HARMONIC FUNCTIONS ALONG BROWNIAN BALLS AND THE LIOUVILLE PROPERTY FOR SOLVABLE LIE GROUPS

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Introduction

In this paper, we consider diffusion processes on topological spaces and their bounded harmonic functions.

It is a direct consequence of the convergence theorem for bounded martingales that for a diffusion bounded harmonic functions converge along almost every trajectory. In section 1, we prove an extension of this general fact. We introduce a notion of distance associated to the diffusion and show in Theorem 1 that convergence takes also place at bounded distance away from the trajectory. For instance, in the case of brownian motion on a Riemannian manifold of bounded sectional curvature, a bounded harmonic function converges a.s. uniformly on a "brownian ball", i.e. a ball of constant (geometric) radius accompanying the trajectory.

In section 3, we illustrate how this general argument can be applied in a homogeneous situation. We give a new proof for the fact that brownian motion on a connected unimodular solvable Lie group with left-invariant metric is Liouville, i.e. every bounded harmonic function is constant. Theorem 2 is an extension of this fact to certain canonical left-invariant diffusions on arbitrary connected solvable Lie groups. It can also be deduced from Raugi's representation theorem for bounded harmonic functions associated to a large class of random walks on locally compact groups with countable base [Rau]. The special case of left-invariant diffusions generated by subelliptic operators on solvable Lie groups, which are the semidirect product of an abelian group A and a nilpotent group N with diagonalizable action of A on N, is studied in [DH] and implies our Theorem 2 for these groups.

We are using standard notation. Only basic facts about stochastic processes are needed. Regarding the theory of martingales, in particular the convergence theorem for bounded martingales, we refer the reader to [B1]. For details about stochastic

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differential equations, diffusion processes and their connection with elliptic differential operators, see [B2], [IW] and [I].

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§1. Convergence of bounded harmonic functions along paths

A diffusion process $P = (P_x)_{x\in M}$ on a topological space M is a continuous M-valued stochastic process satisfying the strong Markov property. For simplicity, we assume that P has infinite life time. In the case that M is a manifold, for instance, the result can immediately be carried over to the general case by sufficiently slowing down the diffusion towards infinity.

Let h be a bounded harmonic function for the diffusion P. Then $(M_t)_{t\geq 0} = (h(w_t))_{t\geq 0}$ is a bounded martingale and therefore converges for P_x-a.e. trajectory ω in the path space $\Omega = \Omega(M)$. (w_t is the canonical evaluation map $\Omega \to M, \omega \mapsto \omega(t)$.) Denote the limit by

$$\lim_{t\to\infty} h(\omega(t)) =: M_\infty(\omega).$$

In other words, h converges a.s. along the trajectories of the diffusion starting at $x \in M$.

We want to show that h does not only converge on but also at a bounded distance away from the trajectories.

First of all, we have to make precise what "distance" means, since we do not assume a metric on our space M. We introduce a stochastic distance associated to the diffusion P. It is most easily formulated using the P-harmonic functions.

Definition: We say that two points $x, y \in M$ have *stochastic distance* $d \in [0,\infty]$ with respect to the diffusion P, if d is the smallest number, s.t.

$$(1)\qquad e^{-d} \quad\leq\quad \frac{h(x)}{h(y)} \quad\leq\quad e^{d} \qquad \forall\, h \in \mathcal{H}_b^+(M).$$

Here $\mathcal{H}_b^+(M)$ denotes the space of bounded positive P-harmonic functions on M.

To obtain a stochastic formulation, consider the restrictions of the diffusion measures P_x and P_y to the tail field $\mathcal{A}_\infty$. (The tail field consists of the events only depending on the asymptotic large time behavior of the trajectories.) If $d < \infty$, the measures P_x and P_y are equivalent because of (1). In this case, (1) translates into the following inequality for the Radon-Nikodym derivatives:

$$(2)\qquad e^{-d} \quad\leq\quad \frac{d(P_x\,|_{\mathcal{A}_\infty})}{d(P_y\,|_{\mathcal{A}_\infty})} \quad\leq\quad e^{d}$$

Note that different points may have zero stochastic distance. An extreme case occurs if all bounded harmonic functions are constant, i.e. if the diffusion process is Liouville. Then the stochastic distance of any two points is zero.

Examples:

(i) Knowing the extremal harmonic functions for brownian motion on hyperbolic space, one can easily verify that the stochastic distance coincides with the geometric distance.

(ii) Consider brownian motion on a complete Riemannian manifold of bounded sectional curvature. Then the stochastic diameter of balls $B_r(x)$ of a certain geometric radius $r > 0$ is bounded in terms of r and the curvature bounds.

