Convex functions on symmetric spaces, side lengths of polygons and the stability inequalities for weighted configurations at infinity

Michael Kapovich, Bernhard Leeb and John Millson

June 21, 2008

Abstract

In a symmetric space of noncompact type $X = G/K$ oriented geodesic segments correspond modulo isometries to vectors in the Euclidean Weyl chamber. We can hence assign vector valued lengths to segments. Our main result is a system of homogeneous linear inequalities, which we call the generalized triangle inequalities or stability inequalities, describing the restrictions on the vector valued side lengths of oriented polygons. It is based on the mod 2 Schubert calculus in the real Grassmannians G/P for maximal parabolic subgroups P.

The side lengths of polygons in Euclidean buildings are studied in the related paper [KLM2]. Applications of the geometric results in both papers to algebraic group theory are given in [KLM3].

Contents

1 Introduction

When studying asymptotic properties of the spectra of certain linear partial differential equations in mathematical physics Hermann Weyl was led in [We] to the question how the spectra of two compact self adjoint operators are related to the spectrum of their sum. The restrictions turn out to be homogeneous linear inequalities involving finite subsets of eigenvalues. It suffices to understand the question in the finite-dimensional case where it can be phrased as follows. Here $\alpha = (\alpha_1, \ldots, \alpha_m)$, $\beta = (\beta_1, \ldots, \beta_m)$ and $\gamma = (\gamma_1, \ldots, \gamma_m)$ denote m-tuples of real numbers arranged in decreasing order and with sum equal to zero.

Eigenvalues of a sum problem. *Give necessary and sufficient conditions on α, β and γ in order that there exist traceless Hermitian matrices $A, B, C \in i \cdot su(m)$ with spectra α, β, γ and satisfying*

$$A + B + C = 0.$$

There is a multiplicative version of this question. We recall that the *singular values* of a matrix A in $GL(m, \mathbb{C})$ are defined as the (positive) square roots of the eigenvalues of the matrix AA^*.

Singular values of a product problem. *Give necessary and sufficient conditions on α, β and γ in order that there exist matrices $A, B, C \in SL(m, \mathbb{C})$ the logarithms of whose singular values are α, β, γ and which satisfy*

$$ABC = 1.$$

We refer to [F2] for detailed information on these questions and their history.

Both questions have natural geometric interpretations and generalizations. Let us consider the group $G = SL(m, \mathbb{C})$, its maximal compact subgroup $K = SU(m)$ and the symmetric space $X = G/K$. Decompose the Lie algebra $\mathfrak{g} = sl(m, \mathbb{C})$ of G according to $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ where $\mathfrak{k} = su(m)$ is the Lie algebra of K and $\mathfrak{p} = i \cdot su(m)$ is the orthogonal complement of $\mathfrak{k}$ relative to the Killing form. $\mathfrak{p}$ is canonically identified with the tangent space $T_o X$ to X at the base point o fixed by K.

The singular values of a group element $g \in G$ form a vector $\sigma(g)$ in the Euclidean Weyl chamber Δ_{euc} and are a complete invariant of the double coset KgK. More geometrically, they can be interpreted as a vector valued distance: Given two points $x_1, x_2 \in X$, $x_i = g_i o$, the singular values $\sigma(g_1^{-1} g_2)$ are the complete invariant of the pair (x_1, x_2) modulo the G-action. We will call it the Δ-*length* of the oriented geodesic segment $\overline{x_1 x_2}$. The Singular values of a product problem thus asks about the possible Δ-side lengths of triangles in X.

In the same vein, the Eigenvalues of a sum problem is a problem for triangles in $\mathfrak{p}$ equipped with the geometry, in the sense of Felix Klein, having as automorphisms the group $Aff(\mathfrak{p})$ generated by the adjoint action of K and all translations. We call $(\mathfrak{p}, Aff(\mathfrak{p}))$ the *infinitesimal symmetric space* associated to X. In this geometry, pairs of points are equivalent if and only if their difference matrices have equal spectra. The spectra of the matrices can again be interpreted as Δ-lengths, and Weyl's question amounts to finding the sharp *triangle inequalities* in this geometry.

This paper is devoted to the

Problem. *Study for an arbitrary connected semisimple real Lie group G of noncompact type with finite center the spaces $\mathcal{P}_n(X)$, $\mathcal{P}_n(\mathfrak{p}) \subset \Delta_{euc}^n$, $n \geq 3$, of possible Δ-side lengths of oriented n-gons in the associated Riemannian symmetric space X and the corresponding infinitesimal symmetric space $\mathfrak{p}$.*

Our main result is an explicit description of $\mathcal{P}_n(X)$ and $\mathcal{P}_n(\mathfrak{p})$ in terms of a finite system of homogeneous linear inequalities parametrized by the Schubert calculus associated to G, see Theorem 1.3 below. In particular, both spaces are finite-sided polyhedral cones and we will refer to them as Δ-*side length polyhedra.* Our approach is based on differential-geometric techniques from the theory of nonpositively curved spaces.

To determine the side length polyhedra we relate oriented polygons in the symmetric space X, respectively, the infinitesimal symmetric space $\mathfrak{p}$ via a Gauss map type construction to weighted configurations on the spherical Tits building at infinity $\partial_{Tits}X$. The question on the possible Δ-side lengths of polygons then translates into a question about weighted configurations as it occurs in geometric invariant theory. Namely, after suitably generalizing the concept of Mumford stability, it turns out that the set of possible Δ-side lengths of polygons coincides with the set of possible Δ-weights (defined below) of semistable configurations (Theorems 5.3 and 5.9), that is, we have to determine the Δ-weights for which there exist semistable configurations.

Since the questions for polygons in X respectively $\mathfrak{p}$ translate into the same question for configurations, we obtain as a byproduct:

Theorem 1.1. $\mathcal{P}_n(X) = \mathcal{P}_n(\mathfrak{p})$.

This generalizes the Thompson Conjecture [Th] which was proven for $GL(m, \mathbb{C})$ in [Kly2] and more generally for complex semisimple groups in [AMW]. Another proof of the most general version of the Thompson conjecture has recently been given in [EL].

In our (logically independent) paper [KLM2] we investigate the Δ-side lengths of polygons in *Euclidean buildings.* The main result asserts that for a thick Euclidean building Y the Δ-side length space $\mathcal{P}_n(Y)$ depends only on the spherical Coxeter complex associated to Y. The proof exploits an analogous relation between polygons in Y and weighted configurations on the Tits boundary $\partial_{Tits}Y$ by ways of a Gauss map. Along the way we show that $\mathcal{P}_n(Y)$ coincides with the space of Δ-weights of semistable weighted configurations on the spherical building $\partial_{Tits}Y$ and, moreover, that the space of Δ-weights of semistable weighted configurations on a thick spherical building B depends only on the spherical Coxeter complex attached to B. Note that every spherical Tits building occurs as the Tits boundary of a Euclidean building, for instance, of the complete Euclidean cone over itself.

The results of [KLM2] imply for polygons in Riemannian symmetric spaces:

Theorem 1.2. *$\mathcal{P}_n(X)$ depends only on the spherical Coxeter complex attached to X.*

In [KLM3] we apply the results of this paper and [KLM2] to algebra. The generalized triangle inequalities give necessary conditions for solving a number of problems in algebraic group theory. For the case $G = GL(m)$ we give a new proof of the Saturation Conjecture first proved in [KT].

Most of the remaining part of the introduction will be devoted to describing the (semi)stability inequalities for Δ-weights of configurations on $\partial_\infty X$. As we said earlier, they coincide with the inequalities for the Δ-side lengths of polygons in X.

A symmetric space of noncompact type X is a complete simply connected Riemannian manifold with nonpositive sectional curvature and as such can be compactified to a closed ball by attaching an ideal boundary (sphere) $\partial_\infty X$. This construction generalizes the compactification of hyperbolic space given by the conformal Poincaré ball model. The natural G-action on X by isometries extends to a continuous action on $\partial_\infty X$. The G-orbits on $\partial_\infty X$ are parameterized by the spherical Weyl chamber, $\partial_\infty X/G \cong \Delta_{sph}$. They are homogeneous G-spaces of the form G/P with P a parabolic subgroup and we call them *generalized flag manifolds*.

A *weighted configuration* on $\partial_\infty X$ is a map $\psi : (\mathbb{Z}/n\mathbb{Z}, \nu) \to \partial_\infty X$ from a finite measure space. We think of the masses $m_i := \nu(i)$ placed in the points $\xi_i := \psi(i)$ at infinity. The Δ-*weights* $h = (h_1, \dots, h_n) \in \Delta^n_{euc}$ of the configuration contain the information on the masses and the G-orbits where they are located: To each orbit $G\xi_i$ corresponds the point $acc(\xi_i)$ in Δ_{sph} where $acc : \partial_\infty X \to \Delta_{sph}$ denotes the natural projection. We view the spherical simplex Δ_{sph} as the set of unit vectors in the complete Euclidean cone Δ_{euc} and define $h_i := m_i \cdot acc(\xi_i)$.

To a weighted configuration on $\partial_\infty X$ one can associate a natural *convex* function on X, the *weighted Busemann function* $\sum_i m_i \cdot b_{\xi_i}$ (well-defined up to an additive constant), compare [DE]. The Busemann function b_{ξ_i} measures the relative distance from the point ξ_i at infinity. We define *stability* and *semistability* of a weighted configuration in terms of asymptotic properties of its Busemann function. These asymptotic properties can in fact be expressed in terms of the Tits angle metric on $\partial_\infty X$ which leads to a notion of stability for weighted configurations on abstract spherical buildings, see also [KLM2]. Our notion of stability agrees with Mumford stability from geometric invariant theory when G is a complex group. Examples can be found in section 6 where we determine explicitly the (semi)stable weighted configurations (more generally, of finite measures) on the Grassmannians associated to the classical groups.

The possible Δ-weights for semistable weighted configurations on $\partial_\infty X$ are given by a finite system of homogeneous linear inequalities. We first describe the *structure* of these inequalities. Let (S, W) denote the spherical Coxeter complex attached to

G and think of the spherical Weyl chamber Δ_{sph} as being embedded in S. For every vertex ζ of Δ_{sph} and every n-tuple of vertices $\eta_1, \ldots, \eta_n \in W\zeta$ we consider the inequality

$$\sum_i m_i \cdot \cos \angle(\tau_i, \eta_i) \leq 0, \tag{1}$$

for $m_i \in \mathbb{R}_0^+$ and $\tau_i \in \Delta_{sph}$ where $\angle$ measures the spherical distance in S. We may rewrite the inequality as follows using standard terminology of Lie theory: Let $\lambda_\zeta \in \Delta_{euc}$ be the fundamental coweight contained in the edge with direction ζ, and let $\lambda_i := w_i \lambda_\zeta$ where $[w_i] \in W/W_\zeta$ such that $w_i \zeta = \eta_i$. With the renaming $h_i = m_i \tau_i$ of the variables (1) becomes the homogeneous linear inequality

$$\sum_i \langle h_i, \lambda_i \rangle \leq 0. \tag{2}$$

The full family of these inequalities has only the trivial solution. The stability inequalities are given by a subset of these inequalities which we single out using the *Schubert calculus.*

For a vertex ζ of Δ_{sph} we denote by $Grass_\zeta \subset \partial_\infty X$ the corresponding maximally singular G-orbit on $\partial_\infty X$. We call it a *generalized Grassmannian* because in the case of $SL(m)$ the $Grass_\zeta$ are the usual Grassmann manifolds. The stabilizers of points in $Grass_\zeta$ are the conjugates of a maximal parabolic subgroup $P \subset G$. The restriction of the G-action to P stratifies $Grass_\zeta$ into *Schubert cells*, one cell C_{η_i} corresponding to each vertex $\eta_i \in S$ in the orbit $W\zeta$ of ζ under the Weyl group W. Hence, if we denote $W_\zeta := Stab_W(\zeta)$ then the Schubert cells correspond to cosets in W/W_ζ. The *Schubert cycles* are defined as the closures $\overline{C}_{\eta_i}$ of the Schubert cells; they are unions of Schubert cells. As real algebraic varieties, the Schubert cycles represent homology classes $[\overline{C}_{\eta_i}] \in H_*(Grass_\zeta; \mathbb{Z}/2\mathbb{Z})$ which we abbreviate to $[C_{\eta_i}]$; in the complex case they even represent *integral* homology classes.

Now we can formulate our main result. We recall that $\mathcal{P}_n(X) = \mathcal{P}_n(\mathfrak{p})$ coincides with the set of Δ-weights of semistable weighted configurations on $\partial_\infty X$.

Theorem 1.3 (Stability inequalities for noncompact semisimple Lie groups)**.** *(i) The set $\mathcal{P}_n(X)$ consists of all $h \in \Delta_{euc}^n$ such that (2) holds whenever the intersection of the Schubert classes $[C_{\eta_1}], \ldots, [C_{\eta_n}]$ in $H_*(Grass_\zeta; \mathbb{Z}/2\mathbb{Z})$ equals $[pt]$.*

(ii) If G is complex, then the set $\mathcal{P}_n(X)$ consists of all $h \in \Delta_{euc}^n$ such that the inequality (2) holds whenever the intersection of the integral Schubert classes $[C_{\eta_1}], \ldots, [C_{\eta_n}]$ in $H_*(Grass_\zeta; \mathbb{Z})$ equals $[pt]$.

Our argument shows moreover that the system obtained by imposing all inequalities (2) whenever the intersection of the Schubert classes $[C_{\eta_1}], \ldots, [C_{\eta_n}]$ is *nonzero* in $H_*(Grass_\zeta;\mathbb{Z}/2\mathbb{Z})$ has the same set of solutions as the smaller system obtained when we require the intersection to be $[pt]$. Note that in the complex case the set of necessary and sufficient inequalities obtained from the *integral Schubert* calculus is in general smaller.

Interestingly, the stability inequalities given by Theorem 1.3 depend on the Schubert calculus whereas their solution set depends only on the Weyl group. This is due to possible redundancies.

In rank one, i.e. when X has strictly negative sectional curvature, we have $\Delta_{euc} \cong \mathbb{R}_0^+$ and the stability inequalities are just the ordinary triangle inequalities. In section 7 we determine the side length spaces for all symmetric spaces of rank two. The rank three case is already quite involved and it is treated in the paper [KuLM].

The polyhedron $\mathcal{P}_3(\mathfrak{p})$ was first determined for $G = SL(m,\mathbb{C})$ by Klyachko [Kly1] who proved that the inequalities corresponding to triples of Schubert classes with intersection a positive multiple of the point class were *sufficient*. The necessity of these inequalities has been know for some time, see [F2, sec. 6, Prop. 2] for a proof of their necessity and the history of this proof. Klyachko's theorem was refined by Belkale [Bel] who showed that it suffices to restrict to those triples of Schubert classes whose intersection is the point class. The determination of $\mathcal{P}_3(\mathfrak{p})$ for general complex simple G was accomplished in [BeSj] using methods from algebraic geometry. However they gave a larger system (than ours) consisting of all inequalities where the intersections of Schubert classes is a *nonzero multiple* of the point class. Thus our Theorem 1.3 for the complex case is a refinement of their result. In the general real case the polyhedron $\mathcal{P}_3(\mathfrak{p})$ was determined in [OSj]. However their inequalities are quite different from ours. They are associated to the *integral* Schubert calculus of the *complexification* $\mathfrak{g}\otimes\mathfrak{C}$ and are efficient for the case of split $\mathfrak{g}$ but become less and less efficient as the real rank of $\mathfrak{g}$ (i.e. the rank of the symmetric space X) decreases. For instance, for the case of real rank one they have a very large number of inequalities when the ordinary triangle inequalities alone will suffice. This is recognized in [OSj] and the problem is posed (Problem 9.5 on page 451) as to whether a formula of the type we found above in terms of the Schubert calculus *modulo* 2 would exist.

Remark 1.4. Recently, a smaller system of inequalities for $\mathcal{P}_n(X)$ was described by Belkale and Kumar [BK]; it is based on a deformation of the (co)homology rings of the generalized Grassmannians. Ressayre [R] then proved that the smaller system is irredundant.

The paper is organized as follows. In section 2 we provide some background from the geometry of spaces of nonpositive curvature, symmetric spaces of noncompact type and of spherical buildings. In section 3 we define and study a notion of stability for measures and weighted configurations on the ideal boundary of symmetric spaces of noncompact type. We provide analogues of some basic results in geometric invariant theory, such as a Harder-Narasimhan Lemma (Theorem 3.22) and prove our main result Theorem 1.3. The *weak* stability inequalities considered in section 3.8 correspond to particularly simple intersections of Schubert cycles and they have a beautiful geometric interpretation in terms of convex hulls. In section 4 we explain in the example of weighted configurations on complex projective space that our notion of stability matches with Mumford stability. In section 5 we discuss the relation between polygons in X and configurations on $\partial_\infty X$ and prove the generalized Thompson Conjecture (Theorem 1.1). In section 6 we will make explicit the stability condition for measures supported on the (generalized) Grassmannians associated to the classical groups. In section 7 we make a detailed study of the polyhedra $\mathcal{P}_n(\mathfrak{p})$ for the rank 2 complex simple groups. We make our system of stability inequalities explicit and for $n = 3$ we describe the minimal subsystems, i.e. the facets of the polyhedron. We check that the irredundant system of stability inequalities consists only of *weak* stability inequalities of the form

$$w^{-1}(h_1 - h_2^\sharp) \leq h_3^\sharp + \cdots + h_n^\sharp, w \in W$$

where the order is the dominance order (defined using the acute cone Δ^*).

Moreover we give the generators (edges) of $\mathcal{P}_3(\mathfrak{p})$. The inequalities for the group G_2 were computed previously in [BeSj, Example 5.2.2]. The paper [KuLM] studies $\mathcal{P}_3(\mathfrak{p})$ for the rank 3 examples and describes the minimal subsystems and the generators of the cone for the root systems C_3 and B_3.

Acknowledgements. We are grateful to the referee for useful comments and references. We would like to thank Andreas Balser for reading an early version of this paper and Chris Woodward for helpful suggestions concerning the computations in section 7. Also we took the multiplication table for the Schubert classes for G_2 from [TW]. In section 7 we used the computer program Porta written by Thomas Christof and Andreas Löbel to find the minimal subsystems and the generators of the cones. Finally we would like to thank George Stantchev for finding the computer program Porta and for much help and advice in implementing it. During the work on this paper the first and the second authors were supported by the NSF grant DMS-05-54349.

2 Preliminaries

In this section, mostly to fix our notation, we will briefly review some basic facts about spaces of nonpositive curvature and especially Riemannian symmetric spaces of noncompact type. We will omit most of the proofs. For more details on spaces with upper curvature bound and in particular spaces with nonpositive curvature, we refer to [Ba, ch. 1-2], [BBI, ch. 4+9], [KlL, ch. 2] and [Le, ch. 2], for the geometry of symmetric spaces of noncompact type to [Ka], [BH, ch. II.10], [Eb, ch. 2+3] and [He, ch. 6], and for the theory of spherical buildings from a geometric viewpoint, i.e. within the framework of spaces with curvature bounded above, to [KlL, ch. 3].

2.1 Metric spaces with upper curvature bounds

Consider a complete geodesic space, that is, a complete metric space Y such that any two points $y_1, y_2 \in Y$ can be joined by a rectifiable curve with length $d(y_1, y_2)$; such curves are called *geodesic segments*. Note that we do not assume Y to be locally compact. Although there is no smooth structure nor a Riemann curvature tensor around, one can still make sense of a sectional curvature *bound* in terms of distance comparison. We will only be interested in *upper* curvature bounds. One says that Y has (globally) curvature $\leq k$ if all triangles in Y are thinner than corresponding triangles in the model plane (or sphere) M^2_k of constant curvature k. Here, a geodesic triangle Δ in Y is a one-dimensional object consisting of three points and geodesic segments joining them. A comparison triangle $\tilde{\Delta}$ for Δ in M^2_k is a triangle with the same side lengths. Every point p on Δ corresponds to a point $\tilde{p}$ on $\tilde{\Delta}$, and we say that Δ is *thinner* than $\tilde{\Delta}$ if for any points p and q on Δ we have $d(p, q) \leq d(\tilde{p}, \tilde{q})$. A metric space with curvature $\leq k$ is also called a *$CAT(k)$-space*. It is a direct consequence of the definition that any two points are connected by a *unique* geodesic if $k \leq 0$, or if $k > 0$ and the points have distance $< \frac{\pi}{\sqrt{k}}$.

By the Rauch Comparison Theorem [CE], a complete manifold locally has curvature $\leq k$ in the distance comparison sense if and only if it has sectional curvature $\leq k$. In the case when $k \leq 0$ (which we are most interested in), it follows from the Rauch's theorem in conjunction with the Cartan-Hadamard theorem that a complete simply-connected manifold has curvature $\leq k$ in the distance comparison sense if and only if it has sectional curvature $\leq k$.

