}$ for all $x \neq x' \in Hx$. We first observe that $\angle_x(m, x') < \pi$ because $d(x', y) < \pi$. The building $\Sigma_x B$ is of type D_5 with Dynkin diagram . Hence the 1-vertex $\overrightarrow{xm}$ and the 3-vertex $\overrightarrow{xx'}$ are either adjacent or connected by a type 153 segment in $\Sigma_x K$. In the first case, m is also adjacent to x' and hence $d(m, x') < \frac{\pi}{2}$. (If z denotes the midpoint of xx', a 3-vertex, then the building $\Sigma_z B$ has type $A_1 \circ A_4$ with Dynkin diagram and splits as a spherical join.) In the other case, the side mx' of the spherical triangle $CH(m, x, x')$ is a singular segment of type 152 and length $\frac{\pi}{2}$. (The link of B of a 5-vertex splits off a type A_4-factor with Dynkin diagram , and a type 2321 segment therein has length π.) This shows that $\mathrm{rad}(Gx, m) \leq \frac{\pi}{2}$.

If $\mathrm{rad}(Gx) < \frac{\pi}{2}$ then G fixes a point in K, which is a contradiction. Hence $\mathrm{rad}(Gx) = \frac{\pi}{2}$ and $m \in Cent(Gx) \cap CH(Gx)$. Since 1-vertices have root type, the convex ball $\overline{B}_{\frac{\pi}{2}}(m)$ is a subcomplex and contains the *simplicial* convex hull $SCH(Gx)$ of Gx. Therefore also $\mathrm{rad}(SCH(Gx)) = \frac{\pi}{2}$ and $m \in Cent(SCH(Gx)) \cap SCH(Gx)$. Applying Corollary 2.20 to $SCH(Gx)$ yields that the action $G \curvearrowright K$ has a fixed point, contradiction. $\square$

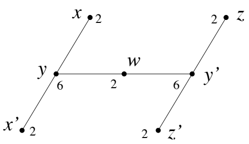
The midpoint of a pair of $2A$-vertices ($6A$-vertices) $x, x' \in K$ with distance $\frac{2\pi}{3}$ is a $6A$-vertex ($2A$-vertex), cf. Lemma 2.16. We define the properties M_i for 2- and 6-vertices in K inductively as follows. Let $M_0 := A$. We say that a 6-vertex (2-vertex) in K has property M_i, $i \geq 1$, if it is the *midpoint of a pair of* $2M_{i-1}$*-vertices (*$6M_{i-1}$*-vertices) in* K *with distance* $\frac{2\pi}{3}$. There are the implications $A = M_0 \Leftarrow \dots \Leftarrow M_i \Leftarrow M_{i+1} \Leftarrow \dots$. We conclude from Lemma 3.3:

Corollary 3.4. *If K contains a $2A$- or a $6A$-vertex, then K contains $2M_i$- and $6M_i$-vertices for all $i \geq 0$.*

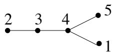
Our strategy will be to investigate the links of M_i-vertices for increasing i and look for larger and larger spheres until we find apartments. We begin with the M_2-vertices.

Lemma 3.5. *The link $\Sigma_w K$ of a $2M_2$-vertex $w \in K$ contains a type* 656565656 *singular 1-sphere.*

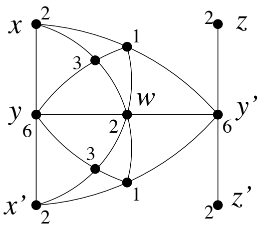
Proof. Consider the following configuration.



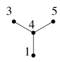
Let the $2M_2$-vertex $w \in K$ be the midpoint of a pair of $6M_1$-vertices $y, y' \in K$, and suppose that y (y') is the midpoint of a pair of $2A$-vertices $x, x' \in K$ ($z, z' \in K$).

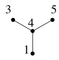
Since $d(x, y') < \pi$, we have $\angle_y(x, w) < \pi$. The building $\Sigma_y B$ is of type D_5 with Dynkin diagram , and any two distinct non-antipodal 2-vertices in it have distance $\frac{\pi}{2}$. Hence $\angle_y(w,x) \leq \frac{\pi}{2}$ and $\angle_y(w,x') \leq \frac{\pi}{2}$. Since $\angle_y(x,x') = \pi$, we have equality $\angle_y(w,x) = \angle_y(w,x') = \frac{\pi}{2}$ and $\overrightarrow{yw}$ is the midpoint of a type 23232 geodesic segment in $\Sigma_y K$ connecting $\overrightarrow{yx}$ and $\overrightarrow{yx'}$.

Because of $d(w,x) < \frac{2\pi}{3}$, the segment wx has type 232, and analogously the segments wx', wz and wz'.



