Random graphs from constrained classes

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Overview

1. **Discussion of various constrained models:**
   - $d$-regular,
   - $H$-free,
   - hereditary classes,
   - trees

2. Classes closed under minors I
   - Combinatorial tools: double counting

3. Classes closed under minors II
   - Analytic tools: generating functions
Classical random graphs

Erdős-Rényi 1960

$G(n, M)$

$n$ labelled vertices, $M$ edges

Uniform model

$$\mathbb{P}[G] = \frac{1}{\binom{N}{M}}, \quad N = \binom{n}{2}$$
Thus we obtain
\[ M(\varepsilon(S)) = \frac{\binom{n}{2} - 2l}{N - 2l}. \]

If \( S_1 \) and \( S_2 \) contain exactly \( s \) common points and \( r \) common edges \((1 \leq r \leq l - 1)\)
we have
\[ M(\varepsilon(S_1) \varepsilon(S_2)) = \frac{\binom{n}{2} - 2l + r}{N - 2l + r} = O\left(\frac{N^{2l-r}}{n^{4-2r}}\right). \]

On the other hand the intersection of \( S_1 \) and \( S_2 \) being a subgraph of \( S_1 \) (and \( S_2 \))
and thus the number of such pairs of subgraphs \( S_1 \) and \( S_2 \) does not exceed
\[ B_{k,l}^2 \sum_{\substack{j \geq \frac{r_k}{l} \ k \ \binom{n}{j} \binom{k}{j}}} = O\left(n^{2k - \frac{r_k}{l}}\right). \]

Thus we obtain
\[ M\left(\sum_{S \in \mathbb{G}^{(n)}_{k,l}} \varepsilon(S) \right)^2 = \]
\[ \sum_{S \in \mathbb{G}^{(n)}_{k,l}} M(\varepsilon(S)) + \frac{n!}{k!2(n - 2k)!} \binom{n}{2} \left(\frac{N}{N - 2l}\right)^2 + O\left(\frac{N^{2l-k}}{n^{2l}} \sum_{i=1}^{l} \binom{n}{2} \left(\frac{N}{N - i}\right)^r\right). \]
The $G(n, p)$ model

$n$ vertices
Each possible edge $(i, j)$ present independently with probability $p$

Non-uniform model

$$\mathbb{P}[G] = p^{e(G)}(1 - p)^{\binom{n}{2} - e(G)}$$

In many cases

$$G(n, p) \approx G(n, M), \quad M = p\binom{n}{2}$$

Main features

- Appearance of giant component at $M = n/2$, or $p = 1/n$
- Thresholds for many basic properties
- Variations: non-homogeneous models, geometric graphs...

Bollobás 1985
Janson, Łuczak, Rucinski 2000
van der Hofstad in preparation
Cubic graphs

\[ \mathcal{G} \] = class of cubic (3-regular) graphs

\[ \mathcal{G}_n \] = graphs in \( \mathcal{G} \) with \( n \) vertices

\[ g_n = |\mathcal{G}_n| \]

Main question

Which are the typical properties of a uniform random graph from \( \mathcal{G}_n \) when \( n \to \infty \)

- Is it connected?
- Does it contain a fixed \( H \) as a subgraph?
- Is it Hamiltonian?

Event \( A \) holds with high probability where

\[ \Pr[G \in \mathcal{G}_n \text{ satisfies } A] \to 1, \quad n \to \infty \]
The pairing model

Cells $C_1, \ldots, C_n$ ($n$ even) each containing 3 distinguishable points

A pairing is a perfect matching among the $3n$ points

By collapsing the cells, each pairing gives rise to a cubic multigraph

Generating a cubic graph at random:

1. Generate a random pairing
2. Collapse the cells
3. If the resulting graph is not simple go to Step 1

Theorem

$\mathbb{P}[\text{Simple}] \rightarrow e^{-2}$
Model $P(n, 3)$: random uniform pairing

Model $G(n, 3)$: random uniform simple cubic graph

$$\pi: P(n, 3) \to G(n, 3)$$

**Fact** Each graph in $G(n, 3)$ has exactly $6^n$ preimages

**Fact** Let $A$ be an event (set of graphs) in $G(n, 3)$ and $A'$ the set of pairings that correspond to graphs in $A$. Then

$$\mathbb{P}_{G(n,3)}[A] = \mathbb{P}_{P(n,3)}(A') \cdot \mathbb{P}[\text{Simple}].$$

Generalizes immediately to

$$P(n, d) \quad \text{and} \quad G(n, d)$$

$$\mathbb{P}[\text{Simple}] \to \exp \left( \frac{1 - d^2}{4} \right)$$
Computing in $P(n, d)$