The presence of a curvature bound allows to define *angles* between segments $\sigma_i : [0, \epsilon) \to Y$ initiating in the same point $y = \sigma_1(0) = \sigma_2(0)$ and parameterized by unit speed. Let $\tilde{\alpha}(t)$ be the angle of a comparison triangle for $\Delta(y, \sigma_1(t), \sigma_2(t))$ in the appropriate model plane at the vertex corresponding to y. If Y has an upper curvature bound then the comparison angle $\tilde{\alpha}(t)$ is monotonically decreasing as $t \searrow 0$. It therefore converges, and we define the angle $\angle_y(\sigma_1, \sigma_2)$ of the segments at y as the limit. In this way, one obtains a pseudo-metric on the space of segments emanating from a point $y \in Y$. Identification of segments with angle zero and metric completion yields the *space of directions* $\Sigma_y Y$. One can show that if Y has an upper curvature bound, then $\Sigma_y Y$ has curvature ≤ 1.

2.2 Spaces of nonpositive curvature

In this section, we assume that Y is a space of nonpositive curvature. We will call spaces of nonpositive curvature also *Hadamard spaces.*

A basic consequence of nonpositive curvature is that the distance function $d : Y \times Y \to \mathbb{R}_0^+$ is *convex.* It follows that geodesic segments between any two points are unique and globally minimizing. In particular, Y is contractible. We will call a complete geodesic $l \subset Y$ also a *line* since it is an isometrically embedded copy of $\mathbb{R}$.

A (parameterized) geodesic *ray* is an isometric embedding $\rho : [0, \infty) \longrightarrow Y$. Two rays ρ_1 and ρ_2 are called *asymptotic* if $t \mapsto d(\rho_1(t), \rho_2(t))$ stays bounded and hence, by convexity, (weakly) decreases. An equivalence class of asymptotic rays is called an *ideal* point or a point *at infinity.* If a ray ρ represents an ideal point ξ, we also say that ρ is *asymptotic* to ξ. We define the *geometric boundary* $\partial_\infty Y$ as the set of ideal points. The topology on Y can be canonically extended to the *cone topology* on $\bar{Y} := Y \cup \partial_\infty Y$ If Y is locally compact, $\bar{Y}$ is a compactification of Y.

There is a natural metric on $\partial_\infty Y$, the *Tits metric.* The Tits distance of two ideal points $\xi_1, \xi_2 \in \partial_\infty Y$ is defined as $\angle_{Tits}(\xi_1, \xi_2) := \sup_{y \in Y} \angle_y(\xi_1, \xi_2)$. It is useful to know that one can compute the distance $\angle_{Tits}(\xi_1, \xi_2)$ by only looking at the angles along a ray ρ asymptotic to one of the ideal points ξ_i; namely $\angle_{\rho(t)}(\xi_1, \xi_2)$ is monotonically increasing and converges to $\angle_{Tits}(\xi_1, \xi_2)$ as $t \to +\infty$. Another way to represent the Tits metric is as follows: If ρ_i are rays asymptotic to ξ_i, then $2 \sin \frac{\angle_{Tits}(\xi_1, \xi_2)}{2} = \lim_{t \to \infty} \frac{d(\rho_1(t), \rho_2(t))}{t}$.

The metric space $\partial_{Tits} Y = (\partial_\infty Y, \angle_{Tits})$ is called the *Tits boundary.* It turns out that $\partial_{Tits} Y$ is always a complete metric length space with curvature ≤ 1. The Tits distance is lower semicontinuous with respect to the cone topology. It induces a topology on $\partial_\infty Y$ which is in general strictly finer than the cone topology and, generically, $\partial_{Tits} Y$ is not compact even when Y is locally compact. More details can be found in [KlL, section 2.3.2].

Two lines in Y are called *parallel* if they have finite Hausdorff distance. Due to a basic rigidity result, the Flat Strip Lemma, any two parallel lines bound an embedded flat strip, that is, a convex subset isometric to the product of the real line with a compact interval. The *parallel set* $P(l)$ of l is defined as the union of all lines parallel to l. There is a canonical isometric splitting $P(l) \cong l \times CS(l)$ and the *cross section* $CS(l)$ is again a Hadamard space.

The convexity of the distance $d(\cdot, \cdot)$ provides natural convex functions. For instance, the distance $d(y, \cdot)$ from a point y is a convex function on Y and, more generally, the distance $d(C, \cdot)$ from a convex subset C.

Related to distance functions are Busemann functions. They measure the relative distance from points at infinity. Their construction goes as follows. For an ideal point $\xi \in \partial_\infty Y$ and a ray $\rho : [0, \infty) \to Y$ asymptotic to it we define the *Busemann function* b_ξ as the pointwise monotone limit

$$b_\xi(y) := \lim_{t\to\infty} (d(y, \rho(t)) - t)$$

of normalized distance functions. It is a basic but remarkable fact that, up to additive constants, b_ξ is independent of the ray ρ representing ξ. As a limit of distance functions, b_ξ is Lipschitz continuous with Lipschitz constant 1. Note that along a ray ρ asymptotic to ξ the Busemann function b_ξ is affine linear with slope one, $b_\xi(\rho(t)) = -t + const.$

The level and sublevel sets of b_ξ are called *horospheres* and *horoballs* centered at ξ. We denote the horosphere passing through y by $Hs(\xi, y)$ and the horoball which it bounds by $Hb(\xi, y)$. The horoballs are convex subsets and their ideal boundaries are convex subsets of $\partial_{Tits}Y$, namely balls of radius $\pi/2$ around the centers of the horoballs: $\partial_\infty Hb(\xi, y) = \{\angle_{Tits}(\xi, \cdot) \leq \frac{\pi}{2}\}$.

Convex functions have directional derivatives. For Busemann functions they are given by the formula

$$\frac{d}{dt^+}(b_\xi \circ \sigma)(t) = -\cos \angle_{\sigma(t)}(\sigma'(t), \xi) \tag{3}$$

where $\sigma : I \to Y$ is a unit speed geodesic segment and the angle on the right-hand side is taken between the positive direction $\sigma'(t) \in \Sigma_{\sigma(t)}Y$ of the segment σ at $\sigma(t)$ and the ray emanating from $\sigma(t)$ asymptotic to ξ.

2.3 Spherical buildings

A *spherical Coxeter complex* (S, W_{sph}) consists of a unit sphere S and a finite subgroup $W_{sph} \subset Isom(S)$ generated by reflections. By a reflection, we mean a reflection at a great sphere of codimension one. W_{sph} is called the *Weyl group* and the fixed point sets of the reflections in W_{sph} are called *walls*. The pattern of walls gives S a natural structure of a cellular (polysimplicial) complex. The top-dimensional cells, the *chambers*, are fundamental domains for the action $W_{sph} \curvearrowright S$. They are spherical simplices if W_{sph} acts without fixed point. If convenient, we identify the *spherical model Weyl chamber* $\Delta_{sph} = S/W_{sph}$ with one of the chambers in S.

A *spherical building* modelled on a spherical Coxeter complex (S, W_{sph}) is a metric space B with curvature ≤ 1 together with a maximal atlas of charts, i.e. isometric embeddings $S \hookrightarrow B$. The image of a chart is an *apartment* in B. We require that any two points are contained in a common apartment and that the coordinate changes between charts are induced by isometries in W_{sph}.

We will usually denote the metric on a spherical building by $\angle_{Tits}$ because in this paper spherical buildings arise as Tits boundaries of symmetric spaces.

Two points $\xi, \eta \in B$ are called *antipodal* if $\angle_{Tits}(\xi, \eta) = \pi$.

The cell structure and the notions of wall, chamber etc. carry over from the Coxeter complex to the building. The building B is called *thick* if every codimension-one face is adjacent to at least three chambers. A non-thick building can always be equipped with a natural structure of a thick building by reducing the Weyl group. If W_{sph} acts without fixed points the chambers are spherical simplices and the building carries a natural structure as a piecewise spherical simplicial complex. We will then refer to the cells as simplices.

There is a canonical 1-Lipschitz continuous *accordion* map $acc : B \to \Delta_{sph}$ folding the building onto the model Weyl chamber so that every chamber projects isometrically. $acc(\xi)$ is called the *type* of the point $\xi \in B$, and a point in B is called *regular* if its type is an interior point of Δ_{sph}.

2.4 Symmetric spaces of noncompact type

A complete simply connected Riemannian manifold X is called a *symmetric space* if in every point $x \in X$ there is a reflection, that is, an isometry σ_x fixing x with $d\sigma_x = -id_x$. We will always assume that X has *noncompact type*, i.e. that it has nonpositive sectional curvature and no Euclidean factor. The identity component G of its isometry group is then a noncompact semisimple Lie group with trivial center, and the point stabilizers K_x in G are its maximal compact subgroups.

Given a line l in X, the products of even numbers of reflections at points on l are called *translations* or *transvections* along l. They form a one-parameter subgroup of isometries in G.

With respect to the cone topology $\overline{X}$ is a closed standard ball, X its interior and $\partial_\infty X$ the boundary sphere. The Tits boundary $\partial_{Tits} X = (\partial_\infty X, \angle_{Tits})$ carries a natural structure as a thick *spherical building* of dimension $rank(X) - 1$. The faces (simplices) of $\partial_{Tits} X$ correspond to parabolic subgroups of G stabilizing them and, as a simplicial complex, $\partial_{Tits} X$ is canonically isomorphic to the spherical Tits building associated to G. The building geometry is interesting if $rank(X) \geq 2$; for $rank(X) = 1$ the Tits metric is discrete with values 0 and π.

The top-dimensional simplices of $\partial_{Tits} X$ are called *(spherical Weyl) chambers.* They can be simultaneously and compatibly identified with a *spherical model Weyl chamber* Δ_{sph}. In fact, each orbit for the natural isometric action $G \curvearrowright \partial_{Tits} X$ meets each chamber in precisely one point and there is a natural projection $acc : \partial_{Tits}X \to \Delta_{sph}$ given by dividing out the G-action. Its restriction to any chamber is an isometry. We call *acc* the *accordion map* because of the way it folds the spherical building onto the model chamber. We refer to the *acc*-image of an ideal point as its (Δ_{sph}-)*type*, cf. section 2.3.

The fixed point set in $\partial_{Tits}X$ of a parabolic subgroup $P \subset G$ is a closed simplex σ_P. The stabilizer of each interior point $\xi \in \sigma_P$ equals P and the map $gP \mapsto g\xi$ defines an embedding of the generalized flag manifold G/P into the ideal boundary. The orbit $G\xi$ is a submanifold of $\partial_\infty X$ with respect to the cone topology. If P is a maximal parabolic subgroup then the fixed point set of P is a vertex (0-dimensional simplex) of $\partial_{Tits}X$ and we have a unique G-equivariant embedding $G/P \hookrightarrow \partial_\infty X$.

A subset $Z \subseteq X$ is called a *totally-geodesic subspace* if, with any two distinct points, it contains the unique line passing through them. Totally-geodesic subspaces are embedded submanifolds and symmetric spaces themselves.

By a *flat* in X we mean a flat totally-geodesic subspace, i.e. a closed convex subset isometric to a Euclidean space. A d-flat is a d-dimensional flat. The Tits metric on $\partial_\infty X$ reflects the pattern of flats in X: The Tits boundary of a flat $f \subset X$ is a *sphere* in $\partial_{Tits}X$, by which we mean a closed convex subset isometric to a unit sphere in a Euclidean space. Since X is a symmetric space, vice versa, every sphere $s \subset \partial_{Tits}X$ arises as the ideal boundary of a flat $f \subset X$, $\partial_{Tits}f = s$, actually of several flats if s is not top-dimensional. The maximal flats in X correspond one-to-one to the top-dimensional spheres, the *apartments*, in $\partial_{Tits}X$. The ideal boundary of a *singular* flat is a subcomplex of $\partial_{Tits}X$.

The natural action of G on the set of all maximal flats is transitive and their dimension is called the *rank* of the symmetric space. We have $rank(X) = dim(\partial_{Tits}X) + 1$.

A non-maximal flat is called *singular* if it arises as the intersection of maximal flats. Each maximal flat F contains finitely many families of parallel codimension-one singular flats. We will also call them *singular hyperplanes* in F. Each singular flat $f \subset F$ can be obtained as the intersection of singular hyperplanes in F. The ideal boundaries of singular flats in F are subcomplexes of the apartment $\partial_{Tits}F$ in $\partial_{Tits}X$.

For a point $x \in F$, there are finitely many singular hyperplanes $f \subset F$ passing through x. They divide F into cones whose closures are called *Euclidean Weyl chambers* with *tip* x. The reflections at the hyperplanes f generate a finite group, the *Weyl group* $W_{F,x}$. It acts on $\partial_{Tits}F$ by isometries and $(\partial_{Tits}F, W_{F,x})$ is the *spherical Coxeter complex* attached to X respectively $\partial_{Tits}X$. It is well-defined up to automorphisms.

The Euclidean Weyl chambers in X can be canonically identified with each other by isometries in G, and hence they can be simultaneously identified with a *Euclidean model Weyl chamber* Δ_{euc}. The ideal boundaries of Euclidean Weyl chambers are spherical Weyl chambers and there is a natural identification $\Delta_{sph} \cong \partial_{Tits}\Delta_{euc}$.

Let f be a flat, not necessarily singular. We call a line l in f *maximally regular* (with respect to f) if it is generic in the sense that its two ideal endpoints are interior points of simplices of $\partial_{Tits}X$ with maximal possible dimension (depending on f). Every maximal flat F containing l must also contain f because the apartment $\partial_{Tits}F$, as a convex subcomplex of $\partial_{Tits}X$, must contain the sphere $\partial_{Tits}f$. Thus the smallest singular flat containing l also contains f.

Any two parallel lines in X bound a flat strip and, since X is a symmetric space, are in fact contained in a 2-flat. The *parallel set* $P(l)$ of a line l is the union of all (maximal) flats containing l. It splits isometrically as $P(l) \cong l \times CS(l)$ and is a totally-geodesic subspace. Its cross section $CS(l)$ is a symmetric space with $rank(CS(l)) = rank(X) - 1$. The cross section $CS(l)$ has no Euclidean de Rham factor if and only if the line is a singular 1-flat, equivalently, iff its ideal endpoints are vertices of $\partial_{Tits}X$.

More generally, we need to consider parallel sets of flats. Given a flat $f \subset X$, its parallel set $P(f)$ is defined as the union of all flats f' which are parallel to f in the sense that f and f' have finite Hausdorff distance, or equivalently, that $\partial_\infty f = \partial_\infty f'$. $P(f)$ is the union of all (maximal) flats containing f and it splits isometrically as $P(f) \cong f \times CS(f)$. The cross section $CS(f)$ is a nonpositively curved symmetric space with $rank(CS(f)) = rank(X) - dim(f)$. It has no Euclidean factor if and only if the flat f is singular. Note that $P(f)$ depends only on the sphere $\partial_{Tits}f$. If l is a maximally regular line in f then every maximal flat which contains l must also contain f and it follows that $P(f) = P(l)$.

The Busemann function b_ξ associated to the ideal point $\xi \in \partial_\infty X$ is smooth. Its gradient is the unit vector field pointing away from ξ, and the differential is given by

$$(db_\xi)_x(v) = -\cos \angle_x(v, \xi), \tag{4}$$

compare formula (3) in the general case of Hadamard spaces. The horospheres $Hs(\xi, \cdot)$ centered at ξ are the level sets of b_ξ and thus orthogonal to the geodesics asymptotic to ξ.

The Busemann functions are convex, but not strictly convex. The Hessian $D^2 b_\xi(x)$ in a point x can be interpreted geometrically as the second fundamental form of the horosphere $Hs(\xi, x)$. The degeneracy of the Hessian is described as follows:

Lemma 2.1. *Let $u, v \in T_xX$ be non-zero tangent vectors, let l_v be the geodesic with initial condition v, and suppose that u points towards the ideal point ξ_u. Then the following are equivalent:*

(i) $D^2_{v,v} b_{\xi_u} = 0$.

(ii) b_{ξ_u} *is affine linear on* l_v.

(iii) u and v span a 2-plane in T_xX with sectional curvature zero, or they are linearly dependent,

(iv) u and v are tangent to a 2-flat or linearly dependent.

Example 2.2 (Busemann functions for the symmetric space associated to $SL(m,\mathbb{C})$)**.** Let V be a finite-dimensional complex vector space equipped with a Hermitian scalar product $((\cdot,\cdot))$. Furthermore, let $G = SL(V)$, K the maximal compact subgroup preserving $((\cdot,\cdot))$ and let $X = G/K$ be the associated symmetric space of noncompact type.

The stabilizer $G_{[v]} \subset G$ of a point $[v]$ in projective space $\mathbb{P}V$ is a maximal parabolic subgroup and there is a unique G-equivariant embedding $\mathbb{P}V \hookrightarrow \partial_\infty X$. We may hence regard $\mathbb{P}V$ as a G-orbit (of vertices) in the spherical building $\partial_{Tits}X$. With respect to the cone topology on $\partial_\infty X$, $\mathbb{P}V$ is an embedded submanifold.

After suitable normalization (rescaling) of the Riemannian metric on X, one can express the Busemann function at the ideal boundary point $[v]$ as

$$b_{[v]}(gK) := \log \|g^{-1}v\| \tag{5}$$

Note that multiplication of v by a scalar changes $b_{[v]}$ by an additive constant. To justify (5) we observe that the right-hand side is invariant under the subgroup $G_v \subset G$ fixing v. Its orbits are the horospheres centered at $[v]$. It remains to verify that the right-hand side is linear with negative slope on some (and hence every) oriented geodesic $c : \mathbb{R} \to X$ asymptotic to $[v]$. The one-parameter group (T_t) of transvections along c has the following form: There is a direct sum decomposition $V = \langle v \rangle \oplus U$ into common eigenspaces for the T_t such that $T_t v = e^{\lambda t}$ and $T_t|_U = e^{-\lambda' t} \cdot id_U$ with $\lambda, \lambda' > 0$ and $\lambda = dim(U) \cdot \lambda'$. Hence $\log \|(T_t)^{-1}v\| = -\lambda t + const$.

2.5 Infinitesimal symmetric spaces

We keep the notation from the previous section. Let $o \in X$ be a base point and $K = Stab_G(o)$ the maximal subgroup fixing it. The tangent space T_oX is canonically identified with the orthogonal complement $\mathfrak{p}$ of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Killing form:

$$T_oX \cong \mathfrak{p}$$

K acts on $\mathfrak{p}$ by the restriction of the adjoint representation. It is an orthogonal action with respect to the Riemannian metric (and the Killing form). We denote by $Aff(\mathfrak{p})$ the group of transformations on $\mathfrak{p}$ generated by K and all translations on $\mathfrak{p}$. We call the geometry, in the sense of Felix Klein, consisting of the space $\mathfrak{p}$ and the group $Aff(\mathfrak{p})$ an *infinitesimal symmetric space*.

A *flat* in $\mathfrak{p}$ is by definition an affine subspace of the form $z + T_o f$ where f is a flat in X passing through o and z is an arbitrary vector in $\mathfrak{p}$. The flat is called *singular* if f is singular. Singular flats are intersections of maximal flats. The maximal flats are of the form $z + \mathfrak{a}$ with $\mathfrak{a}$ a maximal abelian subalgebra contained in $\mathfrak{p}$.

We define the *parallel set* of a line l in $\mathfrak{p}$ as the union of all (maximal) flats containing l. A parallel set is an affine subspace of the form $z + T_oP(c)$ where z is an arbitrary vector and $P(c)$ the parallel set of a geodesic $c \subset X$ through o.

The K-orbits in $\mathfrak{p}$ are parameterized by the Euclidean model Weyl chamber Δ_{euc}. In fact, we can think of Δ_{euc} as sitting in an abelian subalgebra $\mathfrak{a}$ as above, $\Delta_{euc} \subset \mathfrak{a}$, by identifying it with a Weyl chamber. Then each K-orbit $\mathcal{O} \subset \mathfrak{p}$ meets Δ_{euc} in a unique point h. Due to the natural identifications $Aff(\mathfrak{p})\backslash\mathfrak{p} \times \mathfrak{p} \cong K\backslash\mathfrak{p} \cong \Delta_{euc}$ we can assign to an oriented geodesic segment $\overline{z_1z_2}$ in $\mathfrak{p}$ a vector $\sigma(z_1, z_2) \in \Delta_{euc}$ which we call its Δ-*length*.

The exponential map $exp_o : T_oX \to X$ yields a radial projection

$$\mathfrak{p} - \{0\} \to \partial_\infty X$$

assigning to a tangent vector v the ideal point represented by the geodesic ray with initial condition v. This radial projection restricts on the unit sphere $S(\mathfrak{p})$ of $\mathfrak{p}$ to a homeomorphism $S(\mathfrak{p}) \to \partial_\infty X$.

Note that the infinitesimal symmetric space associated to X and all structures which we just defined are, up to canonical isomorphism, independent of the base point o and the corresponding splitting $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.