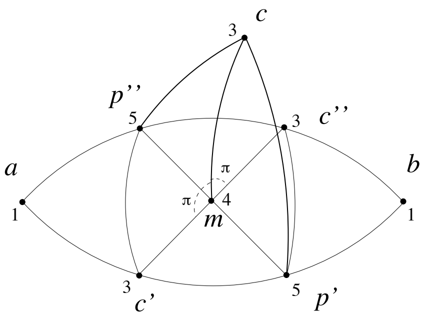
The 3-vertices $\overrightarrow{wx}, \overrightarrow{wx'}$ and the antipodal 6-vertices $\overrightarrow{wy}, \overrightarrow{wy'}$ lie on a type 6316136 singular 1-sphere in $\Sigma_w K$. We see that the link $\Sigma_{yw} K$ of the edge yw contains a pair of antipodal 3-vertices, while $\Sigma_{y'w} K$ contains a pair of antipodal 1-vertices. Exchanging the roles of y and y', we obtain also antipodal 1-vertices in $\Sigma_{yw} K$ and antipodal 3-vertices in $\Sigma_{y'w} K$.

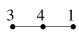
We wish to produce in, say, $\Sigma_{yw} K$ from the pairs of antipodal 1- and 3-vertices a pair of antipodal 5-vertices. Note that the spherical buildings $\Sigma_{yw} B$ and $\Sigma_{y'w} B$ have type D_4 and Dynkin diagram .

Sublemma 3.6. *Let L be a convex subcomplex of a spherical building B' of type D_4 with Dynkin diagram* . *Suppose that L contains a pair of antipodal 1-vertices and a pair of antipodal 3-vertices. Then it contains a singular 1-sphere of type* 1351351.

Proof. We denote the two antipodal 1-vertices by a and b, and the two antipodal 3-vertices by c and d. If c (or d) lies on a minimizing segment γ of type 1351 connecting a and b, then a singular 1-sphere of the desired type is obtained as the convex hull of d (c) and a neighborhood of c (d) in γ.

Suppose now that $d(a,c) + d(c,b) > \pi$. We first describe the convex hull of a, b and c. The segments ac and bc are of type 153. We denote the 5-vertices on them by p'', respectively, p'. Let us consider the segments $ap''c''b$ and $ac'p'b$ of length π. The 5-vertices p'' and p' cannot be antipodal, because they are adjacent to c. So the 5-vertex $\overrightarrow{ap''}$ and the 3-vertex $\overrightarrow{ac'}$ in $\Sigma_a L$ are not antipodal and hence adjacent.



The convex hull of $ap''c''b$ and $ac'p'b$ is then a spherical bigon β. Note that $\Sigma_{p''} B'$ is a building of type A_3 with Dynkin diagram . It contains a segment of length π and type 1343 from $\overrightarrow{p''a}$ to $\overrightarrow{p''c''}$ through $\overrightarrow{p''c'}$. Its 4-vertex corresponds to a 4-vertex m adjacent to p'', c' and c''. Replacing p'' by c'', we see in the same way that m is also adjacent to p'. Hence the bigon β is centered at m. Since c is adjacent to p' and p'', it is also adjacent to their midpoint m. The 3-vertices $\overrightarrow{mc}$, $\overrightarrow{mc'}$ and $\overrightarrow{mc''}$ are pairwise antipodal in $\Sigma_m L$. (Note that $\Sigma_m B'$ is a building of type $A_1 \circ A_1 \circ A_1$.)

The direction $\overrightarrow{md}$ is antipodal to $\overrightarrow{mc}$ and hence antipodal to at least one of the directions $\overrightarrow{mc'}$ and $\overrightarrow{mc''}$. It follows that d is antipodal to c' or c'' and, as in the beginning of the proof, that L contains the desired singular 1-sphere. ☐

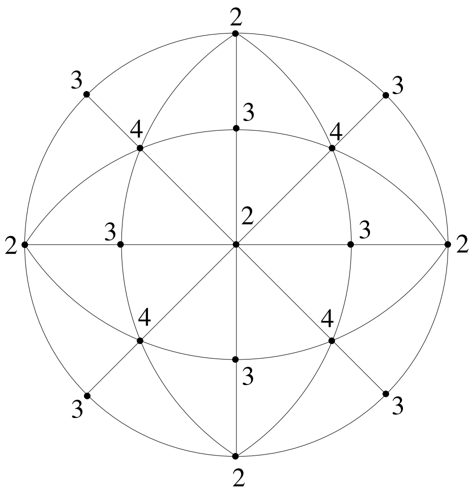
Remark 3.7. The proof shows that we can choose the circle in L to contain the two antipodal 1-vertices or the two antipodal 3-vertices. (If L contains a type 1351351 circle through one of the two antipodal 1-vertices, then it contains another such circle through both of them.)

