Exercises for Stochastic Processes

1. (a) Let $C(\mu, \lambda)$ denote the Cauchy distribution with location parameter μ and scaling parameter λ, i.e. the absolutely continuous probabilty distribution with density function given by

$$f(x) = \frac{1}{\pi} \frac{\lambda}{\lambda^2 + (x-\mu)^2}.$$

Let $X \stackrel{d}{=} C(0, \lambda_1)$ and $Y \stackrel{d}{=} C(0, \lambda_2)$ be independent random variables. Show that $X + Y \stackrel{d}{=} C(0, \lambda_1 + \lambda_2)$.

(Hint: You may use without proof that the characteristic function of $C(0, \lambda)$ is given by $p \mapsto e^{-|p|\lambda}$.)

(b) Consider the following family of maps on $\mathbb{R}$ (equipped with the Borel-σ-algebra):

$$\begin{aligned} P_0(x, A) &= \delta_x(A), \\ P_t(x, A) &= \frac{1}{\pi} \int_A \frac{t}{t^2 + (x-y)^2} dy, \quad t \in (0, \infty). \end{aligned}$$

Show that $(P_t)_{t \geq 0}$ is a normal transition function.

2. Let N_t be a Poisson process ($N_0 = 0$) with intensity parameter λ and let $(X_n)_{n \in \mathbb{N}}$ be a sequence of i. i. d. random variables, which are absolutely continuous with some density function g and independent of N. Consider the *compound Poisson process* with starting point $x \in \mathbb{Z}$:

$$x + Y_t := x + \sum_{i=1}^{N_t} X_i.$$

(a) Show that Y is a Markov process.

(b) Compute $P_t f(x) := \mathbb{E}[f(x + Y_t)]$ for measurable bounded f. Show that $P_t f$ can be written as $e^{tA} f$ for some operator A depending on λ and g (the operator e^{tA} is given by the exponential series in the operator tA).

3. (a) Consider the state space $E = \{0, 1\}$. Show that $(p_t)_{t\geq 0}$ given by

$$p_t(0,0) = \frac{b}{a+b} + \frac{a}{a+b}e^{-t(a+b)}, \quad p_t(0,1) = \frac{a}{a+b} - \frac{a}{a+b}e^{-t(a+b)},$$
$$p_t(1,0) = \frac{b}{a+b} - \frac{b}{a+b}e^{-t(a+b)}, \quad p_t(1,1) = \frac{a}{a+b} + \frac{b}{a+b}e^{-t(a+b)},$$

where $0 < a, b \leq 1$, is a normal transition function.

(b) Consider a finite state, discrete-time Markov chain with transition matrix P. The corresponding transition function p_k is given by $p_k(x,y) = P^k_{x,y}$ (k-th power of P, $P^0 = Id$). Show that

$$\tilde{p}_t(x,y) := e^{-t}\sum_{n=0}^{\infty} \frac{t^n}{n!} p_n(x,y)$$

is a normal transition function.

Deadline: Tuesday, 20.11.2018. Hand in in groups, please!