2.6 A transversality result for homogeneous spaces

In this section, we provide an auxiliary result of differential–topological nature. Let $U_1, \ldots, U_n$ be linear subspaces of a finite-dimensional vector space V. Then one has the inequality

$$codim_V \cap_{i=1}^n U_i \leq \sum_{i=1}^n codim_V U_i. \tag{6}$$

We recall that $U_1, ..., U_n$ are said to intersect transversally if and only if equality holds in (6). Based on this, one says that smooth submanifolds $Z_1, ..., Z_n$ in a smooth manifold Y intersect *transversally* at $z \in Z_1 \cap \cdots \cap Z_n$ if their tangent spaces $T_zZ_1, \ldots, T_zZ_n$ intersect transversally in T_zY. In this case, $Z_1 \cap \cdots \cap Z_n$ is locally near z a submanifold with codimension equal to $\sum_{i=1}^n codim_Y Z_i$. One says that $Z_1, ..., Z_n$ intersect transversally if they intersect transversally everywhere along $Z_1 \cap \cdots \cap Z_n$.

Proposition 2.3. *Let Y be a homogeneous space for the Lie group G, and let $Z_1, \dots, Z_n$ be embedded submanifolds. Then, for almost all $(g_1, \dots, g_n) \in G^n$, the submanifolds $g_1 Z_1, \dots, g_n Z_n$ intersect transversally.*

Proof. The maps $G \times Z_i \longrightarrow Y$ are submersions, and hence the inverse image

$$N := \{(g_1, z_1, \dots, g_n, z_n) : g_1 z_1 = \cdots = g_n z_n\}$$

of the small diagonal in Y^n under the canonical map $q : G \times Z_1 \times \cdots \times G \times Z_n \longrightarrow Y^n$ is a submanifold. We consider the natural projection $p : N \longrightarrow G^n$. Let $g^0 := (g_1^0, \dots, g_n^0)$ be a regular value. According to Sard's theorem, the regular values of p form a subset of full measure in G^n (i.e. the set of singular values has zero measure). It therefore suffices to show that the $g_i^0 Z_i$ intersect transversally.

Let $g_1^0 z_1 = \cdots = g_n^0 z_n =: y \in \cap_{i=1}^n g_i^0 Z_i$. Then $w := (g_1^0, z_1, \dots, g_n^0, z_n) \in p^{-1}(g^0)$ and $ker\ dp_w \cong \cap_{i=1}^n T_y g_i^0 Z_i$. We have that

$$dim(N) = n \cdot dim(G) + \sum_{i=1}^n dim(Z_i) - (n-1) \cdot dim(Y)$$

and, since w is a regular point of p,

$$dim(\cap_{i=1}^n T_y g_i^0 Z_i) = dim(N) - n \cdot dim(G) = dim(Y) - \sum_{i=1}^n codim_Y(Z_i).$$

This yields

$$codim_{T_y Y}(\cap_{i=1}^n T_y g_i^0 Z_i) = \sum_{i=1}^n codim_Y(Z_i),$$

i.e. the $g_i^0 Z_i$ intersect transversally. □

Remark 2.4. In the algebraic category one can prove a more precise result, namely that the intersection is transversal for a Zariski open subset of tuples $(g_1, \dots, g_n)$, compare Kleiman's transversality theorem [Kl].

3 Stable weighted configurations at infinity

We define in sections 3.3 and 3.6 a notion of stability for measures and weighted configurations on the ideal boundary of a symmetric space X of noncompact type. This is done as in [DE] by associating to a measure on $\partial_\infty X$ a natural convex function, a weighted Busemann function. Stability is then defined in terms of its asymptotic properties. As a preparation, we study in section 3.1 properties of convex Lipschitz functions on nonpositively curved spaces and specialize in section 3.2 to weighted Busemann functions on a symmetric space. In sections 3.4 and 3.5 we investigate properties of measures under various stability assumptions. For instance, we show the existence of directions of steepest asymptotic descent for Busemann functions of unstable measures and deduce an analogue of the Harder-Narasimhan Lemma (Theorem 3.22). In section 3.7 we prove Theorem 1.3, the main result of this paper. It provides a finite system of homogeneous linear inequalities describing the possible Δ-weights for semistable configurations.

3.1 Asymptotic slopes of convex functions on nonpositively curved spaces

Let Y be a Hadamard space, i.e. a space of nonpositive curvature. We will now discuss asymptotic properties of Lipschitz continuous convex functions $f : Y \to \mathbb{R}$. Later on they will be applied to convex combinations of Busemann functions on symmetric spaces.

Such a function f is asymptotically linear along any ray $\rho : [0, \infty) \to Y$. We define the *asymptotic slope* of f at the ideal point $\eta \in \partial_\infty Y$ represented by ρ as

$$slope_f(\eta) := \lim_{t \to +\infty} \frac{f(\rho(t))}{t}. \tag{7}$$

That the limit does not depend on the choice of ρ follows, for instance, from the Lipschitz assumption. Since convex functions of one variable have one-sided derivatives, we can rewrite (7) as

$$slope_f(\eta) = \lim_{t \to +\infty} \frac{d}{dt^+}(f \circ \rho)(t)$$

Lemma 3.1. *For any value a of f holds*

$$\partial_\infty \{f \leq a\} = \{slope_f \leq 0\}$$

Proof. The sublevel set $\{f \leq a\} \subset Y$ is non-empty and convex. Let p be a point in it. For any ideal point $\xi \in \partial_\infty \{f \leq a\}$ the ray $\overline{p\xi}$ is contained in $\{f \leq a\}$. Thus f non-increases along it and $slope_f(\xi) \leq 0$. Vice versa, if $\xi \in \partial_\infty Y$ is an ideal point with $slope_f(\xi) \leq 0$ then f is non-increasing along $\overline{p\xi}$. Hence $\overline{p\xi} \subset \{f \leq a\}$ and $\xi \in \partial_\infty \{f \leq a\}$. □

We call $\{slope_f \leq 0\}$ the set of *asymptotic decrease*.

A subset C of a space with curvature ≤ 1 is called *convex* if for any two points $p, q \in C$ with $d(p, q) < \pi$ the unique shortest segment $\overline{pq}$ is contained in C.

Lemma 3.2. *(i) The asymptotic slope function $slope_f : \partial_{Tits}Y \to \mathbb{R}$ is Lipschitz continuous with the same Lipschitz constant as f.*

(ii) The set $\{slope_f \leq 0\} \subset \partial_\infty Y$ is convex with respect to the Tits metric. The function $slope_f$ is convex on $\{slope_f \leq 0\}$ and strictly convex on $\{slope_f < 0\}$.

(iii) The set $\{slope_f < 0\}$ contains no pair of points with distance π. If it is non-empty then $slope_f$ has a unique minimum.

(iv) If Y is locally compact, then f is proper and bounded below if and only if $slope_f > 0$ everywhere on $\partial_\infty Y$.

Proof. (i) Let $\xi_1, \xi_2 \in \partial_{Tits}Y$ and let $\rho_i : [0, +\infty) \to Y$ be rays asymptotic to ξ_i with the same initial point y. Then $d(\rho_1(t), \rho_2(t)) \leq t \cdot 2\sin\frac{\angle_{Tits}(\xi_1,\xi_2)}{2} \leq t \cdot \angle_{Tits}(\xi_1, \xi_2)$. If f is L-Lipschitz, we estimate: $f(\rho_2(t)) \leq f(\rho_1(t)) + Lt \cdot \angle_{Tits}(\xi_1, \xi_2)$, so $\frac{f(\rho_2(t))}{t} \leq \frac{f(\rho_1(t))}{t} + L \cdot \angle_{Tits}(\xi_1, \xi_2)$. Passing to the limit as $t \to +\infty$ yields the assertion.

(ii) Suppose now that

$$\xi_1, \xi_2 \in \partial_\infty\{slope_f \leq 0\}$$

with $\angle_{Tits}(\xi_1, \xi_2) < \pi$. Then the midpoints $m(t)$ of the segments $\overline{\rho_1(t)\rho_2(t)}$ converge to the midpoint μ of $\overline{\xi_1\xi_2}$ in $\partial_{Tits}Y$. Since $f(m(t)) \leq f(y)$, we have $slope_f(\mu) \leq 0$. Thus $\{slope_f \leq 0\} \subset \partial_\infty Y$ is convex.

In order to estimate the asymptotic slope at μ, we observe that $f(\rho_i(t)) \leq slope_f(\xi_i) \cdot t + f(y)$ and thus

$$f(m(t)) \leq \frac{slope_f(\xi_1) + slope_f(\xi_2)}{2} \cdot t + f(y). \tag{8}$$

Furthermore $\lim_{t\to+\infty} \frac{d(\rho_1(t),\rho_2(t))}{t} = 2\sin\frac{\angle_{Tits}(\xi_1,\xi_2)}{2}$. The latter fact implies via triangle comparison that

$$\limsup_{t\to+\infty} \frac{d(y, m(t))}{t} \leq \cos\frac{\angle_{Tits}(\xi_1, \xi_2)}{2}.$$

Using $slope_f(\xi_i) \leq 0$ we deduce

$$slope_f(\mu) \leq \limsup_{t\to+\infty} \frac{f(m(t))}{d(y, m(t))} \leq \frac{slope_f(\xi_1) + slope_f(\xi_2)}{2\cos(\angle_{Tits}(\xi_1, \xi_2)/2)}, \tag{9}$$

and the convexity properties of $slope_f$ follow.

(iii) Assume that $\angle_{Tits}(\xi_1, \xi_2) = \pi$ and $slope_f(\xi_i) < 0$. Then $\frac{d(\rho_1(t),\rho_2(t))}{t} \to 2$ and, by triangle comparison, $\frac{d(y,m(t))}{t} \to 0$ as $t \to +\infty$. Since f is Lipschitz, this implies $\frac{f(m(t))}{t} \to 0$. We obtain a contradiction with (8), hence the first assertion holds.

Suppose that η_n are ideal points with $slope_f(\eta_n) \to \inf slope_f < 0$. Then (9) implies that the sequence (η_n) is Cauchy. Since $\partial_{Tits}Y$ is complete, it follows that there is one and only one minimum for $slope_f$.

(iv) If f were not proper or unbounded below, sublevel sets would be noncompact and hence, by local compactness, contain rays. It follows that there exists an ideal point with asymptotic slope ≤ 0. Conversely, if ξ is an ideal point with $slope_f(\xi) \leq 0$ then f is non-increasing along rays asymptotic to ξ and hence not proper or unbounded below. $\square$

The condition $slope_f \geq 0$ does not imply a lower bound for f. But although there are in general no almost minima there still exist almost critical points. We prove a version in the smooth case.

Definition 3.3. A differentiable function $\phi : M \to \mathbb{R}$ on a Riemannian manifold is said to have *almost critical points* if there exists a sequence (p_n) of points in M such that $\|\nabla\phi(p_n)\| \to 0$.

Lemma 3.4. *Let Y be a Hadamard manifold and let $f : Y \to \mathbb{R}$ be a smooth convex function. If $slope_f \geq 0$ then f has almost critical points.*

Proof. Suppose that f does not have almost critical points. This means that there is a lower bound $\|\nabla f\| \geq \epsilon > 0$ for the length of the gradient of f.

We consider the normalized negative gradient flow for f, that is, the flow for the vector field $V = -\frac{\nabla f}{\|\nabla f\|}$. Its trajectories have unit speed and are complete. For the derivative of f along a trajectory $\gamma : \mathbb{R} \to X$ holds

$$(f \circ \gamma)' = \langle \nabla f, V \rangle \leq -\epsilon.$$

We let $y_n := \gamma(n)$ for $n \in \mathbb{N}$ and fix a base point $o \in X$. Since $f(y_n) \leq f(o) - n\epsilon \to -\infty$, the points y_n diverge to infinity. We connect o to the points y_n by unit speed geodesic segments $\gamma_n : [0, l_n] \to X$. Then $l_n = d(o, y_n) \leq n$. Since X is locally compact, the sequence of segments γ_n subconverges to a ray $\rho : [0, +\infty) \to X$. Using the convexity of f we obtain for $t \geq 0$ the estimate $f(\gamma_n(t)) \leq f(o) - \frac{t}{l_n} n\epsilon \leq f(o) - t\epsilon$ and, by passing to the limit, $f(\rho(t)) \leq f(o) - t\epsilon$. This implies $slope_f(\eta) \leq -\epsilon < 0$ for the ideal point η represented by ρ. $\square$

The next result compares the asymptotic slopes of convex and linear functions on flat spaces.

Lemma 3.5. *Let E be a Euclidean space and $f : E \to \mathbb{R}$ a convex Lipschitz function. Suppose that $slope_f$ assumes negative values and let $\xi \in \partial_{Tits}E$ be the unique minimum of $slope_f$. Then on $\partial_{Tits}E$ we have the inequality:*

$$slope_f \geq slope_f(\xi) \cdot \cos \angle_{Tits}(\xi, \cdot)$$

Proof. We pick a base point o in E and simplify the function f by a rescaling procedure. Consider for $a > 0$ the functions $f_a(x) := \frac{1}{a} \cdot f(ax)$ where ax denotes the image of x under the homothety with scale factor a and center o. As $a \to +\infty$, these functions converge uniformly on compacta to a convex function f_∞ with the same Lipschitz constant as f. Moreover, f_∞ is linear along rays initiating in o and has the same asymptotic slopes as f, i.e. $slope_{f_\infty} \equiv slope_f$ on $\partial_{Tits} E$. We may assume without loss of generality that $f = f_\infty$.

Since ξ is the minimum of $slope_f$, we have

$$f \geq slope_f(\xi) \cdot d(o, \cdot)$$

with equality along the ray ρ_ξ with direction ξ starting in o. Let $\eta \in \partial_{Tits} E$ be another ideal point. We consider the ray $\rho : [0, +\infty) \to E$ towards η initiating in $\rho_\xi(t_0)$ for some $t_0 > 0$. Then $f(\rho(t)) \geq slope_f(\xi) \cdot d(o, \rho(t))$ for $t \geq 0$ with equality in 0. Hence we obtain the estimate

$$\partial_{\dot\rho(0)} f \geq slope_f(\xi) \cdot \partial_{\dot\rho(0)}(d(o, \cdot)) = slope_f(\xi) \cdot \cos \angle_{\rho(0)}(\xi, \eta)$$

for the partial derivative of f in direction of the unit vector $\dot\rho(0)$. Of course,

$$\angle_{\rho(0)}(\xi, \eta) = \angle_{Tits}(\xi, \eta)$$

because E is flat. The convexity of f implies that

$$slope_f(\eta) \geq \partial_{\dot\rho(0)} f \geq slope_f(\xi) \cdot \cos \angle_{Tits}(\xi, \eta).$$

□

3.2 Weighted Busemann functions on symmetric spaces

From now on let X denote a symmetric space of noncompact type. The class of convex functions on X relevant for this paper are finite convex combinations of Busemann functions. (See section 2.2 for the definition of Busemann functions which we will henceforth also refer to as *atomic* Busemann functions.) Since it does not complicate the discussion of their basic properties, we will also consider general "measurable" convex combination, given by integrals of Busemann functions with respect to measures μ on $\partial_\infty X$.

In order to study general weighted Busemann functions we will need

Lemma 3.6. *Suppose that C is a metric compact. Then the space of measures with finite support is dense in the space of all measures of finite total mass on C with respect to the weak topology.*

Proof. We include a proof for the sake of completeness. Let μ be a finite measure on C. Given $N \in \mathbb{N}$ find a finite collection of measurable pairwise disjoint subsets $C_1, ..., C_n \subset C$ whose union equals C and so that $\operatorname{diam}(C_i) \leq 1/N$ for each i. For each i pick $x_i \in C_i$ and consider the atomic measure $m_i \delta_{x_i}$, where $m_i = \mu(C_i)$. Take

$$\mu_N := \sum_{i=1}^{n} m_i \delta_{x_i}.$$

Then for every continuous function f on C,

$$\lim_{N\to\infty} \int_C f(x) d\mu_N = \int_C f(x) d\mu. \quad \square$$

Let $\mathcal{M}(\partial_\infty X)$ be the space of Borel measures on $\partial_\infty X$ with finite total mass equipped with the weak $*$ topology. We recall that $\partial_\infty X$ carries the cone topology and is homeomorphic to a sphere. The natural G-action on $\mathcal{M}(\partial_\infty X)$ is continuous. To a measure $\mu \in \mathcal{M}(\partial_\infty X)$ we assign the *weighted Busemann function*

$$b_\mu := \int_{\partial_\infty X} b_\xi \; d\mu(\xi).$$

It immediately follows from Lemma 3.6 that this function is well-defined up to an additive constant, convex and Lipschitz continuous with Lipschitz constant $\|\mu\|$. Moreover, since for each $k < \infty$, norms of all partial derivatives (of order $\leq k$) of all Busemann functions on X are uniformly bounded, Lemma 3.6 implies that each weighted Busemann function is infinitely differentiable.

Let $o \in X$ be a base point and normalize the Busemann functions by $b_\xi(o) = 0$. Then the map

$$\partial_\infty X \times X \to \mathbb{R}, (\xi, x) \mapsto b_\xi(x)$$

continuous and consequently also the map

$$\mathcal{M}(\partial_\infty X) \times X \longrightarrow \mathbb{R}, \quad (\mu, x) \mapsto b_\mu(x).$$

Moreover, the map

$$\partial_\infty X \to C^1(X), \xi \mapsto b_\xi(x)$$

is continuous in the C^1 topology on $C^1(X)$. In particular, $\nabla b_\mu(x)$ depends continuously on μ.

Lemma 3.7. *Let v be a non-zero tangent vector and l the geodesic with initial condition v. Then the following are equivalent:*

(i) $D^2_{v,v} b_\mu = 0$.

(ii) b_μ is affine linear on l.

(iii) μ is supported on $\partial_\infty P_l$.

Proof. This follows readily from Lemma 2.1, the corresponding result for atomic Busemann functions, because integration yields $D^2_{v,v}b_\mu = \int_{\partial_\infty X} D^2_{v,v}b_\xi \; d\mu(\xi)$. Since $D^2_{v,v}b_\xi \geq 0$, we have $D^2_{v,v}b_\mu = 0$ if and only if $D^2_{v,v}b_\xi = 0$ for μ-almost all ξ. Hence (i)$\Rightarrow$(iii) by Lemma 2.1. Clearly (iii)$\Rightarrow$(ii)$\Rightarrow$(i). $\square$

We denote by $MIN(\mu) \subset X$ the minimum set of b_μ. It is convex but possibly empty. If b_μ attains a minimum then $slope_\mu \geq 0$ everywhere on $\partial_\infty X$. Moreover, Lemma 3.1 implies that

$$\partial_\infty MIN(\mu) = \{slope_\mu = 0\}.$$

Here and later on we abbreviate

$$slope_\mu := slope_{b_\mu}.$$

By Lemma 3.7, $MIN(\mu)$ contains with any two distinct points also the complete geodesic passing through these points. Hence:

Corollary 3.8. *If non-empty, $MIN(\mu)$ is a totally geodesic subspace of X.*

We compute now the asymptotic slopes of weighted Busemann functions. A basic observation is that for an atomic Busemann function $b_\xi : Y \to \mathbb{R}$ the asymptotic slope function on $\partial_{Tits}X$ can be expressed in terms of the Tits geometry. Using formula (3) in section 2.2 for the derivative of Busemann functions one obtains

$$slope_{b_\xi}(\eta) = \lim_{t \to +\infty} \frac{d}{dt^+}(b_\xi \circ \rho)(t) = -\lim_{t \to +\infty} \cos \angle_{\rho(t)}(\xi, \eta)$$

and, since $\angle_{\rho(t)}(\xi, \eta) \nearrow \angle_{Tits}(\xi, \eta)$ as $t \to +\infty$:

$$slope_{b_\xi}(\eta) = -\cos \angle_{Tits}(\xi, \eta).$$

The differential of a weighted Busemann function is obtained by integrating (3):

$$(db_\mu)_x = -\int_{\partial_\infty X} \cos \angle_x(\cdot, \xi) \; d\mu(\xi)$$

With the monotone convergence theorem for integrals we get

$$slope_\mu = -\int_{\partial_\infty X} \cos \angle_{Tits}(\cdot, \xi) \; d\mu(\xi). \tag{10}$$

Notice that the asymptotic slope function is expressed directly in terms of the Tits geometry on the ideal boundary.

We wish to describe the asymptotics of Busemann functions more precisely. For $\xi, \eta \in \partial_\infty X$ and a ray $\rho : [0, +\infty) \to X$ asymptotic to η we have that the convex non-increasing function

$$(b_\xi \circ \rho)(t) + \cos \angle_{Tits}(\xi, \eta) \cdot t$$

converges to a finite limit as $t \to +\infty$. To see this, we consider a flat F with $\xi, \eta \in \partial_\infty F$ and inside it a ray ρ' asymptotic to η. Then $b_\xi \circ \rho'$ is linear with slope $-\cos \angle_{Tits}(\xi, \eta)$ and one can estimate $|(b_\xi \circ \rho)(t) - (b_\xi \circ \rho')(t)| \leq d(\rho(t), \rho'(t)) \leq d(\rho(0), \rho'(0))$ because b_ξ is 1-Lipschitz.

