Exercises for Stochastic Processes

1. Let $X = (X_t)_{t \geq 0}$ be a Gaussian process with stationary increments, such that $X_0 = 0$, $E[X_t] = 0$ and $Var(X_t) = t^2$ for all $t \geq 0$. Show that
$$X_t \overset{d}{=} tN$$
for a standard normal variable N.

2. (a) Let $X = (X_t)_{t \geq 0}$ be a Gaussian process with continuous sample paths. Show that
$$Y_t := \int_0^t X_s \, ds$$
is a Gaussian process as well. You may use the following without proof:

   Let $(N_k)_{k \in \mathbb{N}}$ be a sequence of univariate Gaussian random variables with means $(\mu_k)_{k \in \mathbb{N}}$ and variances $(\sigma_k^2)_{k \in \mathbb{N}}$, such that $N_k \overset{d}{\mapsto} N$ in the sense of weak convergence (convergence in distribution). Then $(\mu_k)_{k \in \mathbb{N}}$ and $(\sigma_k^2)_{k \in \mathbb{N}}$ converge to some values μ and σ^2. Furthermore, N is univariate Gaussian with mean μ and variance σ^2.

   (b) Assume that X is a Brownian motion. Compute the covariance kernel of Y.

3. (a) Let B be a Brownian motion and $0 < s < t$.
   Compute $\mathbb{P}(B_s > 0, B_t > 0)$!

   (b) Let B be a Brownian motion and $s, c > 0$. Show that
$$X_t := B_{s+t} - B_s$$
and
$$Y_t := \frac{B_{ct}}{\sqrt{c}}$$
(each defined for $t \geq 0$) also define Brownian motions!

Deadline: Tuesday, 06.11.2018. Hand in in groups, please!