Number of pairings $= (dn - 1)!! = (dn - 1)(dn - 3) \cdots 3 \cdot 1$

The probability that $k$ disjoint pairs are in a random pair is

$$\frac{(dn - 2k - 1)!!}{(dn - 1)!!} \sim \frac{1}{(dn)^k}$$

$X_k =$ number of $k$-cycles in $P(n, d)$

$$\mathbb{E}X_n \sim \frac{(n)_kd^k(d - 1)^k}{2k} \frac{1}{(dn)^k} \sim \frac{(d - 1)^k}{2k}$$

Theorem $X_n \rightarrow \text{Poisson}(\lambda_k), \quad \lambda_k = \frac{(d - 1)^k}{2k}$

$X_1, \ldots, X_k$ are asymptotically independent Poisson variables

$$\mathbb{P}[X_1 = X_2 = 0] \sim e^{-\lambda_1}e^{-\lambda_2} = \exp\left(\frac{1 - d^2}{4}\right)$$

$$|G_{n,d}| \sim \sqrt{2}e^{(1-d^2)/4}n^{dn/2}\left(\frac{d^d}{e^d(d!)^2}\right)^{n/2}.$$
Properties of random regular graphs

For fixed $d$, whp a random $d$-regular $G$

- Is $d$-connected
- Does not contain $H$ if $e(H) > v(H)$
- Is Hamiltonian
- Has no non-trivial automorphisms
- $\chi(G)$ is concentrated in a unique value $\sim \frac{d}{2 \log d}$.

Extends to graphs with given degrees $(d_1, \ldots, d_n)$ and $d_i$ bounded

Quite more difficult is the case when $d = d(n)$

Conclusion
We have a model $P(n, d)$ in which we can compute and from here deduce properties for the model of interest $G(n, d)$
Tringle-free graphs

\( G = \) class of graphs not containing \( C_3 \)

How to count?
How to generate a random graph?

Bipartite \( \subset \) Triangle-free

Theorem (Erdős-Kleitman-Rothschild 1975)
Almost every triangle-free graph is bipartite

Model for random bipartite graphs \( B(n_1, n_2, p = 1/2) \)

Rough idea of the proof

▶ Partition \( G_n \) into several classes
▶ Show inductively that all classes except one are asymptotically negligible
▶ Use structural properties of these negligible classes to show that all graphs in the remaining set are bipartite

Note Sparse triangle-free graphs are not always bipartite whp
Monotone classes

\( \mathcal{G} \) is monotone if it is closed under taking subgraphs.

The class of \( H \)-free graphs is monotone.

Assume \( \chi(H) = r + 1 \geq 3 \)

Erdős-Stone theorem

\[
\text{ex}(H, n) = \left( 1 - \frac{1}{r} + o(1) \right) \binom{n}{2}
\]

Hence the number of \( H \)-free graphs is \( \geq 2^{\left(1 - \frac{1}{r} + o(1)\right) n^2/2} \)

Erdős-Frankl-Rödl (1986) This is the right order of magnitude

But the error term can hide a huge \( 2^{n^{2-\epsilon}} \) term
Kolaitis-Prömel-Rothschild (1987)
Almost every $K_{r+1}$-free graph is $r$-colorable (or $r$-partite)

An edge $e \in E(G)$ is critical if $\chi(G - e) < \chi(G)$

Prömel-Steger (1992)
Assume $H$ contains a critical edge and $\chi(H) = r + 1 \geq 3$
Then almost every $H$-free graph is $r$-colorable

What if $H$ is bipartite?

$$ex(C_4, n) = \left(\frac{1}{2} + o(1)\right) n^{3/2}$$

The number $g_n$ of $C_4$-free graphs satisfies

$$2^{\frac{1}{2}} n^{3/2} < g_n < 2^{cn^{3/2}}$$

Kleitman-Winston (1982) $0.5 \leq c \leq 1.082$

Tiny improvement on the upper bound (Balogh-Wagner 2015)
Hereditary classes

A class $\mathcal{G}$ is hereditary if it is closed under induced subgraphs

- Interval graphs: intersection graphs of segments in $\mathbb{R}$
- Chordal graphs: every cycle has a chord
- Perfect graphs: $\chi(H) = \omega(H)$ for every induced subgraph $H$

Forb($H$) = graphs not containing $H$ as an induced subgraph

$\text{ex}(C_4, n) = O(n^{3/2})$ but a graph in Forb($C_4$) can have $\Omega(n^2)$ edges

$G$ is split if $V(G) = K_r \cup \overline{K}_s$

Split $\subset$ Forb($C_4$)

Prömel-Steger (1991) Almost every graph in Forb($C_4$) is split

Chordal $= \text{Forb}(C_4, C_5, C_6, \ldots)$

Split $\subset$ Chordal $\subset$ Forb($C_4$)
Prömel-Steger (1992)

\[ |\text{Forb}(H)_n| = 2^{\left(1 - \frac{1}{\chi_c(H) - 1} + o(1)\right)}(\binom{n}{2}) \]

where \( \chi_c(H) \) is the so-called coloring number

\( G \) is **generalized split** if \( V(G) = K_r \cup (K_{s_1} \cup \cdots \cup K_{s_m}) \)

and no edges between the \( K_{s_i} \)

\[ \text{Generalize split} \subset \text{Forb}(C_5) \]

Prömel-Steger (1992)

Almost every graph in \( \text{Forb}(C_5) \) is generalized split

\[ \text{Generalized split} \subset \text{Perfect} \subset \text{Forb}(C_5) \]

McDiarmid-Yolov A perfect graph is Hamiltonian whp
Trees

We can count exactly!