This kind of asymptotic behavior holds more generally along Weyl chambers. Let F be a maximal flat and $V \subset F$ a Weyl chamber. There exists a maximal flat F' asymptotic to ξ and V, i.e. with $\{\xi\} \cup \partial_\infty V \subset \partial_\infty F'$. The restriction of b_ξ to F' is then affine linear. With a (purely) parabolic isometry n which fixes $\partial_\infty V$ and moves F to F', we may write

$$b_\xi(x) = b_\xi(nx) + (b_\xi(x) - b_\xi(nx)). \tag{11}$$

The summand $b_\xi(nx)$ is linear on V whereas $b_\xi(x) - b_\xi(nx)$ is bounded (and convex) on V. Thus the restriction of a Busemann function b_ξ to a Euclidean Weyl chamber is *asymptotically linear* in the sense that it is the sum of a linear and a bounded function.

We generalize to weighted Busemann functions:

Lemma 3.9 (Asymptotic linearity)**.** *The Busemann function b_μ is asymptotically linear on each Euclidean Weyl chamber $V \subset X$ in the sense that its restriction to V decomposes as the sum of a linear function and a bounded function.*

Proof. According to (11) we can decompose each atomic Busemann function b_ξ on V as the sum $b_\xi|_V = l_\xi + s_\xi$ of its linear and bounded part. For measures μ with finite support the claim follows directly.

For arbitrary measures $\mu \in \mathcal{M}(\partial_\infty X)$ one has to argue a bit more carefully. We normalize the functions b_ξ, l_ξ and s_ξ to be zero at the tip of V. The decomposition of b_ξ depends measurably on ξ. Note that l_ξ is 1-Lipschitz and hence s_ξ is 2-Lipschitz. This allows us to integrate and we get $b_\mu = \int l_\xi \, d\mu(\xi) + \int s_\xi \, d\mu(\xi)$. Both summands are Lipschitz and the first one is clearly linear. In view of Lemma 3.6, the second factor is a (uniform on compacts) limit of bounded convex functions. Therefore, it is bounded. $\square$

As a consequence of asymptotic linearity, the function $slope_\mu$ has the property that its values on a simplex $\sigma \subset \partial_{Tits} X$ are determined by the values on the vertices of σ. Namely, if F is a flat in X with $\sigma \subset \partial_{Tits} F$ and if $l : F \to \mathbb{R}$ is an affine linear function with the same asymptotic slopes at the vertices of σ as b_μ then $slope_\mu = slope_l$ on σ. Since $\{slope_l > 0\}$ resp. $\{slope_l \geq 0\}$ is an open resp. closed hemisphere in the round sphere $\partial_{Tits}F$, this implies:

Corollary 3.10. *(i) Suppose that $slope_\mu > 0$ (resp. $slope_\mu \geq 0$, $slope_\mu \leq 0$ or $slope_\mu < 0$) on all vertices of a simplex $\sigma \subset \partial_{Tits}X$. Then the same inequality holds on the entire simplex σ.*

(ii) If $slope_\mu \geq 0$ holds on σ then $\{slope_\mu = 0\} \cap \sigma$ is a face of σ.

3.3 Stability for measures on the ideal boundary

We define stability of a measure $\mu \in \mathcal{M}(\partial_\infty X)$ in terms of its weighted Busemann function b_μ on X and the associated asymptotic slope function on $\partial_{Tits}X$.

Definition 3.11 ((Semi)Stability of measures on $\partial_{Tits}X$)**.** We call a measure $\mu \in \mathcal{M}(\partial_\infty X)$ *stable* if $slope_\mu > 0$, *semistable* if $slope_\mu \geq 0$ and *unstable* if it is not semistable.

Remark 3.12. In fact, formula (10) expresses $slope_\mu$ directly in terms of the intrinsic geometry of $\partial_{Tits}X$ without referring to b_μ. Our definition of stability hence carries over to Borel measures with finite total mass on topological spherical buildings in the sense of Burns and Spatzier [BuSp]. It agrees with [KLM2, Definition 4.1] given in the special case of measures with finite support. In this case the integration in (10) becomes finite summation and makes sense on any spherical building (which may be thought of as a topological spherical building with discrete topology).

If the measure μ is semistable then $\{slope_\mu = 0\}$ is a convex subcomplex of $\partial_{Tits}X$ by Lemma 3.2 (ii) and Corollary 3.10 (ii). The following more subtle variation of the notion of stability will be needed in section 5.3, in particular for Proposition 5.6 and the proof of Theorem 5.9.

Definition 3.13 (Nice semistability of measures on $\partial_{Tits}X$)**.** We call a semistable measure μ *nice semistable* if $\{slope_\mu = 0\}$ is either empty or d-dimensional and contains a unit d-sphere.

Remark 3.14. A d-dimensional convex subcomplex of a spherical building which contains a unit d-sphere carries itself a natural structure as a spherical building. In fact we will show in Lemma 3.18 that for a nice semistable measure μ the set $\{slope_\mu = 0\}$ is the ideal boundary of a totally geodesic subspace.

In view of the slope formula (10) semistability of μ is equivalent to the system of inequalities

$$\int_{\partial_\infty X} \cos \angle_{Tits}(\eta, \cdot)\, d\mu \leq 0 \qquad\qquad \forall\, \eta \in \partial_\infty X \tag{12}$$

and stability to the corresponding system of strict inequalities. Note that according to asymptotic linearity (Corollary 3.10) it suffices to check the inequalities on vertices.

Example 3.15. Suppose that X has rank one, equivalently, that the spherical building $\partial_{Tits} X$ has dimension 0. Then a measure μ on $\partial_{Tits} X$ is stable if and only if it has no atom with mass $\geq \frac{1}{2}|\mu|$, semistable if and only if it has no atom with mass $> \frac{1}{2}|\mu|$, and nice semistable if and only if it is either stable or consists of two atoms with equal mass. This follows from the fact that the Tits metric is discrete (with distances 0 or π) and hence

$$\begin{gathered} slope_\mu(\eta) = -\int_{\partial_\infty X} \cos \angle_{Tits}(\eta, \cdot)\, d\mu = \\ -1 \cdot \mu(\eta) + (|\mu| - \mu(\eta)) = -2 \cdot \mu(\eta) + |\mu|. \end{gathered}$$

In higher rank the stability criterion becomes more complicated. Examples are discussed in section 6 where we work out the case of measures on the Grassmannians associated to the classical groups.

The next result implies that semistability persists under totally geodesic embeddings of symmetric spaces. It is useful when relating the stability conditions for different groups.

Lemma 3.16. *Assume that $C \subset X$ is a closed convex subset and that the measure μ is supported on $\partial_\infty C \subset \partial_\infty X$. Then:*

(i) $\inf b_\mu\big|_C = \inf b_\mu$. In particular, b_μ is bounded below on C if and only if it is bounded below on X.

(ii) If $slope_\mu \geq 0$ on $\partial_\infty C$ then $slope_\mu \geq 0$ on $\partial_\infty X$.

Proof. For every ideal point $\xi \in \partial_\infty C$ holds $b_\xi \geq b_\xi \circ \pi_C$ where $\pi_C : X \to C$ denotes the nearest point projection. Namely, let x be a point in X and let σ be the segment connecting x and its projection $\pi_C(x)$. Then for any point x' on σ we have that the ideal triangle with vertices $x', \pi_C(x)$ and ξ has angle $\frac{\pi}{2}$ at $\pi_C(x)$ and therefore angle $\leq \frac{\pi}{2}$ at x' because the angle sum is $\leq \pi$. Hence b_ξ decreases along σ and we obtain $b_\xi(x) \geq b_\xi(\pi_C(x))$. Integration with respect to μ yields:

$$b_\mu \geq b_\mu \circ \pi_C \tag{13}$$

This implies assertion (i).

Regarding part (ii), suppose that $slope_\mu \geq 0$ on $\partial_\infty C$ and $slope_\mu(\eta) < 0$ for some $\eta \in \partial_\infty X$. Let $\rho : [0, +\infty) \to X$ be a unit speed ray asymptotic to η with $b_\mu(\rho(0)) \leq 0$. Then $b_\mu(\rho(t)) \leq -ct$ with $c := -slope(\eta) > 0$. Let $y_n := \pi_C(\rho(n))$ for $n \in \mathbb{N}_0$. In view of (13) we have $b_\mu(y_n) \leq b_\mu(\rho(n)) \leq -cn$.

Nearest point projections to closed convex sets are 1-Lipschitz and therefore $d(y_0, y_n) \leq n$. On the other hand $d(y_0, y_n) \to \infty$ because $b_\mu(y_n) \to -\infty$. Thus the sequence of segments $\overline{y_0 y_n}$ in C subconverges to a ray $\bar{\rho}$ in C. Using the convexity of Busemann functions, it follows that $b_\mu(\bar{\rho}(t)) \leq b_\mu(y_0) - ct$ and hence $slope_\mu(\bar{\eta}) \leq -c < 0$ at the ideal endpoint $\bar{\eta}$ of $\bar{\rho}$. This is a contradiction because $\bar{\eta} \in \partial_\infty C$. $\square$

3.4 Properties of stable and semistable measures

We investigate now how the various degrees of stability of a measure are reflected in the behavior of the associated weighted Busemann function.

Lemma 3.17. *μ is stable if and only if b_μ is proper and bounded below. In this case b_μ has a unique minimum.*

Proof. Part (iv) of Lemma 3.2 implies that μ is stable if and only if b_μ is proper and bounded below. Since b_μ is convex this is in turn equivalent to $MIN(\mu)$ being compact and non-empty, and by Corollary 3.8 to $MIN(\mu)$ being a point. $\square$

Lemma 3.18. *μ is nice semistable if and only if b_μ attains a minimum.*

Proof. "$\Rightarrow$": Suppose that μ is nice semistable. We are done by Lemma 3.17 if μ is stable. Therefore we assume also that $\{slope_\mu = 0\}$ is non-empty and hence a d-dimensional convex subcomplex which contains a unit d-sphere s. Let $f \subset X$ be a flat with $\partial_\infty f = s$. Furthermore, let l be a maximally regular geodesic inside f. Then any geodesic parallel to l lies in a flat parallel to f and the parallel sets satisfy $P(f) = P(l)$. Lemma 3.7 implies that μ is supported on $\partial_\infty P(f)$. By Lemma 3.16 it suffices to show that the restriction of b_μ to the parallel set $P(f) \cong f \times CS(f)$ attains a minimum. Since b_μ is constant on each flat parallel to f this amounts to finding a minimum on a cross section $\{pt\} \times CS(f)$. We have $slope_\mu > 0$ on $\partial_\infty(\{pt\} \times CS(f))$ because otherwise $\{slope_\mu = 0\}$ would contain a $(d+1)$-dimensional hemisphere, which is absurd. Using Lemma 3.2 (iv) we conclude that b_μ attains a minimum on $\{pt\} \times CS(f)$.

"$\Leftarrow$": If b_μ attains a minimum then μ is semistable and $\{slope_\mu = 0\} = \partial_\infty MIN(\mu)$ is empty or the ideal boundary of a totally geodesic subspace, cf. Corollary 3.8. In the latter case $\{slope_\mu = 0\}$ carries a natural structure as a spherical building and hence contains a top-dimensional unit sphere. $\square$

As a special case of Lemma 3.4 we obtain:

Lemma 3.19. *If μ is semistable then b_μ has almost critical points.*

Lemma 3.20. *If μ is semistable then the closure of its G-orbit in $\mathcal{M}(\partial_\infty X)$ contains a nice semistable measure.*

Proof. By Lemma 3.19 the associated weighted Busemann function b_μ has almost critical points, i.e. there exists a sequence (x_j) of points in X with $\|\nabla b_\mu(x_j)\| \to 0$. We use the G-action on X to move the almost critical points into the base point. Namely let $g_j \in G$ with $g_j x_j = o$. Then $\|\nabla b_{g_j\mu}(o)\| = \|\nabla b_\mu(x_j)\| \to 0$. Due to the compactness of $\mathcal{M}(\partial_\infty X)$ we may assume after passing to a subsequence that $g_j\mu \to \nu$. It follows that $\nabla b_{g_j\mu}(o) \to \nabla b_\nu(o)$ and therefore $\nabla b_\nu(o)$. Hence $\nu \in \overline{G\mu}$ is a nice semistable measure. $\square$

Remark 3.21. (i) One can show that the closure $\overline{G\mu}$ of a semistable orbit contains a *unique* nice semistable G-orbit $G\nu$.

(ii) The Busemann function of a semistable measure μ is in general not bounded below. However, for semistable measures μ with *finite support* one can show that b_μ is bounded below, but this fact will not be needed in this paper.

3.5 Unstable measures and directions of steepest descent

Lemma 3.2 implies that for unstable measures μ there is a *unique* ideal point ξ_{min} of *steepest descent*, i.e. where $slope_\mu$ attains its minimum. We will now look for *vertices* of steepest descent among vertices of a given type. The following uniqueness result is a version of the Harder-Narasimhan Lemma.

Theorem 3.22. *Let $\mu \in \mathcal{M}(\partial_\infty X)$ be unstable, and let ξ_{min} be the unique ideal point of steepest descent for b_μ. Let τ_{min} be the simplex in the Tits boundary spanned by ξ_{min}, i.e. which contains ξ_{min} as an interior point.*

Then for each vertex η of τ_{min} holds: η is the unique minimum of $slope_\mu$ restricted to the orbit $G\eta$, i.e. the unique minimum among vertices of the same type.

Note that $slope_\mu < 0$ on all vertices of τ_{min} due to the asymptotic linearity of Busemann functions on Weyl chambers (Lemma 3.9) and the fact that all simplices in the Tits boundary have diameter $\leq \frac{\pi}{2}$.

Proof. Since $slope_\mu(\xi_{min}) < 0$, there is a unique measure ν supported on the vertices of τ_{min} such that ξ_{min} is the ideal point of steepest ν-descent and $slope_\nu(\xi_{min}) = slope_\mu(\xi_{min})$. On each flat f asymptotic to τ_{min} i.e. with $\tau_{min} \subset \partial_\infty f$, the Busemann function b_ν restricts to a linear function whose negative gradient points towards ξ_{min}. The asymptotic slopes are given by the formula:

$$slope_\nu = slope_\mu(\xi_{min}) \cdot \cos \angle_{Tits}(\xi_{min}, \cdot) \qquad \text{on } \partial_{Tits}X$$

Let f be a minimal flat containing τ_{min} in its ideal boundary, i.e. $dim(f) = dim(\tau_{min}) + 1$ and $\partial_{Tits} f$ is a subcomplex of $\partial_{Tits}X$ with τ_{min} as a top-dimensional simplex. From the asymptotic linearity of Busemann functions on Weyl chambers (Lemma 3.9) follows the existence of a linear function l on f with $slope_l = slope_\mu$ on τ_{min}. Since ξ_{min} is the direction of steepest descent for l, we have that $l = b_\nu|_f$ modulo additive constants. Thus:

$$slope_\mu = slope_\nu \qquad \text{on } \tau_{min}$$

Every ideal point lies in an apartment through ξ_{min}. Therefore by applying Lemma 3.5 to all flats which contain ξ_{min} in their ideal boundary, we obtain the estimate:

$$slope_\mu \geq slope_\nu \qquad \text{on } \partial_{Tits}X$$

As a consequence, it suffices to prove: *(*) η is the unique vertex in the orbit $G\eta$ with minimal Tits distance from ξ_{min}.*

To verify this claim, consider a vertex $\zeta \in G\eta$ in the same orbit. There exists an apartment a in $\partial_{Tits}X$ containing ζ and τ_{min}. Suppose that ζ is separated from ξ_{min} inside a by a wall s. The reflection at s belongs to the Weyl group $W(a)$, and the mirror image ζ' of ζ is a vertex of the same type which is strictly closer to ξ_{min}. Observe that the vertices which cannot be separated from ξ_{min} by a wall are precisely the vertices of Weyl chambers with τ_{min} as a face. Therefore η is the only vertex in $G\eta \cap a$ which cannot be separated from ξ_{min}. Hence η is closer to ξ_{min} than any other vertex in $G\eta \cap a$. This shows (*) and finishes the proof of Theorem 3.22. $\square$

Depending on the geometry of the spherical Weyl chamber and the type of ξ_{min} one can say more. See section 2.4 for the definition of the accordion map $acc : \partial_{Tits}X \to \Delta_{sph}$ and the definition of type.

Addendum 3.23. *Suppose that the vertices of Δ_{sph} closest to the type $acc(\xi_{min})$ of ξ_{min} belong to the face $acc(\tau_{min}) \subset \Delta_{sph}$ spanned by τ_{min}.*

Then there exists a vertex with steepest μ-descent among all vertices. All such vertices are vertices of τ_{min}. In particular, there are only finitely many of them and each G-orbit contains at most one.

Proof. We take up our argument for Theorem 3.22. As we saw, the vertices at minimal Tits distance from ξ_{min} belong to a Weyl chamber containing τ_{min} as a face. By our assumption, they are vertices of τ_{min}. It follows that the vertices of τ_{min} closest to ξ_{min} have minimal μ-slope among vertices. $\square$

Remark 3.24. Other than one might first expect, the assumption of 3.23 does not always hold. This can occur if the Dynkin diagram associated to G branches, i.e. (in the irreducible case) the associated root system is of type D_n, E_6, E_7 or E_8.

We discuss the simplest example, which will not be used elsewhere in the paper.

Example 3.25. Suppose that the Dynkin diagram associated to G is D_4. Then X has rank 4 and the spherical model Weyl chamber Δ_{sph} is a three-dimensional spherical tetrahedron $\eta\xi_1\xi_2\xi_3$ with the following geometry: The face $\xi_1\xi_2\xi_3$, which we regard as the base of the tetrahedron, is an equilateral triangle. The dihedral angles at the edges of the base triangle equal $\frac{\pi}{3}$ whereas the dihedral angles at the other three edges $\overline{\eta\xi_i}$ equal $\frac{\pi}{2}$. From these data one deduces that the edges $\overline{\xi_i\xi_j}$, $i \neq j$, have length $\frac{\pi}{3}$ and the height, i.e. the distance between the vertex η and the center ζ of the base, equals $\frac{\pi}{6}$. This shows that ζ is strictly closer to the opposite vertex η than to the vertices ξ_i of the base.

Consider an atomic measure μ with unit mass concentrated in one ideal point $\hat{\zeta} \in \partial_\infty X$ of type ζ. There are infinitely many vertices of steepest μ-descent among vertices, namely all vertices $\hat{\eta}$ which are vertices of a Weyl chamber σ such that $\hat{\zeta}$ is the center of the face of σ opposite to $\hat{\eta}$.

On the other hand,

Proposition 3.26. *For every measure $\mu \in \mathcal{M}(\partial_\infty X)$, the function $slope_\mu$ is lower semicontinuous on $\partial_\infty X$. As a consequence, the restriction of $slope_\mu$ to each generalized Grassmannian $Grass_\zeta$ has a minimum.*

Proof. The function $(\xi, \eta) \mapsto -\cos\angle_{Tits}(\xi, \eta)$ on $\partial_\infty X \times \partial_\infty X$ is lower semicontinuous. Hence, for any convergent sequence $\eta_n \to \eta$ on $\partial_\infty X$ we have that

$$\liminf_{n\to\infty} -\cos\angle_{Tits}(\xi, \eta_n) \geq -\cos\angle_{Tits}(\xi, \eta)$$

for all $\xi \in \partial_\infty X$. Using Fatou's Lemma, we obtain by integration:

$$\liminf_{n\to\infty} \underbrace{\int_{\partial_\infty X} -\cos\angle_{Tits}(\xi, \eta_n)\, d\mu(\xi)}_{slope_\mu(\eta_n)} \geq \underbrace{\int_{\partial_\infty X} -\cos\angle_{Tits}(\xi, \eta)\, d\mu(\xi)}_{slope_\mu(\eta)}$$

Since $\partial_\infty X$ is a metrizable topological space, this shows that $slope_\mu$ is lower semicontinuous on $\partial_\infty X$.

Since $Grass_\zeta \subset \partial_\infty X$ is compact, the restriction of $slope_\mu$ to $Grass_\zeta$ attains its infimum. $\square$

3.6 Weighted configurations on $\partial_{Tits} X$ and stability

A collection of points $\xi_1, \dots, \xi_n \in \partial_{Tits} X$ and weights $m_1, \dots, m_n \geq 0$ determines a *weighted configuration*

$$\psi : (\mathbb{Z}/n\mathbb{Z}, \nu) \to \partial_{Tits} X$$

on $\partial_{Tits} X$. Here ν is the measure on $\mathbb{Z}/n\mathbb{Z}$ defined by $\nu(i) = m_i$, and $\psi(i) = \xi_i$. By composing ψ with $acc : \partial_{Tits} X \to \Delta_{sph}$ one obtains a map $(\mathbb{Z}/n\mathbb{Z}, \nu) \to \Delta_{sph}$. It corresponds to a point $h = (h_1, \dots, h_n)$ in Δ^n_{euc} which we call the Δ-*weights* of the configuration ψ, i.e. $h_i = m_i \cdot acc(\xi_i)$.

The configuration ψ yields, by pushing forward ν, the measure $\mu = \sum m_i \delta_{\xi_i}$ on $\partial_{Tits} X$. Accordingly, Definition 3.11 carries over from measures to configurations:

Definition 3.27 (Stability of weighted configurations on $\partial_{Tits} X$)**.** The weighted configuration ψ is called stable, semistable, unstable resp. nice semistable if the associated measure μ has this property.