End of proof of Lemma 3.5. Applying Sublemma 3.6 to $L = \Sigma_{yw} K \cong \Sigma_{\overrightarrow{wy}} \Sigma_w K$, we find a type 1351351 singular 1-sphere. We will only use the pair of antipodal 5-vertices on it. These can be regarded in $\Sigma_w K$ as the directions $\overrightarrow{\eta\xi}$ and $\overrightarrow{\eta\xi'}$ to 5-vertices ξ and ξ' adjacent to $\eta := \overrightarrow{wy}$. Then $CH(\eta, \xi, \xi', \overrightarrow{wy'})$ is the desired type 656565656 singular 1-sphere in $\Sigma_w K$. $\square$

Remark 3.8. The 1-sphere in $\Sigma_{yw} K$ provided by Sublemma 3.6 yields in fact a singular 2-sphere in $\Sigma_w K$ which contains the type 656565656 singular 1-sphere.

Now we turn our attention to the M_3-vertices.

Lemma 3.9. *The link $\Sigma_v K$ of a $6M_3$-vertex $v \in K$ contains a singular 2-sphere isomorphic to the B_3-Coxeter complex with Dynkin diagram* 4 3 2.



Proof. Let $v \in K$ be a $6M_3$-vertex which is the midpoint of the pair of $2M_2$-vertices $w, w' \in K$. Let $\gamma \subset \Sigma_w K$ be a singular 1-sphere as provided by Lemma 3.5. The building $\Sigma_w B$ is of type D_5 with Dynkin diagram. The 6-vertices on γ are not antipodal to $\overrightarrow{wv}$, because w' has no antipodes in K. Hence they have distance $\leq \frac{\pi}{2}$ from $\overrightarrow{wv}$. Since they come in (two) pairs of antipodes, equality holds, i.e. the 6-vertices on γ have distance $= \frac{\pi}{2}$ from $\overrightarrow{wv}$, compare the beginning of the proof of Lemma 3.5. It follows that $d(\overrightarrow{wv}, \cdot) \equiv \frac{\pi}{2}$ on γ, and $CH(\gamma \cup \{\overrightarrow{wv}\}) =: h$ is a 2-dimensional hemisphere with center $\overrightarrow{wv}$ and boundary circle γ. It is a simplicial subcomplex of $\Sigma_w K$ composed of twenty four 654-triangles and isomorphic to a hemisphere in the type B_3 Coxeter complex with Dynkin diagram 4 5 6 as described in section 2.2.3.

$\Sigma_{\overrightarrow{wv}} h =: \delta$ is a type 545454545 singular 1-sphere in $\Sigma_{\overrightarrow{wv}} \Sigma_w K \cong \Sigma_{wv} K \cong \Sigma_{\overrightarrow{vw}} \Sigma_v K$. (The building $\Sigma_{wv} B$ has type D_4 with Dynkin diagram.) When regarded as a circle in the latter space, δ is the link at $\overrightarrow{vw}$ of a singular 2-sphere in $\Sigma_v K$ with poles $\overrightarrow{vw}$ and $\overrightarrow{vw'}$. To determine the link δ' of the 2-sphere at the opposite pole $\overrightarrow{vw'}$, we recall that in the D_5-Coxeter complex with Dynkin diagram the possible types for singular segments connecting 2-antipodes are 24342, 2512 and 23232. Hence to a 4-vertex (5-vertex) on δ corresponds a 4-vertex (1-vertex) on δ', i.e. δ' has type 141414141. (This can also be expressed by saying that the natural isomorphism $\Sigma_{wv} K \cong \Sigma_{\overrightarrow{vw}} \Sigma_v K \cong \Sigma_{\overrightarrow{vw'}} \Sigma_v K \cong \Sigma_{w'v} K$ of type D_4 spherical buildings with Dynkin diagram switches the vertex types $1 \leftrightarrow 5$.) We regard δ' as a singular 1-sphere in $\Sigma_{w'v} K \cong \Sigma_{\overrightarrow{vw'}} \Sigma_v K$.

By exchanging the roles of w and w', we obtain likewise a type 141414141 singular 1-sphere in $\Sigma_{wv}K$, besides the type 545454545 singular 1-sphere which we found before. From these, we wish to produce a type 343434343 singular 1-sphere. This leads us to the following continuation of Sublemma 3.6.

