Mathematisches Institut der Universität München
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WS 18/19
Sheet 7

Exercises for Stochastic Processes

1. Show that, for any square-integrable martingale $(M_t)_{t\geq 0}$ and $r < s < t$,
$$\mathbb{E}\left[(M_t - M_s)^2 \mid \mathfrak{F}_r\right] = \mathbb{E}\left[M_t^2 - M_s^2 \mid \mathfrak{F}_r\right].$$

2. Show that Gaussian processes $(X_t)_{t\geq 0}$ that are martingales have independent increments.

3. Let $(B_t)_{t\geq 0}$ be a standard Brownian motion and $(\mathfrak{F}_t)_{t\geq 0}$ the corresponding natural filtration.
   (a) Show that, for $\sigma \geq 0$, the process
   $$\left(e^{\sigma B_t - \frac{\sigma^2 t}{2}}\right)_{t\geq 0}$$
   is a martingale.
   (b) Show that the following processes are martingales:
   - $(B_t^2 - t)$
   - $(B_t^3 - 3tB_t)$
   - $(B_t^4 - 6tB_t^2 + 3t^2)$
   - ...

   and find the general formula for the above sequence.

3. Let $(B_t)_{t\geq 0}$ be a standard Brownian motion and $(\mathfrak{F}_t)_{t\geq 0}$ the corresponding natural filtration.
   Are the following processes martingales with respect to $(\mathfrak{F}_t)_{t\geq 0}$? Prove or disprove:
   (a) $e^{\sigma B_t}$ for $\sigma > 0$,
   (b) $cB_{\frac{t}{c^2}}$ for $c > 1$,
   (c) $tB_t - \int_0^t B_s ds$.

Deadline: Tuesday, 11.12.2018. Hand in in groups, please!