Remark 3.28. Obviously, the definition extends to weighted configurations on abstract spherical buildings. One may extend it further to weighted configurations with infinite support $\psi : (\Omega, \nu) \to B$ on topological spherical buildings B, for instance, on $\partial_{Tits} X$. Here (Ω, ν) denotes a measure space with finite total mass and the map ψ is supposed to be measurable with respect to the Borel σ-algebra on B.

This notion of stability is motivated by Mumford stability in geometric invariant theory. The connection between the two concepts is explained in section 4.

3.7 The stability inequalities for Δ-weights of configurations

We will now address the question which Δ-weights occur for semistable weighted configurations on $\partial_{Tits} X$. We will need the *Schubert calculus.* We refer the reader to appendix in [GT] a detailed discussion of the for Schubert calculus, especially in the case of generalized real flag manifolds. *Schubert calculus* provides a useful description of generators of the groups $H_*(Grass_\zeta; \mathbb{Z}/2\mathbb{Z})$ and $H_*(Grass_\zeta; \mathbb{Z})$ for real and complex generalized Grassmannians as *Schubert classes* and allows one to compute combinatorially the products in the corresponding rings. Since Poincaré duals to Schubert classes are again Schubert classes, this formalism gives a description of the cohomology rings as well.

Think of the model spherical Weyl chamber Δ_{sph} as being embedded in the spherical Coxeter complex (S, W). For a vertex ζ of Δ_{sph}, we denote by $Grass_\zeta$ the corresponding maximally singular G-orbit in $\partial_\infty X$. We call it a *generalized Grassmannian* because in the case of $SL(n)$ the $Grass_\zeta$ are the Grassmann manifolds. The action of a Borel subgroup $B \subset G$ stratifies each $Grass_\zeta$ into *Schubert cells*, one cell C_{η_i} corresponding to each vertex $\eta_i \in S$ in the orbit $W\zeta$ of ζ under the Weyl group W. Hence, if we denote $W_\zeta := Stab_W(\zeta)$ then the Schubert cells correspond to cosets in W/W_ζ. The *Schubert cycles* are defined as the closures $\overline{C}_{\eta_i}$ of the Schubert cells; they are unions of Schubert cells (see e.g [GT]). There is one top-dimensional Schubert cell corresponding to the vertex in S belonging to the chamber opposite to Δ_{sph}. Note that as real algebraic varieties, the Schubert cycles represent homology classes $[\overline{C}_{\eta_i}] \in H_*(Grass_\zeta; \mathbb{Z}/2\mathbb{Z})$ which we abbreviate to $[C_{\eta_i}]$. In the complex case they even represent *integral* homology classes.

It will be useful to have another description of the Schubert cells and Schubert cycles. We recall the definition of the *relative position* of a spherical Weyl chamber σ and a vertex η of $\partial_{Tits}X$. There exists an apartment a in $\partial_{Tits}X$ containing σ and η. Furthermore there exists a unique apartment chart $\phi : a \to S$ which maps σ to Δ_{sph}. We then define the relative position (σ, η) to be the vertex $\phi(\eta)$ of the model apartment S. To see that the relative position is well-defined, we choose an interior point ξ in σ and a minimizing geodesic $\overline{\xi\eta}$. (It is unique if $\angle_{Tits}(\xi, \eta) < \pi$.) We then observe that the ϕ-image of the geodesic $\overline{\xi\eta}$ is determined by its length and its initial direction in $\phi(\xi)$, because geodesics in the unit sphere S do not branch. Thus its endpoint $\phi(\eta)$ is uniquely determined by σ and η.

Notice that G acts transitively on pairs (σ, η) with the same relative position. This follows from the transitivity of the G-action on pairs (σ, a) of chambers and apartments containing them. This implies that the relative position determines the Tits distance:

Lemma 3.29. *Suppose that σ_1, η_1 and σ_2, η_2 have the same relative position $(\sigma_1, \eta_1) = (\sigma_2, \eta_2)$. Suppose further we are given $\xi_1 \in \sigma_1$ and $\xi_2 \in \sigma_2$ with $acc(\xi_1) = acc(\xi_2)$. Then*

$$\angle_{Tits}(\xi_1, \eta_1) = \angle_{Tits}(\xi_2, \eta_2).$$

We now have another description of the Schubert cells as mentioned in the introduction. As above we assume we have chosen a spherical Weyl chamber $\sigma \subset \partial_{Tits}X$ and a vertex ζ of Δ_{sph}. For $\eta_i \in W\zeta$ we then have:

Lemma 3.30. *The Schubert cell C_{η_i} is given by*

$$C_{\eta_i} = \{\eta \in Grass_\zeta : (\sigma, \eta) = \eta_i\}.$$

For all vertices ζ of Δ_{sph} and all n-tuples of vertices $\eta_1, \ldots, \eta_n \in W\zeta$ we consider the inequality

$$\sum_i m_i \cdot \cos\angle(\tau_i, \eta_i) \leq 0, \tag{14}$$

for $m_i \in \mathbb{R}_0^+$ and $\tau_i \in \Delta_{sph}$ where $\angle$ measures the spherical distance in S. We may rewrite the inequality as follows using standard terminology of Lie theory: Let $\lambda_\zeta \in \Delta_{euc}$ be the fundamental coweight contained in the edge with direction ζ, and let $\lambda_i := w_i\lambda_\zeta$ where $[w_i] \in W/W_\zeta$ such that $w_i\zeta = \eta_i$. With the renaming $h_i = m_i\tau_i$ of the variables (14) becomes the homogeneous linear inequality

$$\sum_i \langle h_i, \lambda_i \rangle \leq 0. \tag{15}$$

We will now prove our main result Theorem 1.3 which describes, in terms of the Schubert calculus, a subset of these inequalities which is equivalent to the existence of a semistable weighted configurations for the given Δ-weights. Theorem 1.3 is the combination of the next two theorems.

Theorem 3.31 (Stability inequalities for noncompact semisimple Lie groups)**.** *For $h \in \Delta_{euc}^n$ there exists a semistable weighted configuration with Δ-weights h if and only if (15) holds whenever the intersection product of the Schubert classes $[C_{\eta_1}], \ldots, [C_{\eta_n}]$ in the ring $H_*(Grass_\zeta; \mathbb{Z}/2\mathbb{Z})$ equals $[pt]$.*

Proof. "$\Leftarrow$": Assume that all configurations with Δ-weights h are unstable. Due to the transversality result 2.3, there exist chambers $\sigma_1, \ldots, \sigma_n \subset \partial_{Tits}X$ so that the corresponding n stratifications of the Grassmannians $Grass_\zeta$ by orbits of the Borel subgroups $B_i = Stab_G(\sigma_i)$ are transversal. (This transversality is actually generic.) We choose a configuration with Δ-weights h so that the atoms ξ_i are located on the chambers σ_i.

Now we apply the Harder-Narasimhan Lemma type Theorem 3.22. Since the measure μ on $\partial_\infty X$ associated to the configuration is unstable there exists a vertex ζ of Δ_{sph} such that on the corresponding Grassmannian $Grass_\zeta$ there is a *unique* minimum η_{sing} for $slope_\mu$.

Let $C_i := B_i \cdot \eta_{sing}$ be the Schubert cell passing through η_{sing} for the stratification of $Grass_\zeta$ by B_i-orbits. Note that all points in the intersection $C_1 \cap \cdots \cap C_n$ have the same relative position with respect to all atoms ξ_i and therefore they have equal μ-slopes. Since η_{sing} is the unique minimum of $slope_\mu$ on $Grass_\zeta$ it is hence the unique intersection point of the Schubert cells C_i.

Transversality implies that the corresponding Schubert cycles $\bar{C}_i$ intersect transversally in the unique point η_{sing}. The corresponding inequality (14) resp. (15) in our list is violated because the left sides equal $-slope_\mu(\eta_{sing}) > 0$.

"$\Rightarrow$": Conversely, assume that there exists a semistable configuration ψ on $\partial_{Tits}X$ with Δ-weights h and masses $m_i = \|h_i\|$ located in the ideal points ξ_i of type $\tau_i = \frac{h_i}{\|h_i\|} = acc(\xi)$. Assume further that we have a homologically non-trivial product of n Schubert classes: $[C_{\eta_1}]\cdots[C_{\eta_n}] \neq 0$ in $H_*(Grass_\zeta;\mathbb{Z}/2\mathbb{Z})$. Choose chambers σ_i containing the ξ_i in their closures. (σ_i is unique if ξ_i is regular.) The choice of chambers determines cycles $\bar{C}_{\eta_i}$ representing the Schubert classes. Since their homological intersection is non-trivial we have $\bar{C}_{\eta_1} \cap \cdots \cap \bar{C}_{\eta_n} \neq \emptyset$. Let θ be a point in the intersection. All points on the Schubert *cell* C_{η_i} have the same relative position with respect to the chamber σ_i and therefore $\angle_{Tits}(\xi_i, \cdot)$ is constant along C_{η_i}, namely equal to $\angle_{Tits}(\tau_i, \eta_i)$, cf. Lemma 3.29. The semicontinuity of the Tits distance, compare section 2.2, then implies that $\angle_{Tits}(\xi_i, \cdot) \leq \angle_{Tits}(\tau_i, \eta_i)$ on the *cycle* $\bar{C}_{\eta_i}$. Thus $\angle_{Tits}(\xi_i, \theta) \leq \angle_{Tits}(\tau_i, \eta_i)$. It follows that $\sum_i m_i \cos\angle_{Tits}(\tau_i, \eta_i) \leq \sum_i m_i \cos\angle_{Tits}(\xi_i, \theta) = -slope_\mu(\theta) \leq 0$ where μ is the measure on $\partial_\infty X$ given by the weighted configuration ψ. Hence the inequality (15) holds whenever the corresponding Schubert classes have non-zero homological intersection. $\square$

Remark 3.32. Our argument shows that if a semistable configuration with Δ-weights h exists then all inequalities hold where the homological intersection of Schubert classes is nontrivial but not necessarily a point. This is an in general larger list of inequalities which hence has the same set of solutions in Δ^n_{euc}.

The proof of Theorem 3.31 works in exactly the same way in the *complex* case. The result which one obtains for complex Lie groups is stronger because one can work with *integral* cohomology and obtains a shorter list of inequalities:

Theorem 3.33 (Stability inequalities for semisimple complex groups)**.** *If G is complex, then for $h \in \Delta^n_{euc}$ there exists a semistable weighted configuration with Δ-weights h if and only if (15) holds whenever the intersection product of the integral Schubert classes $[C_{\eta_1}], \ldots, [C_{\eta_n}]$ in $H_*(Grass_\zeta;\mathbb{Z}/2\mathbb{Z})$ equals $[pt]$.*

3.8 The weak stability inequalities

In this section we consider a subsystem of the stability inequalities which corresponds to particularly simple intersections of Schubert cycles. We will see that it has a beautiful geometric interpretation in terms of convex hulls.

Suppose that G is a semisimple complex group. It is well-known, cf. [KuLM, sec. 2.1], that the Schubert cycles in the Grassmannian $Grass_\zeta$ come in pairs of mutually dual cycles, that is, the Poincaré dual of a Schubert cycle is also a Schubert cycle. The pairs of dual cycles can be nicely described using the parametrization of Schubert cycles by vertices in the Weyl group orbit $W\zeta$. To do so, let us denote by w_0 the element (of order 2) in the Weyl group which maps the spherical Weyl chamber Δ_{sph} to the opposite chamber in the model apartment. Note that, the case of irreducible root systems, $w_0 = -1$ if and one if the root system belongs to the list

$$\{A_1, B_n, C_n, D_{2n}, G_2, F_4, E_7, E_8\},$$

see e.g. [Bo], Plates I–IX.

Then it has been shown in [KuLM, Lemma 2.9] that for every vertex $\eta \in W\zeta$ holds

$$[C_\eta] \cdot [C_{w_0\eta}] = [pt] \tag{16}$$

Hence the inequality (15) parametrized by the n-tuple of Schubert cycles C_η, $C_{w_0\eta}$ and $n-2$ times the top-dimensional cycle $C_{w_0\zeta} = Grass_\zeta$ belongs to the system of stability inequalities. We call the subsystem consisting of all these inequalities for all Grassmannians $Grass_\zeta$ the *weak stability inequalities.*

To make them explicit, let $h_1, \ldots, h_n$ denote the Δ-weights of a semistable weighted configuration on the Tits boundary of the symmetric space $X = G/K$, and let λ_ζ denote the fundamental coweight which generates the edge of Δ_{euc} pointing towards the vertex ζ of Δ_{sph}. Then for $\eta = w\zeta$ the weak stability inequality corresponding to the intersection (16) reads:

$$\langle h_1, w\lambda_\zeta\rangle + \langle h_2, w_0 w\lambda_\zeta\rangle + \langle h_3, w_0\lambda_\zeta\rangle + \cdots + \langle h_n, w_0\lambda_\zeta\rangle \leq 0$$

Using the natural isometric involution $h \mapsto -w_0 h =: h^\sharp$ of Δ_{euc} it becomes:

$$\langle w^{-1}(h_1 - h_2^\sharp), \lambda_\zeta\rangle \leq \langle h_3^\sharp + \cdots + h_n^\sharp, \lambda_\zeta\rangle \tag{17}$$

(This involution is the identity for many root systems, see above.)

Let Δ^* denote the (obtuse) cone $\{h \in \mathfrak{a} | \langle h, \lambda_\zeta\rangle \geq 0\, \forall\zeta\}$ dual to the (acute) cone Δ_{euc}, and let "$\leq$" be the order on $\mathfrak{a}$ corresponding to Δ^*, i.e. $h \in \mathfrak{a}$ satisfies $h \geq 0$ if and only if $h \in \Delta^*$. This order is important in representation theory. It is usually referred to as the *dominance order.* The (sub)system of the inequalities (17) for fixed w and varying ζ amounts to the condition $w^{-1}(h_1 - h_2^\sharp) \in (h_3^\sharp + \cdots + h_n^\sharp) - \Delta^*$ and can hence be rewritten as the vector inequality

$$w^{-1}(h_1 - h_2^\sharp) \leq h_3^\sharp + \cdots + h_n^\sharp \tag{18}$$

Theorem 3.34 (Weak stability inequalities)**.** *Let G be a semisimple complex group. Then the Δ-weights $h_1, \ldots, h_n$ of a semistable weighted configuration of n points on the Tits boundary of the symmetric space $X = G/K$ satisfy for each $w \in W$ the inequality*

$$wh_1^\sharp \leq wh_2 + (h_3 + \cdots + h_n). \tag{19}$$

Moreover, this system of vector inequalities is equivalent to the geometric condition

$$h_1^\sharp \in h_2 + convex\ hull(W \cdot (h_3 + \cdots + h_n)). \tag{20}$$

Proof. We proved the first part already. It follows from (18) by applying the order preserving involution $h \mapsto h^\sharp$ of Δ_{euc} and renaming w. (Note that $(wh)^\sharp = -w_0 wh = (w_0 w w_0^{-1}) h^\sharp$.)

For the second part note that the system of inequalities (19) is equivalent to $W(h_1^\sharp - h_2) \subset (h_3 + \cdots + h_n) - \Delta^*$ and hence to

$$h_1^\sharp - h_2 \in \bigcap_{w \in W} w((h_3 + \cdots + h_n) - \Delta^*) = convex\ hull(W \cdot (h_3 + \cdots + h_n))$$

A proof of the last equality (in the case in which $h_3 + \cdots + h_n$ is in the interior of Δ) can be found in [BGW, pp. 138-140]. $\square$

In the case $G = GL(m, \mathbb{C})$ and $n = 3$ the weak stability inequalities are due to Wielandt [Wi] and their geometric interpretation to Lidskii [Li], see the discussion in the first chapter of [F2].

The simplest weak triangle inequality is inequality (19) corresponding to $w = e$, that is to the n-tuples of Schubert cycles in the Grassmannians $Grass_\zeta$ where one cycle is a point and the other $n-1$ are top-dimensional:

$$h_1^\sharp \leq h_2 + \cdots + h_n \tag{21}$$

This inequality is proven in [AM], compare inequality (2.28) therein.

Remark 3.35. In section 7 we will see for the simple complex groups of rank two that the weak stability inequalities are equivalent to the full stability inequalities, i.e. all non-weak stability inequalities are redundant in these cases. However, as the rank increases one would expect that most of the irredundant inequalities are non-weak. Indeed, irredundant non-weak inequalities can already be found in rank three among the inequalities for the group $Sp(6, \mathbb{C})$, see [KuLM, p. 187].

Remark 3.36. These two remarks will apply after section 5 where we relate the Δ-weights of configurations on $\partial_{Tits} X$ with the Δ-side lengths of polygons in X.

(i) Notice that the weak stability inequalities *depend only on the Weyl group* and not on further properties of the Schubert calculus, and therefore also their solution set $\mathcal{W}_n(X) \subset \Delta_{euc}^n$. After proving Theorem 5.9 and combining it with Theorem 1.3 of [KLM2] - compare our discussion in the introduction - we will know that also the solution set $\mathcal{P}_n(X) \subset \Delta_{euc}^n$ to the stability inequalities depends only on the Weyl group. This will imply that $\mathcal{P}_n(X) \subseteq \mathcal{W}_n(X)$, i.e. the weak stability inequalities are a consequence of the stability inequalities not only for a semisimple complex group but for any noncompact semisimple real Lie group G. (We are not claiming that the weak stability inequalities always are a subsystem of the stability inequalities although this seems to be true also.)

(ii) After proving Theorem 5.9 we will know that for the Δ-side lengths α, β, γ of an oriented geodesic triangle in X inequality (19) takes the form $w\alpha^\sharp \leq w\beta + \gamma$. Using the notation $\sigma(x, y)$ for the Δ-distance of the oriented segment $\overline{xy}$ introduced in section 5.1 we see that for any triple of points $x, y, z \in X$ and each $w \in W$ holds

$$w \cdot \sigma(x, z) \leq w \cdot \sigma(x, y) + \sigma(y, z).$$

(We use here that $\sigma(x, z) = \sigma(z, x)^\sharp$.) The special case (21) turns into a nice vector valued generalization of the ordinary triangle inequality:

$$\sigma(x, z) \leq \sigma(x, y) + \sigma(y, z)$$

4 Comparing stability with Mumford stability

To justify our notion of stability for measures on topological spherical Tits buildings, cf. [KLM2, Definition 4.1] and Definitions 3.11 and 3.27, we explain in this section that in the example of weighted n-point configurations on complex projective space $\mathbb{CP}^m$ our notion agrees with Mumford stability in geometric invariant theory.

Mumford introduced his notion of stability in order to construct good quotients for certain algebraic actions on projective varieties. We start by recalling the definition and related concepts from geometric invariant theory, cf. [MFK, N].

Consider first the case of a linear action on projective space: Let V be a finite-dimensional complex vector space, let G be a connected complex reductive group and suppose that $\rho : G \to SL(V)$ is a linear representation with finite kernel. According to Mumford a nonzero vector $v \in V$ is called *unstable* if $0 \in \overline{Gv}$ and *semistable* otherwise. One obtains a notion of stability for the orbits of the projectivized action $G \curvearrowright \mathbb{P}(V)$.

The Hilbert-Mumford criterion asserts that one can test stability on one-parameter subgroups: v is unstable if and only if there exists a one-parameter group $\lambda \subset G$ such that $0 \in \overline{\lambda \cdot v}$, more precisely, if and only if there exists $\alpha \in \mathfrak{p} = i\mathfrak{k}$ such that $\lim_{t\to+\infty} e^{t\alpha}v = 0$. Let $v = \sum v_i$ be a decomposition of v into eigenvectors for α and let $a_i \in \mathbb{R}$ be the corresponding weights, i.e. $e^{t\alpha}v = \sum e^{a_i t}v_i$. Then

$$\lim_{t\to+\infty} e^{-t\mu([v],\alpha)} \cdot e^{t\alpha}v \tag{22}$$

exists and is non-zero where $\mu([v],\alpha) := \max\{a_i | v_i \neq 0\}$ is a so-called *numerical function*. Hence v is unstable if and only if $M([v]) < 0$ where M denotes the derived numerical function

$$M([v]) := \inf_{0 \neq \alpha \in i\mathfrak{k}} \frac{\mu([v],\alpha)}{\|\alpha\|} < 0$$

on $\mathbb{P}(V)$.

If $G \curvearrowright Y$ is an algebraic action of G on an abstract projective variety Y then one needs further data in order to be able to talk about stable orbits for this action. For instance, it would suffice to choose a projective embedding $Y \subseteq \mathbb{P}(V)$ together with a linearization $G \to SL(V)$ of the given action. Then a point $[v] \in Y$, respectively, its G-orbit is called semistable if the vector v is semistable in the above sense.