Cayley’s formula There are \( n^{n-2} \) labelled trees

The number of trees in which \( \text{deg}(v_1) = k \) is equal to

\[
\binom{n-2}{k-1}(n-1)^{n-k-1}
\]

\[
P[\text{deg}(v_1) = k] = \frac{(n-2)(n-1)^{n-k-1}}{n^{n-2}} \sim \frac{e^{-1}}{(k-1)!}
\]

The limiting distribution of vertex degrees is \( 1 + \text{Poisson}(1) \)

Theorem The maximum vertex degree is \( \sim \frac{\log n}{\log \log n} \)

Note For unlabelled trees the degree distribution is geometric and the maximum degree is of order \( \log n \)
Subtrees in random trees

\(X_n = \) number of leaves in random trees

\[\mathbb{E}X_n \sim e^{-1}n, \quad \sigma^2(X_n) \sim e^{-1}(1 - 2e^{-1})n\]

Branching process with offspring distribution \(p_k = 1/k!\)
Conditioning on total progeny of size \(n\), gives a uniform rooted tree
It is natural to expect that

\[\frac{X_n - \mathbb{E}X_n}{\sigma(X_n)} \rightarrow \mathcal{N}(0, 1)\]

Same result holds if
\(X_n = \) number of copies of a fixed pendant subtree
No exact formula for the number $F_n$ of forests but

$$F_n \sim e^{1/2} n^{n-2}$$

$$\mathbb{P}[\text{random forest is connected}] \rightarrow e^{-1/2} \approx 0.607$$

- How many components has a random forest?
- What is the size of the largest tree in a random forest?
Minor-closed classes
Combinatorial tools

Based on Colin McDiarmid et al.
Random forests

Uniform distribution on forests with \( n \) vertices
\( X_n = \) number of components in a random forest

Theorem

1. \( \mathbb{P}[X_n \geq k + 1] \leq \mathbb{P}[\text{Poisson}(1) \geq k] \), for \( k \geq 0 \)
2. \( \mathbb{P}[\text{Random forest is connected}] \geq e^{-1} \)
3. \( \mathbb{E}X_n \leq 2 \)

\( \mathcal{F}_n^k = \) forests with \( k \) components

\[ |\mathcal{F}_{n+1}^k| \leq \frac{1}{k} |\mathcal{F}_n^k| \]

Lemma \( Y \geq 0 \) integer variable, \( \alpha > 0 \). Assume

\[ \mathbb{P}[Y = k + 1] \leq \frac{\alpha}{k + 1} \mathbb{P}[Y = k] \]

Then

\[ \mathbb{P}[Y \geq k] \leq \mathbb{P}[\text{Poisson}(1) \geq k] \]

Note We know that \( \mathbb{P}[\text{Random forest is connected}] \sim e^{-1/2} \)
Addable classes of graphs

\( \mathcal{G} \) is bridge-addable if

\[ G \in \mathcal{G}, \ u \text{ and } v \text{ in different components of } G \implies G + uv \in \mathcal{G} \]

The same proof as before works for any bridge-addable class

Fragment = complement of the largest component

\( F_n = \text{size of the fragment} \)

Theorem

If \( \mathcal{G} \) is bridge-addable then \( \mathbb{E}F_n \leq 2 \)
Conjecture  McDiarmid-Steger-Welsh 2006
For any bridge-addable class $\mathcal{G}$

$$\mathbb{P}[\text{Random graph in } \mathcal{G} \text{ is connected}] \geq e^{-1/2+o(1)}$$

Proved assuming in addition that $\mathcal{G}$ closed under \textbf{deleting} bridges
Addario-Berry, McDiarmid, Reed 2012
Kang, Panagiotou 2013

Full conjecture proved recently
Chapuy, Perarnau 2015
**Growth constants**

$\mathcal{G}$ has a growth constant if the following limit exists

$$\gamma(\mathcal{G}) = \lim_{n \to \infty} \left( |\mathcal{G}_n|/n! \right)^{1/n}$$