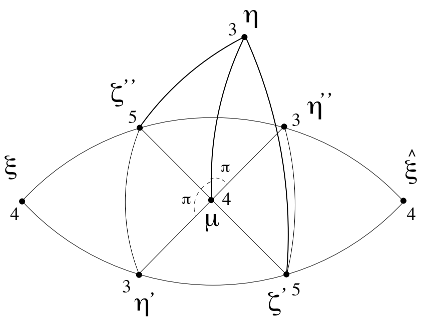
Sublemma 3.10. *Let L be a convex subcomplex of a spherical building B' of type D_4 with Dynkin diagram* 3 4 5 1 *Suppose that L contains a type* 141414141 *singular 1-sphere and a pair of antipodal 5-vertices. Then it contains a singular 2-sphere.*

We recall that in D_4-geometry there is only one type of singular 2-spheres, see section 2.2.3, and hence a singular 2-sphere contains singular 1-spheres of all possible types.

Proof. We denote the type 141414141 singular 1-sphere by ϵ. Let $g, \hat{g} \in \epsilon$ be two antipodal 1-vertices. By Sublemma 3.6 and Remark 3.7 there is a singular 1-sphere ϵ' of type 1351351 containing $g, \hat{g}$. (The roles of 1-,3- and 5-vertices in D_4-geometry are equivalent, as reflected by the symmetries of the Dynkin diagram.)

Since g has an antipode in L, it suffices to find a singular 1-sphere in $\Sigma_g L$. The spherical building $\Sigma_g B'$ has type A_3 and Dynkin diagram 3 4 5. We know already that $\Sigma_g L$ contains the pair of antipodal 4-vertices $\{\xi, \hat{\xi}\} = \Sigma_g \epsilon$ and the pair of antipodes $\{\eta, \hat{\eta}\} = \Sigma_g \epsilon'$ consisting of a 3-vertex η and a 5-vertex $\hat{\eta}$.

To find the singular 1-sphere in $\Sigma_g L$, we proceed as in the proof of Sublemma 3.6. If η or $\hat{\eta}$ lies on a minimizing segment connecting ξ and $\hat{\xi}$, then we are done. Otherwise, let us consider the convex hull of $\xi, \hat{\xi}$ and, say, η. The segments $\xi\eta$ and $\hat{\xi}\eta$ are of type 453. We call the 5-vertices on them ζ'', respectively, ζ'. They are distinct, and we denote by μ the midpoint of the type 545 segment $\zeta''\zeta'$. The segments $\xi\zeta''\hat{\xi}$ and $\xi\zeta'\hat{\xi}$ have types 4534 and 4354, and the spherical bigon bounded by them is right angled. (Note that the spherical building $\Sigma_\xi \Sigma_g B'$ has type $A_1 \circ A_1$ with Dynkin diagram 3 5. It is a complete bipartite graph with edge lengths $\frac{\pi}{2}$.) The 4-vertex μ is the center of the bigon.



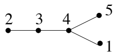
Since η is adjacent to ζ'' and ζ', it is also adjacent to their midpoint μ. The segment $\eta\mu\hat{\eta}$ has type 3435. Let η'' and η' denote the 3-vertices on the type 435 segments $\hat{\xi}\zeta''$, respectively, $\xi\zeta'$. The 3-vertices $\overrightarrow{\mu\eta''}$ and $\overrightarrow{\mu\eta'}$ in $\Sigma_\mu \Sigma_g L$ are antipodal. The 3-vertex $\overrightarrow{\mu\hat{\eta}}$ is antipodal to at least one of them, say to $\overrightarrow{\mu\eta'}$. It follows that η' is an antipode of $\hat{\eta}$ and $CH(\hat{\eta}, \xi, \eta', \zeta')$ is the singular 1-sphere in $\Sigma_g L$ which we are looking for. $\square$

End of proof of Lemma 3.9. Applying Sublemma 3.10 to $L = \Sigma_{vw}K \cong \Sigma_{\overrightarrow{vw}} \Sigma_v K$, we find a singular 2-sphere. We will only use one of the type 343434343 singular 1-spheres contained in it. This circle can be regarded as the link at $\overrightarrow{vw}$ of a singular 2-sphere in $\Sigma_v K$ with poles the 2-vertices $\overrightarrow{vw}$ and $\overrightarrow{vw'}$. It is a 2-sphere of the desired type. $\square$

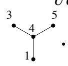
Remark 3.11. The 2-sphere in $\Sigma_{vw}K$ provided by Sublemma 3.10 yields in fact a singular 3-sphere in $\Sigma_v K$ which contains this singular 2-sphere.

Finally, we look at M_4-vertices.