We restrict now to the special case of spaces of weighted configurations on complex projective space. Let us consider the diagonal action of $G = SL(m+1, \mathbb{C})$ on the projective variety $Y \cong \times_{i=1}^{n} \mathbb{CP}^m$ which we regard as the space of n-point configurations $\xi = (\xi_1, \ldots, \xi_n)$ on $\mathbb{CP}^m$. A choice of integral weights $r = (r_1, \ldots, r_n)$ determines a natural projective embedding of Y. Namely, put $W = \mathbb{C}^{m+1}$ and $W^{\otimes r} = \otimes_{i=1}^{n} W^{\otimes r_i}$. The map $\iota : W^n \to W^{\otimes r}$ given by

$$\iota(w) = \iota(w_1, \cdots, w_n) = w_1^{\otimes r_1} \otimes \cdots \otimes w_n^{\otimes r_n}$$

induces the Segre embedding $Y \hookrightarrow \mathbb{P}(W^{\otimes r})$. The natural G-action on $W^{\otimes r}$ linearizes the given action $G \curvearrowright Y$. The configuration $([w_1], \ldots, [w_n]) \in Y$ is Mumford semistable (with respect to the chosen embedding and linearization) if and only if the G-orbit

$$g \mapsto g(w_1^{\otimes r_1} \otimes \cdots \otimes w_n^{\otimes r_n}) = (gw_1)^{\otimes r_1} \otimes \cdots \otimes (gw_n)^{\otimes r_n}$$

does not accumulate at 0, that is, if and only if the orbital distance function

$$\psi_w(g) := \|g(w_1^{\otimes r_1} \otimes \cdots \otimes w_n^{\otimes r_n})\| = \|gw_1\|^{r_1} \cdots \|gw_n\|^{r_n}$$

is bounded away from zero. Here lengths are measured with respect to a fixed Hermitian form on W and the induced Hermitian form on $W^{\otimes r}$.

Let us compare this with our notion of stability. We may view $\xi = ([w_1], \ldots, [w_n])$ as a weighted configuration on the ideal boundary of the symmetric space $X = G/K$, $K = SU(m+1)$, since $\mathbb{CP}^m$ canonically identifies with a maximally singular G-orbit on $\partial_\infty X$. Moreover, we may choose the norm $\|\cdot\|$ on W to be K-invariant. The connection between the orbital distance function ψ_w and weighted Busemann functions on X is based on the fact - cf. Example 2.2 - that after suitable normalization of the metric on X we have $\log \|g^{-1} w_j\| = b_{[w_j]}(gK)$ modulo additive constants. Therefore

$$\log \psi_w(g^{-1}) + const = \sum_{j=1}^{n} r_j \cdot b_{[w_j]}(gK) = b_\mu(gK)$$

where $\mu = \sum_{j=1}^{n} r_j \cdot \delta_{[w_j]}$ is the measure associated to the weighted configuration ξ. Thus the configuration ξ is semistable in Mumford's sense if and only if the weighted Busemann function b_μ is bounded below. If b_μ is bounded below then $slope_\mu \geq 0$ on $\partial_\infty X$, that is, ξ is semistable in our sense, cf. Definition 3.11.

Both notions of (semi)stability are in fact equivalent in this case. It is not hard to show this directly (cf. Remark 3.21). It also follows from the Hilbert-Mumford criterion together with the observation that the numerical function $\mu(\xi, \cdot)$ is essentially the slope function $slope_\mu$. Namely, the existence of the limit (22) implies that

$$b_\mu(e^{t\alpha}K) = \log \psi_w(e^{-t\alpha}) + const = \mu(\xi, -\alpha) \cdot t + O(1)$$

where $\mu(\xi, \alpha)$ is the maximal weight a_i for the action of $\alpha \in i \cdot \mathfrak{su}(m+1)$ on $W^{\otimes r}$ such that the corresponding component of $w^{\otimes r}$ is non-zero. The lines $t \mapsto e^{t\alpha}K$ are the geodesics in X through the base point o fixed by K.

5 Relating polygons and configurations

5.1 The Δ-side lengths of oriented polygons in X and $\mathfrak{p}$

The equivalence classes of oriented geodesic segments in the symmetric space $X = G/K$ modulo the natural G-action by isometries are parameterized by the Euclidean Weyl chamber Δ_{euc} associated to X,

$$G \backslash X \times X \cong \Delta_{euc}.$$

The vector $\sigma(x, y) \in \Delta_{euc}$ corresponding to an oriented segment $\overline{xy}$ can hence be thought of as a vector-valued length. We call it the Δ-*length* of the oriented segment.

We think of $X^{\mathbb{Z}/n\mathbb{Z}}$, $n \geq 3$, as the space of oriented closed n-gons in X. An n-tuple $(x_1, \ldots, x_n)$ is interpreted as the polygon with vertices $x_1, \ldots, x_n$ and the i-th edge $\overline{x_{i-1}x_i}$ is denoted by e_i. (Recall that any two points in X are connected by a unique geodesic segment.) In the sequel, all polygons will be assumed to be *oriented*.

We denote by

$$\sigma : X^{\mathbb{Z}/n\mathbb{Z}} \longrightarrow \Delta_{euc}^n, \quad (x_1, \ldots, x_n) \mapsto (\sigma(x_0, x_1), \ldots, \sigma(x_{n-1}, x_n))$$

the side length map. We are interested in its image

$$\mathcal{P}_n(X) \subset \Delta_{euc}^n.$$

We are also interested in the analogous problems for the infinitesimal symmetric space $T_oX \cong \mathfrak{p}$ associated to G - compare the terminology and notation from section 2.5 - namely to study the image

$$\mathcal{P}_n(\mathfrak{p}) \subset \Delta_{euc}^n$$

of the natural Δ-side length map $\sigma' : \mathfrak{p}^{\mathbb{Z}/n\mathbb{Z}} \longrightarrow \Delta_{euc}^n$.

To solve both problems, we will now translate them into a question about weighted configurations at infinity which has been treated in section 3. Namely, we will prove that the Δ-side length spaces $\mathcal{P}_n(X)$ and $\mathcal{P}_n(\mathfrak{p})$ coincide with the space of possible weights of semistable configurations on $\partial_{Tits}X$. The relation between polygons in X, respectively, in $\mathfrak{p}$ and weighted configurations on $\partial_{Tits}X$ is established by a Gauss map type construction as in [KLM2, sec. 4.2], respectively, by radial projection.

5.2 Relating polygons in $\mathfrak{p}$ and configurations on $\partial_{Tits}X$

An n-tuple $e = (e_1, \ldots, e_n) \in \mathfrak{p}^n$ can be interpreted as an *open n-gon* in $\mathfrak{p}$ with vertices $v_i = \sum_{j \leq i} e_j$, $0 \leq i \leq n$ and edges e_i. The n-gon closes up if and only if the *closing condition*

$$e_1 + \cdots + e_n = 0 \tag{23}$$

holds. (The distinction between open and closed n-gons will be restricted to this section. Elsewhere in this paper n-gons are supposed to be closed.)

An open n-gon e corresponds to a weighted n-point configuration ψ on $\partial_{Tits}X$ by assigning to each non-zero edge e_i the mass $m_i := \|e_i\|$ located at the ideal point ξ_i corresponding to e_i under the radial projection $\mathfrak{p} - \{0\} \to \partial_\infty X$. In the case $e_i = 0$ we set $m_i = 0$ and choose ξ_i arbitrarily. Note that the Δ-weights of ψ equal the Δ-side lengths of e. This is what makes this correspondence between polygons and weighted configurations useful for us.

For a unit vector $v \in \mathfrak{p}$ and the corresponding ideal point $\eta \in \partial_\infty X$ holds $\nabla b_\eta(o) = -v$. Hence

$$e_1 + \cdots + e_n = -\nabla b_\mu(o).$$

The closing condition (23) is therefore equivalent to o being a minimum of b_μ. (Since b_μ is a convex function, its critical points are global minima.) This implies:

Lemma 5.1. *The weighted configurations on $\partial_{Tits}X$ corresponding to closed polygons in $\mathfrak{p}$ are nice semistable.*

Conversely, if ψ is nice semistable with Δ-weights h, we may use the natural G-action on weighted configurations to move a critical point of b_μ - which exists by Lemma 3.18 - to the base point o. Thus:

Lemma 5.2. *Let ψ be a nice semistable weighted configuration on $\partial_{Tits}X$. Then its G-orbit contains a configuration which corresponds to a closed polygon.*

Given a semistable configuration, Lemma 3.20 tells that the closure of its G-orbit contains a nice semistable configuration which then has the same Δ-weights. Hence the Δ-weights of semistable configurations occur also for nice semistable configurations and we conclude:

Theorem 5.3. *For $h \in \Delta^n_{euc}$ there exist closed n-gons in $\mathfrak{p}$ with Δ-side lengths h if and only if there exist semistable weighted configurations on $\partial_{Tits}X$ with Δ-weights h.*

Let us briefly specialize to the case when G is a *complex* semisimple Lie group. We then have $\mathfrak{p} = i\mathfrak{k}$ and may identify $\mathfrak{k}$ and $\mathfrak{p}$ as K-modules. K acts on $\mathfrak{k}$ by the adjoint action.

Given $h = (h_1, \dots, h_n) \in \Delta^n_{euc}$ we let $\mathcal{O}_i$ denote the K-orbit in $\mathfrak{k}$ corresponding to h_i. The product space $\mathcal{O}_1 \times \cdots \times \mathcal{O}_n$ is naturally identified with the space of open n-gons in $\mathfrak{p}$ with fixed Δ-side lengths h. All K-orbits $\mathcal{O}$ in $\mathfrak{k}$ carry natural invariant symplectic structures. It is a standard fact that the momentum maps for the actions $K \curvearrowright \mathcal{O}$ are given by the embeddings $\mathcal{O} \hookrightarrow \mathfrak{k}$ where one identifies $\mathfrak{k}^* \cong \mathfrak{k}$ via the Killing form. Hence:

Lemma 5.4. *The diagonal action $K \curvearrowright \mathcal{O}_1 \times \cdots \times \mathcal{O}_n$ is Hamiltonian with momentum map $\mathcal{O}_1 \times \cdots \times \mathcal{O}_n \to \mathfrak{k} \cong \mathfrak{p}$ given by*

$$m(e) = \sum_{i=1}^{n} e_i.$$

We see that the closing condition (23) amounts in this situation to the momentum zero condition from symplectic geometry, cf. [Ki].

5.3 Relating polygons in X and configurations on $\partial_{Tits}X$

We prove results analogous to those in section 5.2 for polygons in the symmetric space $X = G/K$. They are more difficult to obtain because the closing condition is nonabelian and not directly related to the differential of the weighted Busemann function. We consider only closed polygons.

Let P be a n-gon in X, i.e. a map $P : \mathbb{Z}/n\mathbb{Z} \to X, i \mapsto x_i$. Its side lengths $m_i = d(x_{i-1}, x_i)$ determine a measure ν on $\mathbb{Z}/n\mathbb{Z}$ by putting $\nu(i) = m_i$. and P gives rise to a *Gauss map*

$$\psi : \mathbb{Z}/n\mathbb{Z} \longrightarrow \partial_{Tits}X$$

by assigning to i an ideal point $\xi_i \in \partial_{Tits}X$ so that the ray $\overline{x_{i-1}\xi_i}$ passes through x_i; the point ξ_i is unique unless $x_{i-1} = x_i$. This construction, in the case of hyperbolic plane, already appears in the letter of Gauss to W. Bolyai [G]. Taking into account the measure ν, we view ψ as a weighted configuration on $\partial_{Tits}X$. The Δ-weights of ψ equal the Δ-side lengths of the polygon P.

The next result shows that again the configurations arising from polygons are characterized by a stability property, namely they are nice semistable. That they are semistable is proven in Lemma 4.3 of [KLM2] in a more general situation (and this is actually all we need in the proof of our main results, cf. Theorem 5.9).

Lemma 5.5 (Nice semistability of Gauss map). *The weighted configurations on $\partial_{Tits}X$ arising as Gauss maps of closed polygons in X are nice semistable.*

Proof. For the convenience of the reader we first reproduce the proof of Lemma 4.3 of [KLM2] to show semistability. Let $\gamma_i : [0, m_i] \to X$ be unit speed parameterizations of the sides $\overline{x_{i-1}x_i}$ of the polygon P and let $\eta \in \partial_{Tits}X$ be an ideal point. The derivative of the Busemann function b_η along γ_i is given by

$$\frac{d}{dt}(b_\eta \circ \gamma_i)(t) = -\cos \angle_{\gamma_i(t)}(\xi_i, \eta) \le -\cos \angle_{Tits}(\xi_i, \eta), \tag{24}$$

compare formula (4) in section 2.4. Integrating along γ_i we obtain

$$b_\eta(x_i) - b_\eta(x_{i-1}) \le -m_i \cdot \cos \angle_{Tits}(\xi_i, \eta)$$

and summation over all sides yields

$$0 \le -\sum_{i \in \mathbb{Z}/n\mathbb{Z}} m_i \cdot \cos \angle_{Tits}(\xi_i, \eta) = slope_\mu(\eta).$$

confirming the semistability of the measure $\mu = \psi_*\nu$ and the configuration ψ.

Regarding *nice* semistability, suppose that ψ is not stable, i.e. that $S := \{slope_\mu = 0\}$ is non-empty. For an ideal point $\eta \in S$ we have equality in (24), that is, b_η is linear along every segment γ_i. Denote by l_i the line passing through x_i and asymptotic to η. Lemma 2.1 implies for $i = 1, \dots, n$ that the two lines l_{i-1} and l_i are parallel. It follows that the polygon P is contained in the parallel set $P(l_1) = \cdots = P(l_n)$. Moreover, μ is supported on its ideal boundary, We denote by $\hat{\eta}$ the other ideal endpoint of the lines l_i. Since μ is supported on $\partial_\infty P(l_i)$, the Busemann function b_μ is linear on all lines (parallel to) l_i and hence $slope_\mu(\hat{\eta}) = -slope_\mu(\eta) = 0$, i.e. $\hat{\eta} \in S$.

Recall that S is a convex subcomplex of $\partial_{Tits}X$ because μ is semistable. We may choose η maximally regular in S, that is, as an interior point of a top-dimensional simplex $\sigma \subset S$. Then S contains the convex hull s of σ and $\hat{\eta}$ which is a top-dimensional unit sphere in S, $\dim(s) = \dim(\sigma) = \dim(S)$. Thus μ is nice semistable according to Definition 3.13. $\square$

We are now interested in finding polygons with prescribed Gauss map. Such polygons will correspond to the fixed points of a certain weakly contracting self map of X, compare section 4.3 in [KLM2].

For $\xi \in \partial_{Tits}X$ and $t \geq 0$, we define the map $\phi_{\xi,t} : X \to X$ by sending x to the point at distance t from x on the geodesic ray $\overline{x\xi}$. Since X is nonpositively curved, the function $\delta : t \mapsto d(\phi_{\xi,t}(x), \phi_{\xi,t}(y))$ is convex. It is also bounded because the rays $\overline{x\xi}$ and $\overline{y\xi}$ are asymptotic, and hence it is monotonically non-increasing in t. This means that the maps $\phi_{\xi,t}$ are weakly contracting, i.e. they are 1-Lipschitz. For the weighted configuration ψ we define the weak contraction

$$\Phi_\psi : X \longrightarrow X$$

as the composition $\phi_{\xi_n,m_n} \circ \cdots \circ \phi_{\xi_1,m_1}$. The fixed points of Φ_ψ are the n-th vertices of polygons $P = x_1 \dots x_n$ with Gauss map ψ. The next result is the counterpart of Lemma 5.2.

Proposition 5.6. *If the weighted configuration ψ on $\partial_{Tits}X$ is nice semistable then the weak contraction $\Phi_\psi : X \to X$ has a fixed point.*

Proof. We will use the following auxiliary result which extends Cartan's fixed point theorem for isometric actions on Hadamard spaces with bounded orbits.

Lemma 5.7 ([KLM2, Lemma 4.6])**.** *Let Y be a Hadamard space and $\Phi : Y \to Y$ a 1-Lipschitz self map. If the forward orbits $(\Phi^n y)_{n\geq 0}$ are bounded then Φ has a fixed point in Y.*

It therefore suffices to show that the dynamical system $\Phi_\psi : X \to X$ has a bounded forward orbit $(\Phi_\psi^n(p))_{n\geq 0}$. Suppose that this is false.

Step 1. Our assumption that Φ_ψ does not have a bounded forward orbit implies that Φ_ψ does not map any bounded subset of X into itself. Pick a base point $o \in X$. Since no metric ball centered at o is mapped into itself there is a sequence of points x_n with $d(x_n, o) \to \infty$ which is "pulled away" from o in the sense that

$$d(\Phi_\psi(x_n), o) > d(x_n, o).$$

Since Φ_ψ is 1-Lipschitz we have in fact $d(\Phi_\psi(x), o) > d(x, o)$ for all points x on all segments $\overline{ox_n}$. Since X is locally compact, after passing to a subsequence, the segments $\overline{ox_n}$ Hausdorff converge to a geodesic ray $\rho([0,+\infty)) = \overline{o\xi}$. This ray is "pulled away" from o in the sense that for each $t \geq 0$ holds

$$d(\Phi_\psi(\rho(t)), o) \geq d(\rho(t), o). \tag{25}$$

Step 2. For any unit speed geodesic ray $\rho_0 : [0,+\infty) \to X$ we claim that

$$\lim_{t\to+\infty} d(\Phi_\psi(\rho_0(t)), o) - d(\rho_0(t), o) = -slope_\mu(\eta) \tag{26}$$

where η is the ideal endpoint of ρ_0 and $\mu = \psi_*\nu$ the measure associated to the weighted configuration ψ. To verify this we first look for a ray ρ_1 such that $d(\phi_{\xi_1,m_1}(\rho_0(t)), \rho_1(t)) \to 0$ for $t \to +\infty$. (Note that $\phi_{\xi_1,m_1} \circ \rho_0$ is in general no geodesic ray.) There exists a geodesic line l_1 asymptotic to η such that $\xi_1 \in \partial_\infty P(l_1)$. Inside the parallel set $P(l_1)$ there is a unique ray $\hat\rho_0$ strongly asymptotic to ρ_0. The weak contraction ϕ_{ξ_1,m_1} maps lines parallel to l_1 again to such lines and $\rho_1 := \phi_{\xi_1,m_1} \circ \hat\rho_0$ is a geodesic ray with the desired property. Since

$$b_\eta \circ \phi_{\xi_1,m_1} - b_\eta \equiv -m_1 \cdot \cos \angle_{Tits}(\eta, \xi_1) \qquad \text{on } P(l_1) \tag{27}$$

we have $b_\eta \circ \rho_1(t) - b_\eta \circ \rho_0(t) \to -m_1 \cdot \cos \angle_{Tits}(\eta, \xi_1)$ for $t \to +\infty$.

Proceeding by induction, we find rays $\rho_1, \ldots, \rho_n$ asymptotic to η such that for $i = 1, \ldots, n$ holds

$$d(\phi_{\xi_i,m_i}(\rho_{i-1}(t)), \rho_i(t)) \to 0$$

and

$$b_\eta \circ \rho_i(t) - b_\eta \circ \rho_{i-1}(t) \to -m_i \cdot \cos \angle_{Tits}(\eta, \xi_i)$$

as $t \to +\infty$. It follows using the weak contraction property that

$$d(\Phi_\psi(\rho_0(t)), \rho_n(t)) \to 0$$

and

$$b_\eta \circ \rho_n(t) - b_\eta \circ \rho_0(t) \to slope_\mu(\eta)$$

as $t \to +\infty$. Note that with the normalization $b_\mu(o) = 0$ any ray $\tilde\rho$ asymptotic to η satisfies $d(o, \tilde\rho(t)) + b_\mu(\tilde\rho(t)) \to 0$ as $t \to +\infty$. As a consequence we obtain (26).

Step 3. Suppose first that the configuration ψ is *stable*. Choosing $\rho_0 = \rho$ we obtain in view of $slope_\mu(\xi) > 0$ a contradiction between (26) and (25).

We are left with the case that the configuration ψ is nice semistable but *not stable*. According to Definition 3.13, the d-dimensional convex subcomplex $\{slope_\mu = 0\}$ of $\partial_{Tits}X$ contains a unit d-sphere s. We consider a $(d+1)$-flat $f \subset X$ such that $\partial_\infty f = s$ and inside f a maximally regular geodesic line l. Then $P(f) = P(l)$. Since b_μ is constant on f, Lemma 3.7 implies that the measure μ is supported on $\partial_\infty P(f)$.

For any geodesic $l' \subset f$ we therefore have that μ is supported on $\partial_\infty P(l') \supseteq \partial_\infty P(f)$. Denoting the two ideal endpoints of l' in s by $\eta'_\pm$ we obtain as in (27) that $b_{\eta'_\pm} \circ \Phi_\psi - b_{\eta'_\pm} \equiv slope_\mu(\eta'_\pm) = 0$ on $P(l')$, that is, Φ_ψ preserves each cross section $\{pt\} \times CS(l')$ of $P(l') \cong l' \times CS(l')$. Since this holds for all geodesics l' in f we conclude that Φ_ψ preserves each cross section $\{pt\} \times CS(f)$ of $P(f) \cong f \times CS(f)$.