$\mathcal{G}$ is small if $|\mathcal{G}_n| \leq c^n n!$ for some $c > 0$

$\mathcal{G}$ is addable if bridge-addable and is decomposable, namely:

$$G \in \mathcal{G} \iff \text{each component of } G \text{ is in } \mathcal{G}$$

**Theorem**

A small and addable class has a growth constant

$$|G_{a+b}| \geq \binom{a+b}{b} |C_a||C_b| \frac{1}{2} \geq \binom{a+b}{b} \frac{|G_a||G_b|}{e^2} \frac{1}{2}$$

Apply Fekete’s lemma to $f(n) = |\mathcal{G}_n|/(2e^2 n!)$

$$f(m + n) \geq f(m)f(n) \implies f(n)^{1/n} \to \sup_m f(m)^{1/m} \leq \infty$$
Minor-closed classes

\( \mathcal{G} \) is **minor-closed** if closed under edge deletion and contraction

- Forests
- Planar graphs
- Graphs on a fixed surface
- Graphs with bounded tree-width

**Theorem** Norine-Seymour-Thomas-Wollan, Dvořák-Norine
A proper minor-closed class is small

**Corollary** A proper minor-closed addable class has a growth constant

**Remark** For each surface, the class \( \mathcal{G} \) of graphs embeddable in \( S \) has the same growth constant as the class of planar graphs

**Conjecture** Bernardi-Noy-Welsh 2010
Every proper minor-closed class has a growth constant
Growth constants of some minor-closed classes

<table>
<thead>
<tr>
<th>Class</th>
<th>Growth constant</th>
</tr>
</thead>
<tbody>
<tr>
<td>Forests of trees of bounded height</td>
<td>0</td>
</tr>
<tr>
<td>Forests of paths</td>
<td>1</td>
</tr>
<tr>
<td>Forests of caterpillars</td>
<td>1.73</td>
</tr>
<tr>
<td>Forests</td>
<td>(e = 2.71)</td>
</tr>
<tr>
<td>Outerplanar</td>
<td>7.32</td>
</tr>
<tr>
<td>Series-parallel</td>
<td>9.07</td>
</tr>
<tr>
<td>Planar</td>
<td>27.23</td>
</tr>
<tr>
<td>Graphs on a fixed surface</td>
<td>27.23</td>
</tr>
<tr>
<td>Apex (planar after removing one vertex)</td>
<td>54.46</td>
</tr>
</tbody>
</table>

**Gaps** Intervals without growth constants: \([0, 1]\), \([1, 1.73]\)
Pendant copies

$G$ has a **pendant** copy of $H$ if $G$ has a bridge with a copy of $H$ at one end

**Theorem** McDiarmid-Steger-Welsh 2005
If $\mathcal{G}$ is addable and minor-closed then for each $H \in \mathcal{G}$ a random graph from $\mathcal{G}$ has $\Omega(n)$ pendant copies of $H$

**Consequences** A random graph from $\mathcal{G}$ has whp
- $\Omega(n)$ vertices of degree $k$ for each fixed $k \geq 1$
- Exponentially many automorphisms
Smoothness

A class $\mathcal{G}$ is smooth if the following limits exists

$$\lim_{n \to \infty} \frac{|G_n|}{n|G_{n-1}|} \to \gamma$$

Necessarily $\gamma = \lim(|G_n|/n!)^{1/n}$

In several cases of interest we have estimates of the form

$$|G_n| \sim c \cdot n^\alpha \gamma^n n!$$

which imply smoothness

**Theorem** McDiarmid 2009

A proper addable minor-class is smooth

The class $\mathcal{G}^S$ is smooth but not addable
Components and fragments for addable classes

$\mathcal{G}$ proper addable minor-closed

$\gamma$ growth constant of $\mathcal{G}$, $\rho = \gamma^{-1}$

For $H$ unlabelled in $\mathcal{G}$ let

$$\mu(H) = \frac{\rho^{v(H)}}{\text{aut}(H)}$$

**Fragment**: complement of the largest component

**Theorem**

- The number of components $\kappa(H)$ isomorphic to $H$ is asymptotically Poisson($\mu(H)$)
- The variables $\kappa(H_1), \ldots, \kappa(H_k)$ are asymptotically independent for different $H_i$
- The probability that the fragment is isomorphic to $H$ is $\propto \mu(H)$
Maximum degree

\[ \Delta_n = \text{maximum vertex degree in graphs with } n \text{ vertices} \]

**Theorem** McDiarmid-Reed 2008

For the class of planar graphs there exist \( 0 < c < c' \) such that

\[ c \log n < \Delta_n < c' \log n \quad \text{whp} \]

\( \mathcal{G}_n = \text{planar graphs} \)

\( \mathcal{B}_n \subset \mathcal{G}_n \text{ bad graphs} \)

Suppose for each \( G \in \mathcal{B}_n \) we construct at least \( C(n) \) graphs in \( \mathcal{G}_n \)