Lemma 3.12. *The link $\Sigma_u K$ of a $2M_4$-vertex $u \in K$ contains an apartment and is therefore a top-dimensional subbuilding of $\Sigma_u B$.*

Proof. Let $u \in K$ be a $2M_4$-vertex which is the midpoint of the pair of $6M_3$-vertices $v, v' \in K$. Let $s \subset \Sigma_v K$ be a singular 2-sphere as provided by Lemma 3.9. The building $\Sigma_v B$ is of type D_5 with Dynkin diagram 

. The 2-vertices on s are not antipodal to $\overrightarrow{vu}$, because v' has no antipodes in K. Hence they have distance $\leq \frac{\pi}{2}$ from $\overrightarrow{vu}$. Since they come in pairs of antipodes, equality holds, i.e. the 2-vertices on s have distance $= \frac{\pi}{2}$ from $\overrightarrow{vu}$, compare the beginning of the proofs of Lemmas 3.5 and 3.9. It follows that $d(\overrightarrow{vu}, \cdot) \equiv \frac{\pi}{2}$ on s, and $CH(s \cup \{\overrightarrow{vu}\}) =: h'$ is a 3-dimensional hemisphere with center $\overrightarrow{vu}$ and boundary 2-sphere s.

However, unlike the circle and 2-hemisphere found before, $h' \subset \Sigma_v K$ is *not* a simplicial subcomplex. This can be seen, for instance, from the fact that it contains quadruples of 2-vertices with pairwise distances $\frac{\pi}{2}$, cf. the discussion of Σ_2 in section 2.2.3. Accordingly, the 2-sphere $\Sigma_{\overrightarrow{vu}} h' \subset \Sigma_{\overrightarrow{vu}} \Sigma_v K \cong \Sigma_{uv} K$ is not a subcomplex of the type D_4 building $\Sigma_{uv} B$ with Dynkin diagram 

. (It contains triples of 3-vertices with pairwise distances $\frac{\pi}{2}$.) The simplicial convex hull of $\Sigma_{\overrightarrow{vu}} h'$ is an apartment contained in $\Sigma_{uv} K$. Since u is an interior point of the edge vuv' contained in K, it follows that there are apartments also in $\Sigma_u K$. $\square$

Let again I denote the property of *being an interior point of* K. Clearly, $I \Rightarrow A$ because by assumption K is no subbuilding. Lemma 3.12 says that $M_4 \Rightarrow I$ for 2- and 6-vertices.

Lemma 3.13. *There are no $2I$- and $6I$-vertices in K.*

Proof. Otherwise, we suppose without loss of generality that K contains $2I$-vertices. By Lemma 3.3, there exist two $2I$-vertices in K with distance $\frac{2\pi}{3}$. Then among the 6-vertices in K adjacent to one of them are antipodes of the other, a contradiction. $\square$

Corollary 3.14. *All 2- and 6-vertices in K have antipodes in K.*

Proof. Otherwise, Corollary 3.4 and Lemma 3.12 imply that K contains $2I$-vertices, which contradicts Lemma 3.13. $\square$

Now the main work is done and we enter the endgame of our argument.

Lemma 3.15. *All 1-vertices in K have antipodes in K.*

Proof. Suppose that K contains $1A$-vertices. We argue as in case 2 in the proof of Theorem 3.1. The set of all $1A$-vertices in K, which have distance $\leq \frac{\pi}{2}$ from any other $1A$-vertex in K, must be empty in view of Corollary 2.19, because it is G-invariant and has diameter $\leq \frac{\pi}{2}$. Hence, there exists a pair of $1A$-vertices $x, x' \in K$ with distance $> \frac{\pi}{2}$, i.e. with distance $\frac{2\pi}{3}$. The midpoint y is another $1A$-vertex, and there exists yet another $1A$-vertex $z \in K$ with $d(y, z) = \frac{2\pi}{3}$.

Since z is no antipode of x or x', we have $0 < \angle_y(x, z), \angle_y(x', z) < \pi$. At least one of these angles is $\geq \frac{\pi}{2}$, say, the first one. The building $\Sigma_y B$ has type A_5 with Dynkin diagram

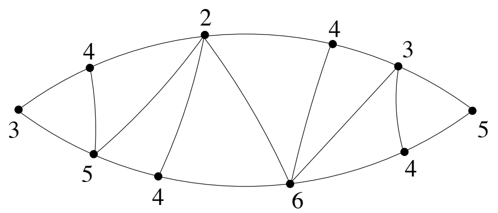
. The 4-vertices $\overrightarrow{yx}, \overrightarrow{yz} \in \Sigma_y K$ must have distance $\arccos(-\frac{1}{3})$ and their simplicial convex hull is a rhombus whose other diagonal is a 26-edge, compare the discussion of Σ_1 in section 2.2.3. In particular, there exists a 2-vertex $w \in K$ adjacent to y. By Corollary 3.14, it has an antipode

$\hat{w} \in K$. The segment $wy\hat{w}$ is of type 21656. Since the 6-vertex on it adjacent to y has an antipode in K, too, it follows that y has an antipode in K, a contradiction. $\square$

Lemma 3.16. *All 3- and 5-vertices in K have antipodes in K.*

Proof. By duality, it is enough to treat the case of 3-vertices.