Let Z be one of the cross sections. As in step 1, there exists a ray ρ in Z satisfying (25). Note that $slope_\mu > 0$ on $\partial_\infty Z$. Otherwise $\{slope_\mu = 0\}$ would contain a hemisphere of dimension $dim(s) + 1$ which is impossible because $dim(\{slope_\mu = 0\}) = dim(s)$, compare the proof of Lemma 3.18. As in the case when μ is stable, see the beginning of this step, we obtain a contradiction. This concludes the proof of the Proposition. $\square$

Remark 5.8. If the configuration ψ is stable then Φ_ψ has a *unique* fixed point. Namely, if x and x' are different fixed points on a line l then Φ_ψ restricts on l to an isometry. It follows that μ is supported on $\partial_\infty P(l)$ and hence not stable.

Analogous to Theorem 5.3 we obtain:

Theorem 5.9. *For $h \in \Delta^n_{euc}$ there exist closed n-gons in X with Δ-side lengths h if and only if there exist semistable weighted configurations on $\partial_{Tits} X$ with Δ-weights h.*

Proof. This is a direct consequence of Lemma 5.5, Proposition 5.6 and the fact already used above that the existence of semistable configurations on $\partial_{Tits} X$ with Δ-weights h implies the existence of nice semistable ones, cf. Lemma 3.20. $\square$

Combining Theorems 5.3 and 5.9 we obtain Theorem 1.1 stated in the introduction. It generalizes the Thompson Conjecture [Th] which was formulated for the case of $G = GL(n, \mathbb{C})$. Special cases were obtained in [Kly2] and [AMW]. Another proof in the general case has recently been given in [EL].

6 The stable measures on the Grassmannians of the classical groups

In section 3.3 we defined stability for measures of finite total mass on the ideal boundary of symmetric spaces of noncompact type. In this section we will make the stability condition explicit in the case of the classical groups. We will restrict ourselves to measures supported on a maximally singular orbit, that is, to measures supported on the (generalized) Grassmannians associated to the classical groups.

6.1 The special linear groups

Let $G = SL(n, \mathbb{C})$ and let $X = SL(n, \mathbb{C})/SU(n)$ be the associated symmetric space. Recall that the Tits boundary $\partial_{Tits}X$, as a spherical building, is combinatorially equivalent to the complex of flags of proper non-zero linear subspaces of $\mathbb{C}^n$. The vertices of $\partial_{Tits}X$ correspond to the linear subspaces and the simplices to the partial flags of such, compare [Br, p. 120].

Let μ be a measure with finite total mass supported on an orbit of vertices $G\eta$ in $\partial_{Tits}X$. Such an orbit is identified with the Grassmannian $G_q(\mathbb{C}^n)$ of q-planes for some q, $1 \leq q \leq n-1$. (This identification is a homeomorphism with respect to the cone topology, cf. section 2.4.) The main issue in determining $slope_\mu$ and evaluating the system of inequalities (12) given in section 3.3 is to compute the distances between vertices in $\partial_{Tits}X$. We denote by $[U]$ the vertex in $\partial_{Tits}X$ corresponding to the subspace U, and for non-trivial linear subspaces $U, V \subset \mathbb{C}^n$ we introduce the auxiliary function

$$\dim_U(V) := \frac{\dim(U \cap V)}{\dim(U)} \qquad .$$

Lemma 6.1. *The distance between the vertices of $\partial_{Tits}X$ corresponding to proper non-zero linear subspaces $U, V \subset \mathbb{C}^n$ is given by*

$$\cos \angle_{Tits}([U], [V]) = C \cdot (\dim_U(V) - \frac{1}{n} \dim(V)) \tag{28}$$

where $C = C(\dim(U), \dim(V), n)$ is a positive *constant.*

Proof. Let $p = \dim(U)$, $q = \dim(V)$, $s = \dim(U \cap V)$ and choose a basis $e_1 \dots e_n$ of $\mathbb{C}^n$ so that $e_1 \dots e_p$ is a basis of U and $e_{p-s+1} \dots e_{p+q-s}$ is a basis of V. The splitting $\mathbb{C}^n = \langle e_1 \rangle \oplus \dots \oplus \langle e_n \rangle$ determines a maximal flat F in X (of dimension $n-1$) in the sense that the vertices of the apartment $\partial_{Tits}F \subset \partial_{Tits}X$ correspond to the non-trivial linear subspaces of $\mathbb{C}^n$ spanned by some of the e_i. Let us denote by $\hat{e}_i$ the unit vector field in F pointing towards $[\langle e_i \rangle] \in \partial_{Tits}F$. Then, since $\hat{e}_1 + \dots + \hat{e}_n = 0$, we find by symmetry that $\hat{e}_i \cdot \hat{e}_j = -\frac{1}{n-1}$ for $i \neq j$. Note that the vector field $\hat{e}_{i_1} + \dots + \hat{e}_{i_k}$ points towards the vertex $[\langle e_{i_1} \dots e_{i_k} \rangle] \in \partial_{Tits}F$. It follows that the Tits distance between the vertices $[U]$ and $[V]$ is given by the angle between the vector fields $\hat{e}_1 + \dots + \hat{e}_p$ and $\hat{e}_{p-s+1} + \dots + \hat{e}_{p+q-s}$ whose cosine equals

$$\sqrt{\frac{n-1}{p(n-p)}} \sqrt{\frac{n-1}{q(n-q)}} \, (pq \frac{-1}{n-1} + s \frac{n}{n-1}) = const(p, q, n) \cdot (\frac{s}{p} - \frac{q}{n})$$

whence (28). □

The slope function $slope_\mu$ then takes the following form, cf. (10):

$$slope_\mu([U]) = -\int_{G_q(\mathbb{C}^n)} \cos\angle_{Tits}([U],[V])\, d\mu([V]).$$

Hence $slope_\mu([U]) \geq 0$ if and only if

$$\int_{G_q(\mathbb{C}^n)} \dim_U(V)\, d\mu([V]) \leq \frac{q}{n}\|\mu\|. \tag{29}$$

We conclude using Corollary 3.10:

Proposition 6.2. *A measure μ with finite total mass on the Grassmannian $G_q(\mathbb{C}^n)$ is semistable if and only if (29) holds for all proper non-zero linear subspaces $U \subset \mathbb{C}^n$. It is stable if and only if all inequalities hold strictly.*

Example 6.3. Let μ be a measure with finite total mass on complex projective space $\mathbb{CP}^m = G_1(\mathbb{C}^{m+1})$. Then μ is semistable if and only if each d-dimensional projective subspace carries at most $\frac{d+1}{m+1}$ times the total mass. For instance, a measure on the complex projective line is semistable if and only if each atom carries at most half of the total mass.

6.2 The orthogonal and symplectic groups

Let us now consider the other families of classical groups $G = SO(n,\mathbb{C})$ and $G = Sp(2n,\mathbb{C})$. To streamline the notation we note that in either case G is the group preserving a non-degenerate bilinear form b on a finite-dimensional complex vector space W and a complex volume form. We denote by $Y = G/K$ the symmetric space associated to G. The spherical Tits building $\partial_{Tits}Y$ is combinatorially equivalent to the flag complex of non-zero b-isotropic subspaces of W, cf. [Br, pp. 123-126]. (In the case of $SO(2m,\mathbb{C})$ one may prefer to consider the natural *thick* building structure on $\partial_{Tits}Y$ combinatorially equivalent to the flag complex for the "oriflamme geometry", but for our considerations this does not make a difference.)

As before in the case of the special linear group the main task is to compute the distances between vertices of $\partial_{Tits}Y$. For an isotropic subspace $U \subset W$ we let (U) denote the corresponding vertex in $\partial_{Tits}Y$.

Lemma 6.4. *The distance between the vertices of $\partial_{Tits}Y$ corresponding to non-zero b-isotropic subspaces $U, V \subset W$ is given by*

$$\cos\angle_{Tits}((U),(V)) = C \cdot (\dim_U(V) + \dim_U(V^\perp) - 1)$$

where $C = C(\dim(U), \dim(V), \dim(W))$ is positive constant.

Proof. We will use the inclusion of G into the appropriate special linear group. It induces an isometric embedding $\iota : \partial_{Tits} Y \hookrightarrow \partial_{Tits} X$ of Tits boundaries,

$$\cos \angle_{Tits}((U),(V)) = \cos \angle_{Tits}(\iota((U)), \iota((V))).$$

Recall that the form b induces an isometric automorphism (polarity) of $\partial_{Tits} X$ which we will also denote b. It satisfies $b([U]) = [U^\perp]$. The point $\iota((U))$ is the midpoint of the edge in $\partial_{Tits} X$ joining the vertices $[U]$ and $[U^\perp]$. (These are interchanged by b, and they are connected by an edge because U is isotropic.)

There exists an apartment $a \subset \partial_{Tits} X$ containing the edge joining $[U]$ and $[U^\perp]$ and the edge joining $[V]$ and $[V^\perp]$. (We allow the possibility that $U = U^\perp$ or $V = V^\perp$.) Note that $\angle_{Tits}([U],[U^\perp])$ and $\angle_{Tits}([V],[V^\perp])$ depend only on $dim(W)$ and $dim(U)$ resp. $dim(V)$, cf. (28). Hence

$$\begin{aligned}\cos \angle_{Tits}(\iota((U)), \iota((V))) = C \cdot (&\cos \angle_{Tits}([U],[V]) + \cos \angle_{Tits}([U],[V^\perp]) \\ &+ \cos \angle_{Tits}([U^\perp],[V]) + \cos \angle_{Tits}([U^\perp],[V^\perp]))\end{aligned}$$

with a constant $C = C(dim(U), dim(V), dim(W)) > 0$. Since b is isometric, we have

$$\cos \angle_{Tits}([U^\perp],[V]) = \cos \angle_{Tits}([U],[V^\perp])$$

and

$$\cos \angle_{Tits}([U^\perp],[V^\perp]) = \cos \angle_{Tits}([U],[V]).$$

So

$$\cos \angle_{Tits}(\iota((U)), \iota((V))) = 2C \cdot (\cos \angle_{Tits}([U],[V]) + \cos \angle_{Tits}([U],[V^\perp]))$$

We now can appeal to our computation of Tits distances for the special linear group and deduce the assertion from (28). ☐

The Grassmannian $G^o_q(W,b)$ of b-isotropic q-planes in W embeds as a maximally singular G-orbit in $\partial_{Tits} Y$. Let μ be a measure on $G^o_q(W,b)$ with finite total mass. The slope function $slope_\mu$ takes on vertices of $\partial_{Tits} Y$ the form

$$slope_\mu((U)) = -C \cdot \int_{G^o_q(W,b)} (dim_U(V) + \dim_U(V^\perp) - 1)\, d\mu((V))$$

with $C = C(dim(U), q, dim(W)) > 0$. Using Corollary 3.10, we obtain:

Proposition 6.5. *Let b be a non-degenerate symmetric or alternating form on a finite-dimensional complex vector space W and let $G^o_q(W,b)$ be the Grassmannian of b-isotropic q-planes. A measure μ with finite total mass supported on $G^o_q(W,b)$ is semistable if and only if for every non-zero b-isotropic subspace $U \subset W$ holds*

$$\int_{G^o_q(W,b)} (\dim_U(V) + \dim_U(V^\perp))\, d\mu([V]) \leq \|\mu\|.$$

It is stable if and only if all inequalities hold strictly.

7 The Δ-side length polyhedra for the rank two root systems

In this section, we make the stability inequalities given in Theorem 1.3 explicit for the semisimple complex Lie groups of rank two. Since the Δ-side length polyhedron depends only on the spherical Coxeter complex, respectively, on the root system $\mathcal{R}$, cf. Theorem 1.2, we will also denote it by $\mathcal{P}_n(\mathcal{R})$. For the root system $\mathcal{R}$ there are three possible cases, $A_1 \times A_1$, A_2, $B_2 = C_2$ and G_2, and the corresponding semisimple complex Lie algebras $\mathfrak{g}$ are $sl(2,\mathbb{C}) \times sl(2,\mathbb{C}), sl(3,\mathbb{C}), so(5,\mathbb{C})$ and g_2.

For $n = 3$, i.e. in the case of triangles, we shall see that the systems of inequalities for $B_2 = C_2$ and G_2 are redundant. Using the computer program Porta we will describe the irredundant subsystems. It turns out that in all rank two cases the redundant inequalities are precisely the non-weak triangle inequalities in the sense of section 3.8. We will also give generators for the polyhedral cones $\mathcal{P}_3(\mathcal{R})$ (again using Porta).

We first, briefly, discuss the rank 1 case. Then the Weyl group is $W = \mathbb{Z}/2$, $\Delta = \mathbb{R}_+$; the only Grassmannian is $\mathbb{CP}^1$, the only nontrivial product of Schubert classes is

$$[pt] \cdot [\mathbb{CP}^1] \cdot ... \cdot [\mathbb{CP}^1] = [pt].$$

The class $[pt]$ corresponds to $w = 1 \in W$ and the class $[\mathbb{CP}^1]$ corresponds to the generator -1 of W. Therefore, the stability inequalities take the form

$$m_j + \sum_{i \neq j, i=1}^{n} (-m_i) \leq 0,$$

i.e.,

$$0 \leq m_j \leq \frac{1}{2} \sum_{i=1}^{n} m_i, \quad j = 1, ..., n.$$

These are, of course, the ordinary triangle inequalities for polygons in $\mathbb{R}^2$.

Another simple case is of the root system $\mathcal{R} = A_1 \times A_1$. The Weyl chamber $\Delta = (\mathbb{R}_+)^2$ is self-dual: $\Delta^* = \Delta$. By functoriality of the stability inequalities,

$$\mathcal{P}_n(\mathcal{R}) = \mathcal{P}_n(A_1) \times \mathcal{P}_n(A_1).$$

Therefore, in view of the triangle inequalities for A_1, we see that the stability inequalities for $\mathcal{P}_n(\mathcal{R})$ take the form:

$$h_i \leq \sum_{i=1, i \neq j}^{n} h_j, \quad h_i \in \Delta, \quad i = 1, ..., n,$$

where the order $\leq$ is the dominance order. Hence, all the stability inequalities in this case are weak. We now proceed to the irreducible rank 2 root systems.

7.1 Setting up the notation

Let G be a simple complex Lie group with finite center and Lie algebra isomorphic to $\mathfrak{g}$.

For the root systems A_2, $B_2 = C_2$ and G_2 the Weyl group W is a dihedral group of order 6, 8 and 12 respectively. In each case there are two (generalized) Grassmannians G/P_i associated to G. They correspond to the two vertices ζ_1 and ζ_2 of the spherical Weyl chamber (arc) Δ_{sph}. After identifying the spherical model Weyl chamber Δ_{sph} with a chamber in the Tits boundary $\partial_{Tits}X$ of the associated symmetric space $X = G/K$ (which is the spherical Tits building attached to G) we may choose the maximal parabolic subgroup P_i as the stabilizer of ζ_i in G, compare the discussion in the introduction preceding Theorem 1.3 and the notation used there.

If the order of the dihedral group is $2m$ with $m = 3, 4$ or 6 then the complex dimension of each Grassmannian is $m - 1$ and there are m Bruhat cells, one of each dimension between 0 and $m - 1$. It follows that the rational cohomology rings are polynomial algebras on a two-dimensional generator (the hyperplane section class).

Let w_i be the reflection in W fixing ζ_i and let λ_{P_i} be the unique fundamental weight fixed by w_i. The Schubert cycles in G/P_i are in one-to-one correspondence with the vertices of the spherical Coxeter complex in the W-orbit of ζ_i, respectively, with the maximally singular weights in the W-orbit of λ_{P_i}, We measure the word length in W with respect to the generating set $\{w_1, w_2\}$ and can thus speak of the length of a coset in $W^{P_i} := W/\{e, w_i\}$. For $j = 0, \ldots, m-1$ there is a unique coset of length j, a corresponding weight λ_j in the orbit $W\lambda_{P_i}$ and a corresponding Schubert cycle C_{λ_j}. We will abuse notation and also let C_{λ_j} denote the homology class carried by the Schubert cycle C_{λ_j}. Then C_{λ_j} is a generator of the infinite cyclic group $H_{2j}(G/P_i; \mathbb{Z})$. We let γ_{m-j-1} be the cohomology class which is the Poincaré dual of C_{w_j}. Thus γ_j is a generator of $H^{2j}(G/P_i; \mathbb{Z})$.

The system of stability inequalities divides into two subsystems, one for each Grassmannian. The inequalities for each of these subsystems are parameterized by a subset of the ordered partitions of $m - 1$ into n nonnegative integers. The partition $m - 1 = j_1 + \cdots + j_n$ gives rise to an inequality in the G/P_i-subsystem if and only if the product $\gamma_{j_1} \cdots \gamma_{j_n}$ is the fundamental class generating $H^{2m-2}(G/P_i; \mathbb{Z})$. Note that the weak triangle inequalities in the sense of section 3.8 correspond to those decompositions $\gamma_{m-1} = \gamma_j \cdot \gamma_k \cdot \gamma_l$ of the fundamental class where at least one factor has degree zero. The symmetric group S_n acts naturally on the set of inequalities by permuting the Δ-side lengths. Thus unordered partitions give rise to S_n-orbits of inequalities and in our examples below we will write down one representing inequality from each orbit.

Note that for $m-1 = j+k+l$ we have $\gamma_j \cdot \gamma_k = c^i_{jk}\, \gamma_{m-l-1}$ with certain nonnegative integers c^i_{jk} which we will refer to as the *structure constants* of the ring $H^*(G/P_i;\mathbb{Z})$. In the case $n = 3$ the inequality corresponding to the partition (j, k, l) occurs if and only if $c^i_{jk} = 1$. The cohomology rings $H^*(G/P_i;\mathbb{Z})$ may be easily calculated using Chevalley's formula, see Lemma 8.1 of [FW] or Theorem 6.1 of [TW] as was pointed out to us by Chris Woodward. In addition we have taken the cohomology rings for G_2 from [TW], page 20.

7.2 The polyhedron for A_2

We consider the group $G = SL(3,\mathbb{C})$. The Euclidean Weyl chamber is given by

$$\Delta_{euc} = \{(x, y, z) : x + y + z = 0 \wedge x \geq y \geq z\}.$$

The fundamental weights are

$$\lambda_{P_1}(x, y, z) = x \quad \text{and} \quad \lambda_{P_2}(x, y, z) = -z.$$

One Grassmannian is $\mathbb{CP}^2$ and the other is the dual projective space $(\mathbb{CP}^2)^\vee$.

In $\mathbb{CP}^2$ the 0-, 1- and 2-dimensional Schubert cycles $[pt]$, $[\mathbb{CP}^1]$ and $[\mathbb{CP}^2]$ correspond to the maximally singular coweights $(2,-1,-1)$, $(-1, 2,-1)$ and $(-1,-1, 2)$ and hence to the weights x, y and z. Similarly, the 0-, 1- and 2-dimensional Schubert cycles in $(\mathbb{CP}^2)^\vee$ correspond to the maximally singular coweights $(1, 1,-2)$, $(1,-2, 1)$ and $(-2, 1, 1)$, respectively, to the weights $-x$, $-y$ and $-z$. In both cases, all the structure constants are 1 and we get one inequality for each unordered partition of 2.

As we mentioned before the symmetric group S_n acts naturally on the set of inequalities by permuting the Δ-side lengths. The S_n-orbits of inequalities correspond to the ordered partitions of 2 and we will write down one representing inequality for each ordered partition.

The inequalities associated to $\mathbb{CP}^2$:

$$x_1 + z_2 + \cdots + z_n \leq 0$$
$$y_1 + y_2 + z_3 + \cdots + z_n \leq 0$$

The two inequalities correspond to the partitions $2+0+\cdots+0$ and $1+1+0+\cdots+0$ of 2, that is, to the one point intersections of Schubert cycles $[pt]\cdot[\mathbb{CP}^2]\cdots[\mathbb{CP}^2] = [pt]$ and $[\mathbb{CP}^1]\cdot[\mathbb{CP}^1]\cdot[\mathbb{CP}^2]\cdots[\mathbb{CP}^2] = [pt]$, respectively, to the decompositions $\gamma_2\gamma_0^{n-1} = \gamma_1^2\gamma_0^{n-2}$ of the fundamental class γ_2. Similarly, we have

The inequalities associated to $(\mathbb{CP}^2)^\vee$:

$$z_1 + x_2 + \cdots + x_n \geq 0$$
$$y_1 + y_2 + x_3 + \cdots + x_n \geq 0$$

In the case $n = 3$ all inequalities are weak triangle inequalities and moreover the system of inequalities is known to be irredundant, see [KTW].

The following 8 vectors are a set of generators of the polyhedral cone $\mathcal{P}_3(A_2)$ in the 6-dimensional space $\mathfrak{a}^3$.