(iii) Let M be a smooth manifold and the diffusion be generated by a second order elliptic differential operator with continuous coefficients. Since there is a Harnack inequality for small open sets, we see by a covering argument that relatively compact open sets have finite stochastic diameter. If the operator is moreover invariant under the action of some group, then the stochastic diameter is invariant as well.

We now make precise, what we mean by "convergence at a bounded distance away from the trajectory".

By a *neighbourhood system of bounded stochastic diameter*, we mean a family $\big(U(x)\big)_{x\in M}$ of open sets, such that $U(x)$ contains x and has diameter $\leq d$ with respect to the stochastic distance for some fixed positive number d. Think of $\big(U(\omega(t))\big)_{t\geq 0}$ as accompanying the trajectory ω. E.g. in the case of brownian motion on a Riemannian manifold, we call $\big(B_r(\omega(t))\big)_{t\geq 0}$ the *brownian ball* of radius r accompanying the trajectory ω.

The nonconstancy on the accompanying set of the bounded harmonic function h is measured by

$$\Delta_t := sup\{|h(x) - h(w_t)| : x \in U(w_t)\}$$

Lemma 1. *For every $x \in M$:*

$$\lim_{t\to\infty} \Delta_t = 0 \qquad\qquad P_x - a.s.$$

As an immediate consequence, we obtain the

Theorem 1. *Let $(P_x)_{x\in M}$ be a diffusion on a topological space M and h a bounded harmonic function for this diffusion. Then h converges P_x-a.s. on the set $\big(U(\omega(t))\big)$ accompanying the trajectory ω uniformly in the following sense:*

For every $\epsilon > 0$, there is a $t(\omega,\epsilon) > 0$, s.t.:

$$t \geq t(\omega,\epsilon) \quad \wedge \quad x \in U(\omega(t)) \qquad \Longrightarrow \qquad |h(x) - M_\infty(\omega)| \leq \epsilon$$

Proof: (of lemma)

We estimate the deviation $h(y) - h(x)$ for $y \in U(x)$. The martingale convergence theorem implies:

$$h(y) = \int_\Omega M_\infty(\omega)\, P_y(d\omega)$$

By the Cauchy-Schwartz inequality:

$$\begin{aligned}\big(h(y) - h(x)\big)^2 &= \Big(\int_\Omega (M_\infty(\omega) - h(x))\, P_y(d\omega)\Big)^2 \\ &\leq \int_\Omega (M_\infty(\omega) - h(x))^2\, P_y(d\omega)\end{aligned}$$

Since the stochastic diameter of the $U(x)$ is bounded by assumption by some constant $d > 0$, we may use the corresponding Harnack inequality (2) to estimate the right-hand side:

$$\begin{aligned}\ldots \qquad &\leq e^d \cdot \int_\Omega (M_\infty(\omega) - h(x))^2\, P_x(d\omega) \\ &= e^d \cdot E_x\big[(M_\infty - M_0)^2\big]\end{aligned}$$

Taking the supremum of the left-hand side:

$$\Delta_0^2 \leq e^d \cdot E_x\big[(M_\infty - M_0)^2\big]$$

We transfer this inequality to arbitrary time $t > 0$ via the Markov property:

$$\Delta_t^2 \leq e^d \cdot E_x\big[(M_\infty - M_t)^2 \big| \mathcal{F}_t\big] \qquad P_x\text{-a.e.}$$

The right-hand side is a positive supermartingale, as the decomposition

$$E_x\big[(M_\infty - M_t)^2 \big| \mathcal{F}_t\big] \quad = \quad \underbrace{E_x\big[M_\infty^2 \big| \mathcal{F}_t\big]}_{\text{martingale}} - \underbrace{M_t^2}_{\text{submartingale}}$$

shows, and therefore converges. Its limit has to be zero *a.e.*, because

$$E_x\big[(M_\infty - M_t)^2\big] = E_x\big[M_\infty^2\big] - E_x\big[M_t^2\big] \to 0 \qquad \text{as } t \to \infty$$

We conclude

$$\lim_{t\to\infty} \Delta_t^2 = 0 \qquad P_x\text{-a.s.},$$

q.e.d.

Remark: In our argument we did not use the boundedness of h. It works as well under the weaker assumption that $(h(w_t))$ is a L^2-martingale.