And each graph \( \in \mathcal{G}_n \) is produced at most \( R(n) \) times

\[ C(n) |\mathcal{B}_n| \leq R(n) |\mathcal{G}_n| \]

\[ \frac{|\mathcal{B}_n|}{|\mathcal{G}_n|} \leq \frac{R(n)}{C(n)} \rightarrow 0 \]

Then almost all graphs in \( \mathcal{G}_n \) are good (not bad)

In the proofs we use \( C(n)/R(n) \rightarrow \infty \)
Lower bound

$G$ is bad if $\Delta(G) < \lceil c \log n \rceil$

Recall $G$ has $\geq \alpha n$ pendant vertices whp

$h = \lceil c \log n \rceil$

- Take ordered list $v_1, \ldots, v_h$ of pendant vertices
- Remove them and build a fan with apex $v_1$
- Attach the fan to $G$ through a bridge $uv_1$

$$C(n) \geq (\alpha n)_h(n - h) \geq \left( \frac{\alpha n}{2} \right)^h n$$

How many times can a graph be produced?

- $v_1$ is the only vertex with degree $h$
- $v_2, \ldots, v_n$ are the neighbors of $v_1$ inducing a path
- Guess the neighbors of $v_1, \ldots, v_h$ in $G$

$$R(n) \leq (n - h)^h \leq n^h$$

Hence

$$\frac{C(n)}{R(n)} \geq n \left( \frac{\alpha}{2} \right)^{c \log n} \rightarrow \infty \quad \text{if } c \text{ is small enough}$$
Upper bound

$G$ is bad if $\Delta(G) > \lceil C \log n \rceil$

Lemma A vertex is adjacent to $\leq 2 \frac{\log n}{\log \log n}$ pendant vertices whp

$\deg(v) = d > \lceil C \log n \rceil$, $a = \lfloor 2 \log n \rfloor$

- $v_1, \ldots, v_d$ neighbors of $v$
- $u_1, \ldots, u_a$ list of pendant vertices not adjacent to $v$
- Choose $a$ neighbors of $v$ as $v_{i_1} < \cdots v_{i_a}$
- Build a fan with $v, u_1, \ldots, u_a$ and join $u_j$ to $v_{i_1}, \ldots, v_{i_{j+1}-1}$

$$C(n) \geq (\alpha n - 2 \log n / \log \log n)_a \binom{d}{a} \geq \left( \frac{\alpha n}{2} \cdot \frac{d}{a} \right)^a$$

Repetitions

- Guess vertex $v$
- $u_1, \ldots, u_a$ are the neighbors of $v$ in the right order
- $v_1, \ldots, v_h$ are the other neighbors of the $u_j$
- Guess the neighbors of $u_1, \ldots, u_a$ in $G$

$$R(n) \leq (2n)n^a$$

$$\frac{C(n)}{R(n)} \geq \frac{1}{2n} \left( \frac{\alpha C}{4} \right)^{2 \log n} \rightarrow \infty \quad \text{if } C \text{ is large enough}$$
Same argument shows:

$\mathcal{G}$ minor-closed and forbidden minors 3-connected

$\implies \Delta_n \geq c \log n$

**Conjecture** Giménez, Mitsche, N.

$\Delta_n \leq c' \log n$
Conclusions

General results that apply in a wide setting to minor-closed and addable classes

Precise results on number and structure of connected components

Partial results on subgraphs, degrees, automorphisms

No results on the number of edges

Theorem (Sparsity)
For each proper minor-closed class $G$ there exists $c = c(G)$ such that for each $G \in G$

$$e(G) < cn$$

Other parameters: diameter, largest 2-connected component
Minor-closed classes II
Analytic tools

Based on M.N. and Omer Giménez, Michael Drmota, et al.
Trees revisited

\( T_n = \text{number of trees} \)
\( R_n = nT_n = \text{number of rooted trees} \)

Generating functions (formal power series)

\[
R(z) = \sum_{n \geq 1} R_n \frac{z^n}{n!}
\]

\[
R(z) = z \left( 1 + R(z) + \frac{R(z)^2}{2} + \cdots + \frac{R(z)^k}{k!} + \cdots \right) = ze^{R(z)}
\]

Unique solution with \( R_n \) positive
Lagrange’s inversion gives (easily) \( R_n = n^{n-1} \)

\[
T_n = \frac{R_n}{n} = n^{n-2}, \quad R(z) = zT'(z) = \sum nT_n \frac{z^n}{n!}
\]

\( F_n = \text{number of forests}, \quad F(z) = \sum F_n \frac{z^n}{n!} \)

\[
F(z) = e^{T(z)}
\]
Generating functions as analytic functions

\[ f(z) = \sum f_n z^n \text{ analytic at } 0 \]

\[ \rho = \frac{1}{\limsup |f_n|^{1/n}} \text{ radius of convergence} \]

\[ f_n \approx \rho^{-n} \]