Suppose that $x \in K$ is a $3A$-vertex. If K contains a 1-vertex y adjacent to x, then it also contains an antipode $\hat{y}$ of y (Lemma 3.15). The segment $yx\hat{y}$ contains a 6-vertex z adjacent to x. It has an antipode $\hat{z}$ in K and, by Lemma 2.16, x has an antipode in K, a contradiction. The same reasoning shows that K cannot contain 6- or 2-vertices adjacent to x, i.e. it contains at most 4- and 5-vertices adjacent to x.



Hence, for any point $p \in K$ the direction $\overrightarrow{xp}$ lies on a 45-edge in $\Sigma_x B$. As a consequence, p is contained in a spherical bigon $\beta \subset B$ with x as one of its tips and with $\Sigma_x \beta$ a 45-edge. (Of course, $\beta \not\subset K$.) It has the combinatorial structure as shown in the figure. (This is easily verified by taking into account that the links of 4-vertices in B have type $A_2 \circ A_1 \circ A_2$ with Dynkin diagram $\overset{2}{\bullet}\!-\!\overset{3}{\bullet}\ \ \overset{1}{\bullet}\ \ \overset{5}{\bullet}\!-\!\overset{6}{\bullet}$.) If K contains another 3-vertex x', then xx' can only be a type 34243 singular segment. Since the 2-vertex on it has an antipode in K (Corollary 3.14), it follows that also x has an antipode in K, a contradiction.

Thus x is the only 3-vertex in K and must be fixed by H. Since $G \curvearrowright K$ does not have a fixed point, we have $G \supsetneq H$ and Gx consists of x and a $5A$-point. They cannot be antipodal and their unique midpoint is fixed by G, a contradiction. $\square$

Lemma 3.17. *All 4-vertices in K have antipodes in K.*

Proof. Suppose that $x \in K$ is a $4A$-vertex. Then all vertices in K adjacent to x have antipodes in K. If K contains vertices adjacent to x, then the same reasoning as in the beginning of the proof of Lemma 3.16 shows that x has an antipode, too, a contradiction. Hence $dim(K) = 0$, which is also a contradiction. $\square$

We proved that all vertices in K have antipodes in K. With Proposition 2.15 it follows that K is a subbuilding. This contradicts our assumption that K is a counterexample to the Center Conjecture, and we obtain our second main result:

Theorem 3.18. *The Center Conjecture 1.3 holds for spherical buildings of type E_6.*

3.4 The case of classical types

The Center Conjecture for the spherical buildings of classical types (A_n, B_n and D_n) was first proven by Mühlherr and Tits in [MT06] using combinatorial methods and the incidence geometries of the respective buildings. We present in this section a proof from the point of view of CAT(1) spaces and using methods of comparison geometry, compare [Ra09b, ch. 4.1].

L. Kramer informed us about a similar argument in the A_n-case showing that a convex subcomplex $K \subset B$ is a subbuilding or $Stab_{Inn(B)}(K)$ fixes a point in K [Kr09].

We will use the information in section 2.2.4 regarding the geometry of the Coxeter complexes.

Let K be a convex subcomplex of a spherical building B of classical type and suppose that it is not a subbuilding. By Proposition 2.15, there are vertices in K without antipodes in K. Let $t = \max\{\, i \mid \exists\, iA\text{-vertex in } K\}$ (As before, we call an i-vertex of K without antipodes in K an *iA-vertex*.)

Lemma 3.19. *Let B be of type A_n or B_n and let $x \in K$ be a tA-vertex. Then there is no vertex of type $> t$ in K adjacent to x*

Proof. Let $t' > t$ and suppose that there exists a t'-vertex $y \in K$ adjacent to x. The maximality of t implies that y has an antipode $\widehat{y} \in K$. Notice that $\Sigma_x B$ splits off a factor of type A_{n-t} and its Dynkin diagram has labels $t+1, \ldots, n$. This implies that the direction $\overrightarrow{x\widehat{y}}$ has type $t'' > t$. It follows that the segment $x\widehat{y} \subset K$ has a t''-vertex z in its interior and, by Lemma 2.16, z cannot have antipodes in K, contradicting the maximality of t. $\square$

3.4.1 The A_n-case

Theorem 3.20. *The Center Conjecture 1.3 holds for spherical buildings of type A_n.*

Proof. We assume that $n \geq 2$, because otherwise the assertion is trivial.

Let K be a convex subcomplex of a spherical building B of type A_n and suppose that it is not a subbuilding. Let $t_1 = \min\{\, i \mid \exists\, iA\text{-vertex in } K\}$ and $t_2 = \max\{\, i \mid \exists\, iA\text{-vertex in } K\}$. Let $x_i \in K$ be a t_iA-vertex. By Lemma 3.19, there is no vertex of type $> t_2$ in K adjacent to x_2 and, analogously, no vertex of type $< t_1$ in K adjacent to x_1.