Remark: One might want to modify the above definition of stochastic distance by allowing all positive harmonic functions, not only the bounded ones. This amounts to considering the whole family of h-processes P^h obtained from P by conditioning with positive harmonic functions h. Therefore, we have to give (1) a form which is invariant under conditioning:

$$(1') \qquad e^{-2d} \quad \leq \quad \frac{h_1(x)}{h_2(x)} \cdot \frac{h_2(y)}{h_1(y)} \quad \leq \quad e^{2d} \qquad \forall\, h_1, h_2 \in \mathcal{H}^+(M),$$

where $\mathcal{H}^+(M)$ denotes the space of positive P-harmonic functions on M. Recall that the P^{h_0}-harmonic functions are exactly the functions h/h_0 with h P-harmonic. The corresponding stochastic statement is:

$$(2') \qquad e^{-2d} \quad \leq \quad \frac{d(P_x^h\,|_{\mathcal{A}_\infty})}{d(P_y^h\,|_{\mathcal{A}_\infty})} \quad \leq \quad e^{2d} \qquad \forall\, h \in \mathcal{H}^+(M)$$

(Sketch of proof for $(2') \Rightarrow (1')$: Insert for h positive linear combinations $a\cdot h_1 + b\cdot h_2$. Then h_1/h is bounded h-harmonic and (2') implies:

$$e^{-2d} \quad \leq \quad \frac{h_1(x)}{h(x)} \Big/ \frac{h_1(y)}{h(y)} \quad \leq \quad e^{2d}$$

Now let a/b tend to zero and (1') follows.)
Theorem 1 holds as well with this modified notion of stochastic distance.

The modified (pseudo-)distance separates points better than the original one. But still the extremal case might occur that all positive harmonic functions are constant, as in the case of brownian motion on nilpotent Lie groups with left-invariant metric [Mg]. Then any two points have distance zero.

§2. The canonical diffusion

We will use Theorem 1 in section 3 to prove the Liouville property for certain diffusion processes on solvable Lie groups. In the present section, we describe these processes.

Let G be a connected Lie group equipped with a left invariant metric $\langle\cdot,\cdot\rangle$. We define the second order elliptic operator

$$L^G := \sum_i (E_i)^2 \tag{1}$$

where E_i is an orthonormal basis of the Lie algebra g. L^G does only depend on the metric, but not on the particular basis chosen. $\frac{1}{2}L^G$ generates a diffusion process on G given by the stochastic differential equation

$$dX_t = \sum_i E_i(X_t) \circ dW_t^i, \tag{2}$$

where (W_t) is standard brownian motion on $\mathbb{R}^{\dim G}$. We call (X_t) the *canonical diffusion* on G.

L^G is related to but in general different from the Laplace operator Δ^G on G:

$$\begin{aligned} L^G &= \sum_i (D^2_{E_i,E_i} + D_{E_i}E_i) \\ &= \Delta^G + \sum_i D_{E_i}E_i \end{aligned}$$

As the following computation will show, the deviation from the Laplace operator is given by the trace of the adjoint representation, which depends only on the group structure. For $X \in g$ we have:

$$\begin{aligned} \langle \sum_i D_{E_i}E_i, X\rangle &= -\sum_i \langle E_i, D_{E_i}X\rangle \\ &= -\sum_i \underbrace{\langle E_i, D_X E_i\rangle}_{=0} + \sum_i \langle E_i, [X, E_i]\rangle \\ &= \operatorname{trace}\operatorname{ad} X \end{aligned} \tag{3}$$

The 1-form trace ad on g corresponds via the scalar product to a left-invariant vector field, which we denote by A. (It is in fact the gradient of the function log det Ad on G.) We get

$$L^G = \Delta^G + A. \tag{4}$$

The above discussion shows:

Fact. *The canonical diffusion is brownian motion if and only if G is unimodular.*

We consider the canonical diffusion, because it is compatible with homomorphisms which are submersions:

Proposition 1. *A surjective homomorphism $\phi : G \to H$ of Lie groups equipped with left invariant metrics, which is a Riemannian submersion, maps the canonical diffusion on G to the canonical diffusion on H.*

Proof: Choose a left invariant orthonormal basis $E_1, ..., E_n$ on G, so that $E_1, ..., E_m$ respectively $E_{m+1}, ..., E_n$ are orthogonal respectively tangent to $\ker \phi$. The vectorfields $E'_i := d\phi \cdot E_i$ are leftinvariant on H. $E'_{m+1}, ..., E'_n$ vanish and, since ϕ is a Riemannian submersion, $E'_1, ..., E'_m$ form an orthonormal basis on H.

Itô's formula applied to (2) yields for the process (Y_t), where $Y_t := \phi(X_t)$:

$$\begin{aligned} dY_t &= \sum_{i=1}^{n} (d\phi \cdot E_i(X_t)) \circ dW_t^i \\ &= \sum_{i=1}^{m} E'_i(Y_t) \circ dW_t^i \end{aligned}$$

Thus (Y_t) is the canonical diffusion on H, q.e.d.