Pringsheim's theorem
If \( f_n \geq 0 \) then \( \rho \) is a singularity of \( f(z) \)

Subexponential term depends on the type of the singularity
<table>
<thead>
<tr>
<th>$f(z)$</th>
<th>$f_n$</th>
<th>$\rho$</th>
<th>Type</th>
<th>$\sim$ (up to constant)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{1}{1 - z - z^2}$</td>
<td>Fib$\ _n$</td>
<td>$\frac{\sqrt{5} - 1}{2}$</td>
<td>Pole</td>
<td>$\left(\frac{1 + \sqrt{5}}{2}\right)^n$</td>
</tr>
<tr>
<td>$\frac{1}{1 - z} \log \frac{1}{1 - z}$</td>
<td>$H_n = \sum_{i=1}^{n} \frac{1}{i}$</td>
<td>1</td>
<td>Log</td>
<td>$\log n$</td>
</tr>
<tr>
<td>$\frac{1}{\sqrt{1 - 4z}}$</td>
<td>$\binom{2n}{n}$</td>
<td>$1/4$</td>
<td>Branch</td>
<td>$n^{-1/2}4^n$</td>
</tr>
<tr>
<td>$\frac{1}{2z} \sqrt{1 - 4z}$</td>
<td>$\frac{1}{n + 1} \binom{2n}{n}$</td>
<td>$1/4$</td>
<td>Branch</td>
<td>$n^{-3/2}4^n$</td>
</tr>
<tr>
<td>$e^z$</td>
<td>$\frac{1}{n!}$</td>
<td>$\infty$</td>
<td>Essential</td>
<td>$n^{-1/2}n^{-n}e^n$</td>
</tr>
</tbody>
</table>
\( R(z) = ze^{R(z)} \)

Consider \( R(z) \) as a complex function

By the implicit function theorem \( R(z) \) ceases to be analytic when

\[
R(z) = ze^{R(z)}, \quad 1 = ze^{R(z)}
\]

Solution \( z = e^{-1}, \ R = 1 \)

\( e^{-1} = \text{radius of convergence of } R(z) \)

\[
T(z) = R(z) - \frac{R(z)^2}{2}
\]

\( T(e^{-1}) = 1/2 \)

Together with \( F(z) = e^{T(z)} \) this implies

\[
F_n \sim e^{1/2} T_n
\]

\( \mathbb{P}[\text{Random forest is connected}] \to e^{-1/2} \)
Probability of connectivity

\( G \) decomposable class
\( C \) connected graphs in \( G \)

\[
G(z) = \sum G_n \frac{z^n}{n!}, \quad C(z) = \sum C_n \frac{z^n}{n!}
\]

\[
G(z) = e^{C(z)}
\]

\( \rho = \) radius of convergence of \( G(z) \)
Assume \( C(\rho) < \infty \)

\[
G_n \sim e^{C(\rho)} G_n
\]

\[ P[\text{Random graph in } G \text{ is connected}] \rightarrow p = e^{-C(\rho)} \]

<table>
<thead>
<tr>
<th>Class of graphs</th>
<th>( p )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Forests</td>
<td>( e^{-1} \approx 0.607 )</td>
</tr>
<tr>
<td>Planar</td>
<td>0.963</td>
</tr>
<tr>
<td>Series-parallel</td>
<td>0.889</td>
</tr>
<tr>
<td>Forests of caterpillars</td>
<td>0</td>
</tr>
</tbody>
</table>
Asymptotic estimates

\[ z = R(z)e^{-R(z)} \]
\[ ye^{-y} \text{ inverse function of } R(z) \]
\[ \rho = e^{-1} \text{ singularity of } R(z), \quad R(\rho) = 1 \]

Taylor

\[ ye^{-y} = e^{-1} - \frac{1}{2e}(y - 1)^2 + O((y - 1)^3) \]

Set \( z = ye^{-y} \) and solve for \( y = R(z) \)

\[ R(z) = 1 - \sqrt{2}\sqrt{1 - ez} + O(1 - ez) \]

Singularity analysis Flajolet-Odlyzko

\[ f(z) \sim (1 - z)^{-\alpha} \quad \implies \quad f_n \sim \frac{n^{\alpha-1}}{\Gamma(\alpha)} \]

\[ f_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z^{n+1}} \, dz \]

\[ R_n \sim (-\sqrt{2}) \frac{n^{-3/2}}{-2\sqrt{\pi}} e^n n! = \frac{n!e^n}{n\sqrt{2\pi n}} \sim n^{n-1} \]
Number of components

\[ F_{n,k} = \text{number of forests with } k \text{ components} \]

\[ F(z, u) = \sum F_{n,k} \frac{z^n}{n!} u^k \]

\[ F(z, u) = 1 + uT(z) + u^2 \frac{T(z)^2}{2!} + \cdots = e^{uT(z)} \]

\[ p_n(u) = \sum_k \frac{F_{n,k}}{F_n} u^k = \left[ z^n \right] F(z, u) \rightarrow ue^{(u-1)/2}, \quad n \rightarrow \infty \]

\[ X_n = \text{number of components in random forests} \]
\[ p_n(u) \text{ is the probability GF of } X_n \]

