Exercises for Stochastic Processes

1. Let $(\mathfrak{F}_t)_{t\geq 0}$ be an arbitrary filtration. Show that $(\mathfrak{F}_{t+})_{t\geq 0}$, given by

$$\mathfrak{F}_{t+} := \cap_{s>t}\mathfrak{F}_s, \quad t \geq 0,$$

is a right-continuous filtration.

2. Let $(\mathfrak{F}_t)_{t\geq 0}$ be a right-continuous filtration.

   (a) Show that $\tau : \Omega \to \mathbb{R}_+ \cup \{\infty\}$ is a $(\mathfrak{F}_t)$-stopping time if and only if $\{\tau < t\} \in \mathfrak{F}_t$ for all $t \geq 0$.

   (b) Let $(\tau_n)_{n\in\mathbb{N}}$ be a sequence of $(\mathfrak{F}_t)$-stopping times. Show that $\sup_n \tau_n$, $\inf_n \tau_n$, $\limsup_n \tau_n$, $\liminf_n \tau_n$ and, if existent, $\lim_n \tau_n$ are $(\mathfrak{F}_t)$-stopping times.

3. Let B be a Brownian motion. Show that the random set of times at which B has local maxima is a.s. dense in $\mathbb{R}_+$.

   (Hint:
   First prove the following: A continuous function on $[a, b]$ that is monotone in no subinterval of $[a, b]$ has a local maximum in (a, b). Then show that the paths of Brownian motion are monotone in no interval a.s.)

4. Let B be a standard Brownian motion. Compute the distribution function of the stopping time $\tau_1 := \inf\{t \geq 1 \mid B_t = 0\}$.

   (Hint:
   You may use the following result without proof: Let $\tau^x := \inf\{t \geq 0 : x + B_t = 0\}$, then for $x \neq 0$

$$\mathbb{P}(\tau^x < t) = \int_0^t \frac{\mid x \mid}{\sqrt{2\pi z^3}} exp\{\frac{-x^2}{2z}\}dz.$$

   Use the Markov property to express the distribution of τ_1 in terms of the distribution of τ^x.)

Deadline: Tuesday, 27.11.2018. Hand in in groups, please!