§3. The Liouville property for the canonical diffusion

Recall that the *Liouville property* states that every bounded harmonic function is constant, or equivalently in stochastic terms, that the diffusion satisfies a 0-1 law on the stationary σ-algebra $\mathcal{A}_\vartheta$ of shift-invariant events.

Theorem 2. *The canonical diffusion on a connected solvable Lie group G is Liouville.*

Proof: First we find a small normal subgroup to perform the induction step. (In case G is nilpotent, the center would do.) This is due to the solvability of G: $g \otimes_{\mathbb{R}} \mathbb{C}$ acts on itself via the adjoint representation and by the theorem of Lie-Kolchin [Hum], there is a complex 1-dimensional invariant subspace U. Since the action is defined over $\mathbb{R}$, this implies the existence of an ideal $h \subset g$, of dimension ≤ 2. Let H be the corresponding connected subgroup.

Since the Adjoint action of G on U is given by

$$\mathrm{Ad}(g) \cdot Y = \chi(g)\, Y \qquad \forall\, Y \in U, g \in G$$

with a character $\chi : G \to \mathbb{C}^*$, there is a constant $C \geq 1$, so that

$$\| \mathrm{Ad}(g) \cdot X \| \leq C \cdot |\chi(g)| \cdot \|X\| \qquad \forall\, X \in h, g \in G \tag{1}$$

This allows us to estimate the distance of paths to their translates by elements of H. Let $x = \exp(X)$ with $X \in h$:

$$\begin{aligned} \mathrm{dist}(w_t, x \cdot w_t) &= \mathrm{dist}(e, w_t^{-1} x w_t) \\ &\leq \| \mathrm{Ad}(w_t^{-1}) \cdot X \| \\ &\leq C \cdot |\chi(w_t)|^{-1} \cdot \|X\| \qquad \text{by (1)} \end{aligned}$$

As a consequence of Proposition 1 applied to the homomorphism

$$\log |\chi| : G \to \mathbb{R}$$

we know that the image of the canonical diffusion under $\log|\chi|$ is brownian motion on $\mathbb{R}$. This means that P_g-a.s. $|\chi(w_t)|$ becomes arbitrarily large after arbitrarily long time, and we conclude that for P_g-a.e. path ω:

$$\forall T \quad \exists t \geq T: \qquad \mathrm{dist}(\omega(t), x \cdot \omega(t)) \leq \delta,$$

where δ is some fixed positive number, which can be chosen arbitrarily. I.e. for $r > \frac{1}{2}\delta$ the balls $B_r(\omega(t))$ and $B_r(x \cdot \omega(t))$ accompanying the trajoctories ω respectively $x \cdot \omega$ will intersect after arbitrarily large time.

Now let h be a bounded harmonic function. Note that left translation by $x \in G$ transforms the diffusion starting at $g \in G$, P_x, to the diffusion starting at $x \cdot g$, $P_{x \cdot g}$. By Theorem 1, h converges P_g-a.s. on both of the accompanying balls, and because these intersect after arbitrarily large time, the limits have to coincide:

$$\lim_{t\to\infty} h(\omega(t)) = \lim_{t\to\infty} h(x \cdot \omega(t)) \qquad P_g\text{-a.s.}$$

Integration yields

$$h(g) = h(xg) \qquad \forall g \in G, x \in \exp(h),$$

and since H is connected and h continuous:

$$h(g) = h(xg) \qquad \forall g \in G, x \in \overline{H}.$$

Thus, h is $\overline{H}$-invariant of the form

$$h = h_1 \circ \pi,$$

where $\pi : G \to G/\overline{H} =: G_1$ is the natural projection. Equip the connected solvable Lie group G_1 with the unique left invariant metric, so that π is a Riemannian submersion. By proposition 1, π maps the canonical diffusion on G to the canonical diffusion on G_1 and h_1 is harmonic with respect to it.

The dimension decreases: $\dim G_1 < \dim G$. We proceed by induction and finally conclude that h is constant, q.e.d.

We point out the special case, where the canonical diffusion coincides with brownian motion:

Liouville property of connected unimodular solvable Lie groups. *Every bounded harmonic function on a connected unimodular solvable Lie group is constant.*

Remark: The unimodularity condition is necessary. For instance, there are plenty of bounded harmonic functions on the standard hyperbolic plane, but nevertheless it can be realized as a 2-dimensional solvable Lie group with left-invariant metric. To do this, take the group of all isometries preserving orientation and a certain point at infinity and equip it with the Riemannian metric enherited by its simply transitive action on hyperbolic plane.

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