Mathematisches Institut der Universität München
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WS 18/19
Sheet 6

Exercises for Stochastic Processes

1. Let $(\mathfrak{F}_t)_{t\geq 0}$ be a right-continuous filtration and let $\tau, (\tau_n)_{n\in\mathbb{N}}$ be $(\mathfrak{F}_t)$-stopping times.
   (a) Show that
   $$\mathfrak{F}_\tau := \{A \mid \forall t \geq 0 : A \cap \{\tau \leq t\} \in \mathfrak{F}_t\}$$
   is a σ-algebra.
   (b) Show that τ is F_τ-measurable.
   (c) Show that, if $\tau_1 \leq \tau_2$, then $\mathfrak{F}_{\tau_1} \subset \mathfrak{F}_{\tau_2}$.
   (d) Show that, if $\tau_n \downarrow \tau$, then $\mathfrak{F}_\tau = \bigcap_n \mathfrak{F}_{\tau_n}$.

2. The "tail σ-algebra" w.r.t. standard Brownian motion $B_t(\omega) = \omega(t)$ on $C[0,\infty)$ is defined as
   $$\mathfrak{T} := \bigcap_{t>0} \sigma\big(\{B_s \mid s \geq t\}\big)\,.$$
   Show that, for any $A \in \mathfrak{T}$, $\mathbb{P}(A) \in \{0,1\}$.

3. Let $(\mathfrak{F}_t)_{t\geq 0}$ be a right-continuous filtration, B an $(\mathfrak{F}_t)$-Brownian motion and τ a finite $(\mathfrak{F}_t)$-stopping time. Show that $X_t := B_{\tau+t} - B_\tau$ defines a Brownian motion, which is independent of $\mathfrak{F}_\tau$.

4. Consider the Markov family $\mathbb{X}$ given by the space $\Omega := \{\omega : \mathbb{R}_+ \to \mathbb{R} \mid \omega(t) = a + bt \textit{ for some } a, b \in \mathbb{R}\}$ equipped with the σ-algebra $\mathfrak{F} := \sigma(X_s : s \geq 0)$, the process $X_t(\omega) := \omega(t)$ for $\omega \in \Omega$, the canonical family of time-shift to the right $(\theta_t)_{t\geq 0}$, the filtration $(\mathfrak{F}_t)_{t\geq 0} := (\sigma(X_s : s \leq t))_{t\geq 0}$ and the family of probability measures $\mathbb{P}_x$ on Ω. For $x \in \mathbb{R}$, let $\mathbb{P}_x$ be defined by: $\mathbb{P}_x(\{t \mapsto -t\}) = 1$ for $x < 0$, $\mathbb{P}_x(\{t \mapsto t\}) = 1$ for $x > 0$ and $\mathbb{P}_0(\{t \mapsto -t\}) = \mathbb{P}_0(\{t \mapsto t\}) = \frac{1}{2}$.
   (a) Show that $\mathbb{X}$ satisfies the Markov property.
   (b) Compute the corresponding transition function $(P_t)_{t\geq 0}$.
   (c) Prove that, in general, $x \mapsto (P_t f)(x)$ is not in $C_b(\mathbb{R})$ for $f \in C_b(\mathbb{R})$.

Deadline: Tuesday, 04.12.2018. Hand in in groups, please!