If $t_1 = t_2$, we may choose $x_1 = x_2$. It follows that $dim(K) = 0$ and the Center Conjecture holds trivially in this case. We therefore assume in the following that $t_1 < t_2$.

Consider the segment x_1x_2 as embedded in the vector space realization of the Coxeter complex of type A_n described in section 2.2.4, such that $x_1 = v_{t_1}$ (we work with vectors representing vertices) and such that the initial part of x_1x_2 is contained in the fundamental Weyl chamber Δ. Then x_2 lies in the convex hull of Δ and the antipode of x_1 in the Coxeter complex, a bigon which is given by all inequalities (2.5) except (t_1). The coordinates of x_2 are a permutation of the coordinates of v_{t_2}. It follows from the observation above, that the face of Δ spanned by the initial part of x_1x_2 contains no vertices of types $1, \ldots, t_1 - 1$. This implies for $x_2 = (a_0, \ldots, a_n) \in \mathbb{R}^{n+1}$ that $a_0 = \cdots = a_{t_1-1}$ and $a_{t_1} \leq \cdots \leq a_n$. Note that x_2 is adjacent to x_1 if and only if in addition $a_{t_1-1} \leq a_{t_1}$ holds. If x_2 is not adjacent to x_1, it follows that $a_0 = t_2$ and

$$x_2 = (\underbrace{t_2, \ldots, t_2}_{t_1}, \underbrace{-(n+1-t_2), \ldots, -(n+1-t_2)}_{t_2}, \underbrace{t_2, \ldots, t_2}_{n+1-t_1-t_2}).$$

In particular, $n + 1 - t_2 \geq t_1$. Since x_1 and x_2 are not antipodal, we even have the strict inequality

$$n + 1 > t_1 + t_2. \tag{3.21}$$

Consider now the embedding of x_1x_2 into the Coxeter complex such that $x_2 = v_{t_2}$ and the initial part of x_2x_1 is contained in Δ. The observation above implies now that the face of Δ spanned by the initial part of x_2x_1 contains no vertices of types $t_2 + 1, \ldots, n$. This implies for

$x_1 = (b_0, \dots, b_n) \in \mathbb{R}^{n+1}$ that $b_0 \leq \dots \leq b_{t_2-1}$ and $b_{t_2} = \dots = b_n$. If x_1 is not adjacent to x_2, equivalently, if $b_{t_2-1} > b_{t_2}$ it follows that

$$x_1 = (\underbrace{-(n+1-t_1), \dots, -(n+1-t_1)}_{t_1+t_2-(n+1)}, \underbrace{t_1, \dots, t_1}_{n+1-t_1}, \underbrace{-(n+1-t_1), \dots, -(n+1-t_1)}_{n+1-t_2})$$

and $t_1 \geq n+1-t_2$. But this inequality contradicts (3.21). Hence, x_1 and x_2 must be adjacent.

We saw that any t_1A-vertex is adjacent to any t_2A-vertex. Their distance is $\leq \operatorname{diam}(\Delta) =: \delta < \frac{\pi}{2}$. This implies that the set of t_iA-vertices has circumradius $\leq \delta$ and therefore a unique circumcenter c_i. Moreover, c_i is contained in the closed convex hull of the set of t_iA-vertices, in particular $c_i \in K$. It follows that c_i has distance $\leq \delta$ from every $t_{3-i}A$-vertex, and $d(c_1, c_2) \leq \delta$.

The Dynkin diagram for A_n has only one nontrivial symmetry which exchanges the labels $i \leftrightarrow (n+1-i)$. Hence a building automorphism in $Stab_{Aut(B)}(K) - Stab_{Inn(B)}(K)$ must switch the labels $t_1 \leftrightarrow t_2$ (according to their definition) and exchange $c_1 \leftrightarrow c_2$, whereas the automorphisms in $Stab_{Inn(B)}(K)$ fix both c_1 and c_2. It follows that the midpoint $m(c_1, c_2) \in K$ of c_1 and c_2 is fixed by the entire group $Stab_{Aut(B)}(K)$. ☐

3.4.2 The cases B_n and D_n

Theorem 3.22. *The Center Conjecture 1.3 holds for spherical buildings of type B_n.*

Proof. If $n = 2$, then $\dim(K) \leq 1$ and the Center Conjecture holds. So, let K be a convex subcomplex of a spherical building B of type B_n for $n \geq 3$ and suppose that it is not a subbuilding. Let $t = \max\{\, i \mid \exists\, iA\text{-vertex in } K\}$. Let $x \in K$ be a tA-vertex. By Lemma 3.19, there are no vertices of type $> t$ in K adjacent to x.