By the continuity theorem

\[ X_n \rightarrow 1 + \text{Poisson}(1/2) \]
- What about more complex classes?
- What about other parameters?
Graph decompositions

1. Decomposing a graph into connected components
2. Decomposing a connected graph into 2-connected components
3. Decomposing a 2-connected graph into 3-connected components.

With respect to generating functions, 1 and 2 are easy, 3 needs more work
The algebra of graph decompositions

\( \mathcal{G} \) class of graphs closed under 1- and 2-connected components

\( \mathcal{C} = \) connected graphs in \( \mathcal{G} \)

\( \mathcal{B} = \) connected graphs in \( \mathcal{G} \)

\( G(z), C(z), B(z) \) associated GFs

\[
G(z) = e^{C(z)}
\]

\( C^\bullet = \) connected graphs rooted at a vertex

\[
C^\bullet(z) = zC'(z) = \sum nC_n \frac{z^n}{n!}
\]

\[
C^\bullet(z) = ze^{B'(C^\bullet(z))}
\]

**Approach** If we know \( B(z) \), we have access to \( C(z) \)
Outerplanar graphs

G is **outerplanar** if planar and all vertices are in the outer face

**Fact** A 2-connected outerplanar graph consists of a cycle plus a collection of non-crossing chords = **polygon dissection**

Counting polygon dissections is classical (back to Cayley and Kirkman) and we obtain

\[ B'(z) = \frac{1 + 5z - \sqrt{1 - 6z + z^2}}{8} \]

\[ B(z) = \frac{z^2}{2!} + \frac{z^3}{3!} + 9\frac{z^4}{4!} + \cdots \]

Singularity at \( z = 3 - 2\sqrt{2} \approx 0.172 \)

\[ C^\bullet(z) = ze^{B'(C^\bullet(z))} \]

Two possible sources of singularities for \( C^\bullet(z) \):

- Singularities inherited from \( B(z) \)
- A branch point when solving \( C^\bullet - ze^{B'(C^\bullet)} \)
A branch point when solving $C^\bullet - ze^{B'(C^\bullet)}$
is a common solution of

$$F(z, w) = 0, \quad \frac{\partial F}{\partial w}(z, w) = 0.$$ 

where $F(z, w) = w - ze^{B'(w)}$

Positive solution

$$z \approx 0.137, \quad w = 0.171 < 3 - 2\sqrt{2}$$

Hence $\rho \approx 0.137$ is the dominant singularity of $C^\bullet(z)$

**Theorem** Bodirsky-Giménez-Kang-N.
The number of outerplanar graphs is asymptotically

$$c \cdot n^{-5/2} \gamma^n n!, \quad \gamma = \rho^{-1} \approx 7.321$$

$\mathbb{P}[\text{Random outerplanar graph is connected}] \rightarrow e^{-C(\rho)} \approx 0.862.$
Number of edges

\[ G_{n,k} = \text{number of graphs with } n \text{ vertices and } k \text{ edges} \]

\[ G(z, y) = \sum_{n,k} G_{n,k} \frac{z^n}{n!} y^k \]

Enriched equations

\[ G(z, y) = e^{C(z,y)} \]

\[ C^\bullet(z, y) = ze^{B'(z^\bullet, y)} \]

Outerplanar graphs

\[ B'(z, y) = \frac{1 + zy(3 + 2y) - \sqrt{1 - 2yz - 4y^2z + y^2z^2}}{4(1 + y)} \]

\[ B'(z, 1) = \frac{1 + 5z - \sqrt{1 - 6z + z^2}}{8} \]
From $B'(z, y)$ we have access to $C(z, y)$ and to $G(z, y)$

$\rho(y) = \text{singularity of } z \mapsto G(z, y)$

$$[z^n] G(z, y) \sim c(y)n^{-5/2}\rho(y)^{-n}$$

$$p_n(y) = \sum_k \frac{G_{n,k} y^k}{G_n} = \frac{[z^n] G(z, y)}{[z^n] G(z)} \sim A(y)B(y)^n$$

$$A(y) = c(y)/c(1), \quad B(y) = \rho(1)/\rho(y)$$

$$p_n(y) = A(y)B(y)^n \left(1 + O\left(\frac{1}{n}\right)\right)$$

$X_n = \text{number of edges in outerplanar graphs}$

$$\phi_{X_n^*}(t) \to e^{-t^2/2}$$

**Theorem** BGKN

$X_n$ is asymptotically normal with

$$\mathbb{E}X_n \sim -\frac{\rho'(1)}{\rho(1)} n \approx 1.56n, \quad \sigma^2(X_n) \sim 0.22n$$

$$\mathbb{P}[|X_n - \mathbb{E}X_n| > \epsilon n] \text{ is exponentially small}$$
Planar graphs

