Exercises for Stochastic Processes

In the following you may use without a proof that, for any continuous-time normal Markov chain, the time of the first jump is exponentially distributed with some parameter $c(x)$ depending on the starting point $x \in E$ (possibly 0 or ∞). This result will be shown in the lecture.

1. We drop normality and consider Markov chains with random starting points: For some probability measure π on E and some normal family of Markov chains $(X_t, \mathbb{P}_x)$ let Y be a Markov chain with law $\mathbb{P}$ given by

$$\mathbb{P}(A) := \sum_{x \in E} \pi(x) \mathbb{P}_x(A).$$

   Is the first jump time of Y exponentially distributed in general? Prove or disprove.

2. Let X be a *pure birth Markov chain* with rates λ_i on $\mathbb{N}_0$ introduced in the lecture.

   (a) Show that for $\lambda > 0$

$$p_t(i,j) = \binom{j-1}{j-i} e^{-i\lambda t}(1 - e^{-\lambda t})^{j-i}$$

   is a normal transition function on $\mathbb{N}_0$.

   (b) Prove that $(p_t)_{t \geq 0}$ corresponds to the pure birth chain with $\lambda_i = i\lambda$ (*linear pure birth chain*). You may use the explicit form of the corresponding Q-matrix given in the lecture.

   (c) Prove the suggested necessary and sufficient condition for an occurrence of a finite-time explosion in terms of the associated sequence of exponential waiting times, i.e., show that for

$$T_i := \inf\{t \geq 0 : X_t = i\}, \qquad i \in \mathbb{N}_0,$$
$$T_\infty := \sup_{i \geq 0} T_i,$$

   the following holds:

$$\sum_{i \geq 0} \frac{1}{\lambda_i} < \infty \qquad \Rightarrow \qquad T_\infty < \infty \qquad a.s.$$
$$\sum_{i \geq 0} \frac{1}{\lambda_i} = \infty \qquad \Rightarrow \qquad T_\infty = \infty \qquad a.s.$$

3. Let (X_n) be a sequence of independent continuous-time Markov chains on $\{0,1\}$ with Q-matrices $\begin{pmatrix} -\beta_n & \beta_n \\ \delta_n & -\delta_n \end{pmatrix}$. Assume that $\sum \frac{\beta_n}{\beta_n+\delta_n} < \infty$. Define

$$X(t) := (X_1(t), X_2(t), \dots)$$

and

$$E := \left\{ x \in \{0,1\}^{\mathbb{N}} \;\middle|\; \sum x_n < \infty \right\}.$$

(a) Show that E is countable and $\mathbb{P}(X(t) \in E \mid X(0) \in E) = 1$.

(b) Show that $p_t(x,y) := \mathbb{P}(X(t) = y \mid X(0) = x)$ is a transition function on E.

(c) Assume that, moreover, $\sum \beta_n = \infty$. Show that $c(x) = \infty$ for all $x \in E$, where $-c(x) := q(x,x) = \lim_{h\to 0} \frac{1-p_h(x,x)}{h}$.

(d) Show that $\mathbb{P}_x(X(t) = x \text{ for all } t < \epsilon) = 0$ holds for any $x \in E$ and $\epsilon > 0$.

Deadline: Tuesday, 08.01.2019. Hand in in groups, please!