Let $x' \in K$ be another tA-vertex. Consider the segment xx' as embedded in the vector space realization of the Coxeter complex of type B_n described in section 2.2.4. Assume that

$$x = v_t = (0, \dots, 0, \underbrace{1, \dots, 1}_{n+1-t})$$

and that the initial part of xx' is contained in the fundamental Weyl chamber Δ. Then the coordinates of x' satisfy all inequalities (2.7) except (t). They agree up to permutation and signs with the coordinates of x. The observation above implies that the face of Δ spanned by the initial part of xx' contains no vertices of types $t+1, \dots, n$. For $x' = (a_1, \dots, a_n)$ this means that $a_t = \dots = a_n$ (besides $0 \leq a_1 \leq \dots \leq a_{t-1}$ which we will not use). If $a_t = 1$, then $x = x'$; if $a_t = 0$, then $d(x, x') = \frac{\pi}{2}$; and if $a_t = -1$, then x and x' are antipodal. The last case cannot occur, and hence $d(x, x') \leq \frac{\pi}{2}$. It follows that the $Stab_{Aut(B)}(K)$-invariant set of tA-vertices in K has diameter $\leq \frac{\pi}{2}$. Therefore, $Stab_{Aut(B)}(K)$ fixes a point in K by Corollary 2.19. ☐

Theorem 3.23. *The Center Conjecture 1.3 holds for spherical buildings of type D_n.*

Proof. For $n \geq 5$, the D_n-case of the Center Conjecture follows from the B_n-case, because a spherical building of type D_n can be regarded as a (thin) spherical building of type B_n. In the same vein, the D_4-case follows from the F_4-case, compare (2.10) and (2.11) in section 2.2.4. ☐

A direct proof of the $D_{n\geq 5}$-case can be given as well. We skip it here because it is very similar to the argument in the B_n-case. To keep our treatment of the classical types self-contained, we include a direct proof in the D_4-case.

Direct proof in the D_4-case. Let K be a convex subcomplex of a spherical building B of type D_4 and suppose that K is a counterexample to the Center Conjecture.

Suppose first, that K contains $3A$-vertices. Recall that the 3-vertices in D_4 are the vertices of root type. Their possible mutual distances are $0, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \pi$. The midpoint of a segment connecting two 3-vertices at distance $\frac{\pi}{3}$ lies in the interior of a simplex of type 124 adjacent to both 3-vertices.

Arguing as in the beginning of case 2 in the proof of Theorem 3.1, since K is a counterexample, there exist $3A$-vertices $x, x' \in K$ at distance $\frac{2\pi}{3}$. The simplicial convex hull of the segment xx' is 3-dimensional and the midpoint y_1 of xx' is an interior $3A$-vertex of K. Let $y_2 \in K$ be a $3I$-vertex at distance $\frac{2\pi}{3}$ to y_1. Since y_i is interior, we can find $z_i \in K$, with $d(z_i, y_i) = \frac{\pi}{6}$, such that $\overrightarrow{y_i z_i}$ is antipodal to $\overrightarrow{y_i y_{3-i}}$ in $\Sigma_{y_i} K$ for $i = 1, 2$. In particular, z_1 and z_2 are antipodal. Notice that z_i lies in the interior of a simplex of type 124. It follows that K contains a 2-sphere, contradicting Corollary 2.22. Hence all 3-vertices in K have antipodes in K.

Since K is a counterexample, there is a vertex $w \in K$ without antipodes in K, cf. Proposition 2.15. It has type $i \in \{1, 2, 4\}$. If there exists a 3-vertex $v \in K$ adjacent to w, then it has an antipode $\hat{v}$ in K. The interior of the segment $w\hat{v}$ contains another 3-vertex v'. By Lemma 2.16, v' is a $3A$-vertex, contradiction. Thus w cannot be adjacent to a 3-vertex in K. This implies that w is the only i-vertex in K, because any two distinct nonantipodal i-vertices are connected by a segment of type $i3i$.

It follows that the non-empty $Stab_{Aut(B)}(K)$-invariant set V of A-vertices in K consists of vertices of pairwise different types $\neq 3$. In particular, $|V| \leq 3$ and the possible distances between vertices in V are $\frac{\pi}{3}$ and $\frac{2\pi}{3}$. If $\mathrm{diam}(V) > \frac{\pi}{2}$, we may assume without loss of generality that V contains a $1A$-vertex u_1 and a $2A$-vertex u_2 with distance $\frac{2\pi}{3}$. The segment u_1u_2 then has type 142. Its midpoint is a $4A$-vertex u_4 and $V = \{u_1, u_2, u_4\}$. We conclude that always $\mathrm{rad}(V) < \frac{\pi}{2}$. Hence V has a unique circumcenter in K and it is fixed by $Stab_{Aut(B)}(K)$. $\square$

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