Whitney  A 3-connected planar graph has a unique embedding in $S^2$

\{3-connected planar graphs\} $\longleftrightarrow$ \{3-connected planar maps\}

A map is a graph embedded in the (oriented) sphere

Counting maps  Tutte 1960’s
Presently a rich theory with connections to algebra and physics

$M_{n,k} =$ number of 3-connected maps with $n$ edges and $k$ vertices

$M(z, w) = \sum M_{n,k} z^n w^k$

Mullin-Schellenberg 1968
$M(z, w)$ is an explicit algebraic function of degree four

Gao-Wormald 2002
Asymptotic enumeration of 2-connected planar graphs
\[ C^\bullet(z) = ze^{B'(C^\bullet(z))} \]

Two possible sources of singularities for \( C^\bullet(z) \):
- Singularities inherited from \( B(z) \)
- A branch point when solving \( C^\bullet - ze^{B'(C^\bullet)} \)

Giménez-N. 2005

Explicit (but long) expression for \( B(z) \)

No branch point

Singularity \( \rho \) of \( C^\bullet(z) \) is such that

\[ C^\bullet(\rho) = \rho_B \]

Asymptotic enumeration of planar graphs

\[ c \cdot n^{-7/2} \gamma^n n!, \quad \gamma = \rho^{-1} \]

Limit laws for components (Poisson), edges (Gaussian)…
Extensions

- Graphs with given average degree
- Extremal parameters
- Graphs on surfaces
- Logical limit laws
Given average degree

\[ G_{n,m} = \text{number of planar graphs with } n \text{ vertices and } m \text{ edges} \]

When \( m \leq n \) (average degree \( \leq 2 \)) Kang, Łuczak

When \( m = qn \), \( q \in (1, 3) \) Giménez, N.

\[ G_{n,qn} \sim c(q)n^{-4}\gamma(q)^nn! \]

\[ \lim_{q \to 1^+} \gamma(q) = e \]
\[ \lim_{q \to 3^-} \gamma(q) = \frac{256}{27} \approx 9.5 \]
Extremal parameters

Drmota, Giménez, N., Panagioutou, Steger
The maximum degree $\Delta_n$ in random planar graphs satisfies

$$|\Delta_n - c \log n| = O(\log \log n)$$

$c \approx 2.53$
Asymptotic estimates plus Boltzmann samplers

Chapuy, Fusy, Giménez, N.
The diameter $D_n$ of a random planar graph satisfies

$$n^{1/4-\epsilon} < D_n < n^{1/4+\epsilon}$$

Scaling limits of quadrangulations plus decomposition into $k$-connected components
Extremal parameters II

Gao, Wormald 1999
The largest block in a random map has size $n/3$
The second largest block has size $O(n^{2/3+\epsilon})$

Banderier, Flajolet, Schaeffer, Soria 2001
The size of the largest block follows a stable law (Airy law) of parameter $3/2$

Panagiotou, Steger 2010
Giménez, N., Rué
$X_n = \text{size of largest block in random planar graphs}$

$$\mathbb{E}X_n \sim 0.96n$$

Second largest block is of size $O(n^{2/3+\epsilon})$

GNR Airy law + Similar results for largest 3-connected component

Remark For outerplanar graphs only small blocks
The number of graphs embeddable in $S^g$ is asymptotically

\[ c(g)n^{\frac{5}{2}g - 1} \gamma^n n! \]

\[ \gamma \approx 27.23 \text{ growth constant of planar graphs} \]

**CFGMN**

A random graph of genus $g$ has the same limit laws for basic parameters as a random planar graph.

The largest component has genus $g$ and the remaining components are planar.
Logical limit laws

The zero-one law for random graphs:

For every graph property $A$ expressible in first order logic and constant $p$

$$\lim P[G(n, p) \text{ satisfies } A] \to 0 \text{ or } 1$$

Heinig, Müller, N., Taraz 2014

$G$ addable minor closed, $C$ connected graphs in $G$

1. The zero-one law holds in $C$

2. The convergence law holds in $G$

$$\lim P[G \in G_n \text{ satisfies } A] \text{ exists}$$

Moreover, it holds in monadic second order logic = first order logic plus quantification over sets of vertices

Atserias, Kreutzer, N.

The previous statement does not hold for graphs of genus $g > 0$

Unique non-planar 3-connected component