(2,-1,-1) (2,-1,-1) (2,-1,-1)	(1,1,-2) (2,-1,-1) (0,0,0)
(0,0,0) (1,1,-2) (2,-1,-1)	(2,-1,-1) (0,0,0) (1,1,-2)
(0,0,0) (2,-1,-1) (1,1,-2)	(2,-1,-1) (1,1,-2) (0,0,0)
(1,1,-2) (0,0,0) (2,-1,-1)	(1,1,-2) (1,1,-2) (1,1,-2)

7.3 The polyhedron for $B_2 = C_2$.

We consider the group $G = SO(5)$. (Note that $SO(5)$ is isomorphic to $PSp(4)$.)

After a suitable change of coordinates we may write the invariant quadratic form on $\mathbb{C}^5$ as $q(u) = u_1u_5 + u_2u_4 + u_3^2$. The diagonal matrices with real eigenvalues $x, y, 0, -x, -y$ then form a Cartan subalgebra $\mathfrak{a}$ in $\mathfrak{g}$ and the Euclidean Weyl chamber is given by

$$\Delta_{euc} = \{(x, y) : x \geq y \geq 0\}.$$

The fundamental weights are

$$\lambda_{P_1}(x, y) = x \quad \text{and} \quad \lambda_{P_2}(x, y) = x + y.$$

The Grassmannian G/P_1 is the space of isotropic lines in $\mathbb{C}^5$. This is the smooth quadric three-fold $\mathcal{Q}_3$ in $\mathbb{CP}^4$ given by the equation $q(u) = 0$. The other Grassmannian G/P_2 is the space of totally-isotropic two-planes in $\mathbb{C}^5$.

For the Grassmannian G/P_1 the Schubert cycles of dimension $0, 1, 2$ and 3 correspond to the maximally singular coweights $(1, 0), (0, 1), (0, -1)$ and $(-1, 0)$, respectively, to the weights $\lambda = x, y, -y$ and $-x$:

λ	dim C_λ	PD C_λ
x	0	γ_3
y	1	γ_2
$-y$	2	γ_1
$-x$	3	1

The Schubert cycles and their homological intersections can still be determined by hand: The 2-dimensional Schubert cycle is a hyperplane section and the 1-dimensional Schubert cycle is an embedded projective line. In particular, the self intersection of the 2-cycle is *twice* the 1-cycle. The cohomology ring is given by the following table:

$H^*(\mathcal{Q}_3)$	1	γ_1	γ_2	γ_3
1	1	γ_1	γ_2	γ_3
γ_1	γ_1	$2\gamma_2$	γ_3	0
γ_2	γ_2	γ_3	0	0
γ_3	γ_3	0	0	0

We see that the structure constant $c_{1,1}^1$ equals 2, so any partition of 3 which involves the pair $(1,1)$ does not give rise to an inequality. Indeed, the only decompositions of the fundamental class into products of Schubert classes are $\gamma_3 = \gamma_2 \cdot \gamma_1$.

The inequalities associated to $G/P_1 = \mathcal{Q}_3$:

$$x_1 \leq x_2 + \cdots + x_n$$
$$y_1 - y_2 \leq x_3 + \cdots + x_n$$

For the Grassmannian G/P_2 the Schubert cycles of dimension $0, 1, 2$ and 3 correspond to the maximally singular coweights $(1,1)$, $(1,-1)$, $(-1,1)$ and $(-1,-1)$, respectively, to the weights $\lambda = x+y, x-y, -x+y$ and $-x-y$:

λ	dim C_λ	PD C_λ
$x+y$	0	γ_3
$x-y$	1	γ_2
$-x+y$	2	γ_1
$-x-y$	3	1

The cohomology ring of G/P_2 is easily determined due to the exceptional isomorphism $SO(5,\mathbb{C}) \cong PSp(4,C)$. (Recall that $Sp(4,\mathbb{C})$, the automorphism group of a complex symplectic 2-form ω on $\mathbb{C}^4$, acts on the 6-dimensional vector space $\Lambda^2(\mathbb{C}^4)^*$ of alternating bilinear forms on $\mathbb{C}^4$. The induced action $Sp(4,\mathbb{C}) \curvearrowright \Lambda^2(\mathbb{C}^4)^*$ preserves a natural non-degenerate quadratic form and the line generated by ω.) The Grassmannians associated to the groups $SO(5,\mathbb{C})$ and $Sp(4,C)$ are the same and one of the Grassmannians for $Sp(4,C)$ is $\mathbb{CP}^3$. Since $G/P_1 \cong \mathcal{Q}_3 \not\cong \mathbb{CP}^3$ we conclude that $G/P_2 \cong \mathbb{CP}^3$. Thus the structure constants for the cohomology ring of G/P_2 are given by the following table:

$H^*(G/P_2)$	1	γ_1	γ_2	γ_3
1	1	γ_1	γ_2	γ_3
γ_1	γ_1	γ_2	γ_3	0
γ_2	γ_2	γ_3	0	0
γ_3	γ_3	0	0	0

The possible decompositions of the fundamental class into products of Schubert classes are $\gamma_3 = \gamma_2 \cdot \gamma_1 = \gamma_1^3$.

The inequalities associated to G/P_2:

$$
\begin{aligned}
x_1 + y_1 &\le x_2 + y_2 + \cdots + x_n + y_n \\
x_1 - y_1 - x_2 + y_2 &\le x_3 + y_3 + \cdots + x_n + y_n \\
-x_1 + y_1 - x_2 + y_2 - x_3 + y_3 &\le x_4 + y_4 + \cdots + x_n + y_n
\end{aligned}
$$

Observe that this last inequality is redundant because it follows from $y \le x$ which is one of the defining inequalities of Δ_{euc}.

In the case $n = 3$, according to Porta the system obtained when the last (S_3-invariant) inequality is removed is irredundant. We observe that the redundant triangle inequalities are exactly the non-weak ones.

The following 12 vectors are a set of generators of the polyhedral cone $\mathcal{P}_3(B_2) = \mathcal{P}_3(C_2)$ in the 6-dimensional space $\mathfrak{a}^3$.

(1,1) (1,1) (2,0)	(1,0) (0,0) (1,0)	(1,1) (1,1) (0,0)
(1,1) (2,0) (1,1)	(1,0) (1,0) (0,0)	(1,0) (1,0) (1,1)
(2,0) (1,1) (1,1)	(0,0) (1,1) (1,1)	(1,0) (1,1) (1,0)
(0,0) (1,0) (1,0)	(1,1) (0,0) (1,1)	(1,1) (1,0) (1,0)

7.4 The polyhedron for G_2

In this case both Grassmannians have dimension 5.

We use non-rectangular linear coordinates on $\mathfrak{a}$ such that the Euclidean Weyl chamber is given by

$$\Delta_{euc} = \{(x, y) : x \ge 0, y \ge 0\}$$

and such that the standard basis vectors are the fundamental coweights. We require moreover that $\|(1,0)\| = \sqrt{3} \cdot \|(0,1)\|$. With respect to these coordinates the natural Euclidean metric on $\mathfrak{a}$ takes, up to scale, the form $3dx \otimes dx + \frac{3}{2} dx \otimes dy + \frac{3}{2} dy \otimes dx + dy \otimes dy$.

Let G/P_1 be the Grassmannian corresponding to the fundamental coweight $(1,0)$. The Schubert cycles of dimension $0, \ldots, 5$ correspond to the Weyl group orbit of maximally singular coweights $(1,0)$, $(-1,3)$, $(2,-3)$, $(-2,3)$, $(1,-3)$, $(-1,0)$, respectively, via the scalar product on $\mathfrak{a}$ up to a scale factor to the orbit of weights $2x+y$, $x+y$, x, $-x$, $-x-y$, $-2x-y$.

λ	dim C_λ	PD C_λ
$2x+y$	0	γ_5
$x+y$	1	γ_4
x	2	γ_3
$-x$	3	γ_2
$-x-y$	4	γ_1
$-2x-y$	5	1

The structure constants for the cohomology ring of G/P_1 are given by the following table [TW]:

$H^*(G/P_1)$	1	γ_1	γ_2	γ_3	γ_4	γ_5
1	1	γ_1	γ_2	γ_3	γ_4	γ_5
γ_1	γ_1	γ_2	$2\gamma_3$	γ_4	γ_5	0
γ_2	γ_2	$2\gamma_3$	$2\gamma_4$	γ_5	0	0
γ_3	γ_3	γ_4	γ_5	0	0	0
γ_4	γ_4	γ_5	0	0	0	0
γ_5	γ_5	0	0	0	0	0

The possible decompositions of the fundamental class into products of Schubert classes are $\gamma_5 = \gamma_4 \cdot \gamma_1 = \gamma_3 \cdot \gamma_2 = \gamma_3 \cdot \gamma_1^2$.

The inequalities associated to G/P_1:

$$\begin{aligned}
2x_1 + y_1 &\leq 2x_2 + y_2 + \cdots + 2x_n + y_n \\
x_1 + y_1 - x_2 - y_2 &\leq 2x_3 + y_3 + \cdots + 2x_n + y_n \\
x_1 - x_2 &\leq 2x_3 + y_3 + \cdots + 2x_n + y_n \\
x_1 - x_2 - y_2 - x_3 - y_3 &\leq 2x_4 + y_4 + \cdots + 2x_n + y_n
\end{aligned}$$

Note that the fourth group of inequalities is redundant. Indeed, we may obtain it from the first inequality by adding the inequality $0 \leq y_1 + y_2 + y_3 + 2x_4 + y_4 + \cdots + 2x_n + y_n$ which is implied by the inequalities defining Δ_{euc}.

We now describe the subsystem corresponding to G/P_2, the Grassmannian corresponding to the fundamental coweight $(0,1)$. The Schubert cycles of dimension $0,\ldots,5$ correspond to the Weyl group orbit of maximally singular coweights $(0,1)$, $(1,-1)$, $(-1,2)$, $(1,-2)$, $(-1,1)$, $(0,-1)$, respectively, to the orbit of weights $3x+2y$, $3x+y$, y, $-y$, $-3x-y$, $-3x-2y$.

λ	dim C_λ	PD C_λ
$3x+2y$	0	γ_5
$3x+y$	1	γ_4
y	2	γ_3
$-y$	3	γ_2
$-3x-y$	4	γ_1
$-3x-2y$	5	1

The structure constants for the cohomology ring of G/P_2 are given by the following table [TW]:

$H^*(G/P_2)$	1	γ_1	γ_2	γ_3	γ_4	γ_5
1	1	γ_1	γ_2	γ_3	γ_4	γ_5
γ_1	γ_1	$3\gamma_2$	$2\gamma_3$	$3\gamma_4$	γ_5	0
γ_2	γ_2	$2\gamma_3$	$2\gamma_4$	γ_5	0	0
γ_3	γ_3	$3\gamma_4$	γ_5	0	0	0
γ_4	γ_4	γ_5	0	0	0	0
γ_5	γ_5	0	0	0	0	0

The possible decompositions of the fundamental class into products of Schubert classes are $\gamma_5 = \gamma_4 \cdot \gamma_1 = \gamma_3 \cdot \gamma_2$.

The inequalities associated to G/P_2:

$$\begin{aligned}
3x_1 + 2y_1 &\leq 3x_2 + 2y_2 + \cdots + 3x_n + 2y_n \\
3x_1 + y_1 - 3x_2 - y_2 &\leq 3x_3 + 2y_3 + \cdots + 3x_n + 2y_n \\
y_1 - y_2 &\leq 3x_3 + 2y_3 + \cdots + 3x_n + 2y_n
\end{aligned}$$

Our system of inequalities becomes that of [BeSj, p. 458] once one replaces y_j in our inequalities with $3y_j$ and takes into account that they have extra inequalities corresponding to intersections of Schubert classes that are nonzero multiples of the point class that are not equal to 1.

In the case $n = 3$ we find using the program Porta that the system obtained by removing the S_3-orbit of redundant inequalities mentioned previously is an irredundant system. We observe that also in this case the redundant triangle inequalities are exactly the non-weak ones.

The following 24 vectors are a set of generators of the polyhedral cone $\mathcal{P}_3(G_2)$ in the 6-dimensional space $\mathfrak{a}^3$.

(0,3) (1,0) (2,0)	(1,0) (0,1) (0,2)	(0,1) (0,0) (0,1)
(0,3) (2,0) (1,0)	(1,0) (0,2) (0,1)	(0,1) (0,1) (0,0)
(1,0) (0,3) (2,0)	(0,2) (0,1) (1,0)	(0,0) (1,0) (1,0)
(1,0) (2,0) (0,3)	(0,2) (1,0) (0,1)	(1,0) (0,0) (1,0)
(2,0) (0,3) (1,0)	(0,3) (1,0) (1,0)	(1,0) (1,0) (0,0)
(2,0) (1,0) (0,3)	(1,0) (0,3) (1,0)	(0,1) (0,1) (1,0)
(0,1) (0,2) (1,0)	(1,0) (1,0) (0,3)	(0,1) (1,0) (0,1)
(0,1) (1,0) (0,2)	(0,0) (0,1) (0,1)	(1,0) (0,1) (0,1)

References

[AM] H. Abels and G. Margulis, *Coarsely geodesic metrics on reductive groups*, in: Modern dynamical systems and applications, Cambridge UP, 2004, 163–183.

[AMW] A. Alekseev, E. Meinrenken and C. Woodward, *Linearization of Poisson actions and singular values of matrix products*, Ann. Inst. Four. 51 (2001), 1691–1717.

[Ba] W. Ballmann, *Lectures on spaces of nonpositive curvature*, Birkhäuser 1995.

[Bs] A. Balser, *Polygons with prescribed Gauss map in Hadamard spaces and Euclidean buildings*, Canadian Math. Bull. 49 (2006), no. 3, 321–336.

[Bel] P. Belkale, *Local systems on $\mathbb{P}^1 - S$ for S a finite set*, Compos. Math. 129 no. 1 (2001), 67–86.

[BK] P. Belkale and S. Kumar, *Eigenvalue problem and a new product in cohomology of flag varieties*, Invent. Math. 166 (2006), no. 1, 185–228.

[BeSj] A. Berenstein and R. Sjamaar, *Projections of coadjoint orbits, moment polytopes, and the Hilbert-Mumford criterion*, J. Amer. Math. Soc. 13 (2000), 433–466.

[BGW] A.V. Borovik, I.M. Gelfand and N. White, *Coxeter matroids*, Birkhäuser Progress in Math. vol. 216, 2003.

[Bo] N. Bourbaki, "Lie groups and Lie algebras", Chap. 4, 5, 6. Springer Verlag, 2002.

[BH] M. Bridson and A. Haefliger, *Metric spaces of non-positive curvature*, Springer 1999.

[Br] K. Brown, *Buildings*, Springer 1988.

[BBI] D. Burago, Y. Burago, S. Ivanov, *A course in metric geometry*, Graduate Studies in Mathematics vol. 33, AMS, 2001.

[BuSp] K. Burns, R. Spatzier, *On topological Tits buildings and their classification*, Publ. IHES 65 (1987), 5–34.

[CE] J. Cheeger, D. Ebin, "Comparison theorems in Riemannian geometry." North-Holland Mathematical Library, Vol. 9. North-Holland Publishing Co., New York, 1975.

[DE] A. Douady and C. Earle, *Conformally natural extension of homeomorphisms of the circle,* Acta Math. 157 (1986), 23–48.

[Eb] P. Eberlein, *Geometry of nonpositively curved manifolds,* Chicago 1996.

[EL] S. Evens and J-H. Lu, *Thompson's conjecture for real semi-simple Lie groups*, in: *The breadth of symplectic and Poisson geometry*, Festschrift in honor of Alan Weinstein, Birkhäuser, Progress in Mathematics 232 (2005), 121-172.

[F1] W. Fulton, *Eigenvalue sums of Hermitian matrices (after A. Klyachko),* Séminaire Bourbaki, exposé 845, 1998.

[F2] W. Fulton, *Eigenvalues, invariant factors, highest weights, and Schubert calculus*, Bull. Amer. Math. Soc. 37 (2000), 209–249

[FW] W. Fulton and C. Woodward, *On the quantum product of Schubert classes*, J. Algebraic Geom. 13 (2004), no. 4, 641–661.

[G] C.F. Gauß, *Letter to W. Bolyai*, Collected works, vol. 8, pp. 222–223.

[GT] C. Gorodski and G. Thorbergsson, *Cycles of Bott–Samelson type for taut representations*, Ann. Global Anal. Geom. 21 (2002), no. 3, 287–302.

[HL] P. Heinzer and F. Loose, *Reduction of complex Hamiltonian G-spaces*, GAFA vol. 4 no. 3 (1994), 288–297.

[He] S. Helgason, *Differential geometry and symmetric spaces*, 1962.

[HR] U. Helmke and J. Rosenthal, *Eigenvalue inequalities and Schubert calculus*, Math. Nachrichten 171 (1995), 207–225.

[Hi] H. Hiller, *Geometry of Coxeter groups*, Pitman Research Notes in Mathematics, vol. 54 (1982).

[KLM2] M. Kapovich, B. Leeb and J. Millson, *Polygons in buildings and their refined side lengths*, preprint, November 2005.

[KLM3] M. Kapovich, B. Leeb and J. Millson, *The generalized triangle inequalities in symmetric spaces and buildings with applications to algebra*, Memoirs of the AMS, vol. 192, 2008.

[Ka] F.I. Karpelevič, *The geometry of geodesics and the eigenfunctions of the Beltrami-Laplace operator on symmetric spaces.* Trudy Moskov. Mat. Obšč. 14 48–185 (Russian); translated as Trans. Moscow Math. Soc. 1965 1967 pp. 51–199. Amer. Math. Soc., Providence, R.I., 1967.

[KN] G. Kempf and L. Ness, *The length of vectors in representation spaces,* in: Algebraic Geometry, Proceedings, Copenhagen 1978, Lecture Notes in Mathematics 732 (1979), 233–243.

[Ki] F.C. Kirwan, *Cohomology of quotients in symplectic and algebraic geometry*, Princeton 1984.

[Kl] S.L. Kleiman, *The transversality of a general translate,* Compositio Math. 28 (1974), 287–297.

[KlL] B. Kleiner and B. Leeb, *Rigidity of quasi-isometries for symmetric spaces and Euclidean buildings*, Pub. IHES 86 (1997), 115–197.

[Kly1] A. Klyachko, *Stable bundles, representation theory and Hermitian operators*, Selecta Mathematica 4 (1998), 419–445.

[Kly2] A. Klyachko, *Random walks on symmetric spaces and inequalities for matrix spectra*, Lin. Alg. Appl. 319 no. 1-3 (2000), 37–59.

[KT] A. Knutson and T. Tao, *The honeycomb model of $GL(n)$ tensor products I: Proof of the saturation conjecture*, J. Amer. Math. Soc. 12 (1999), 1055–1090.

[KTW] A. Knutson, T. Tao and C. Woodward *The honeycomb model of $GL(n)$ tensor products II: Puzzles determine the facets of the Littlewood-Richardson cone*, J. Amer. Math. Soc. 17 (2004), no. 1, 19–48.

[KuLM] S. Kumar, B. Leeb and J. Millson, *The generalized triangle inequalities for rank 3 symmetric spaces of noncompact type*, Contemporary Math. 332 (2003), Explorations in Complex and Riemannian Geometry, (volume dedicated to Robert Greene), 171–195.

[Le] B. Leeb, *A characterization of irreducible symmetric spaces and Euclidean buildings of higher rank by their asymptotic geometry*, Bonner Mathematische Schriften 326, Bonn 2000.

[Li] B.V. Lidskii, *Spectral polyhedron of a sum of two Hermitian matrices*, Funct. Anal. Appl. 16 (1982), 139–140.

[MZ] J. Millson and B. Zombro, *A Kähler structure on the moduli space of isometric maps of a circle into Euclidean space,* Invent. Math. 123, (1996), 35–59.

[MFK] D. Mumford, J. Fogarty and F. Kirwan, *Geometric Invariant Theory,* 3rd enlarged ed., Springer 1994.

[N] L. Ness, *A stratification of the null cone via the moment map,* Amer. J. Math. 106, (1984), 1281–1329.

[OSj] L. O'Shea and R. Sjamaar, *Moment maps and Riemannian symmetric pairs*, Math. Ann. 317 (2000), 415–457.

[R] N. Ressayre, *Geometric invariant theory and generalized eigenvalue problem*, preprint 2007. ArXiv:0704.2127

[TW] C. Teleman and C. Woodward, *Parabolic bundles, products of conjugacy classes and Gromov-Witten invariants*, Ann. Inst. Fourier 53 no. 3 (2003), 713–748.

[Th] R.C. Thompson, *Matrix spectral inequalities*, John Hopkins University 1988.

[We] H. Weyl, *Das asymptotische Verteilungsgesetz der Eigenwerte linearer partieller Differentialgleichungen*, Math. Ann. 71 (1912), 441–479.

[Wi] H. Wielandt, *An extremum property of sums of eigenvalues*, Proc. AMS 6 (1955), 106–110.

Michael Kapovich, Department of Mathematics, University of California, Davis, CA 95616, USA, kapovich@math.ucdavis.edu

Bernhard Leeb, Mathematisches Institut, Universität München, Theresienstrasse 39, D-80333 München, Germany, b.l@lmu.de

John Millson, Department of Mathematics, University of Maryland, College Park, MD 20742, USA, jjm@math.umd.edu