Exercises for Stochastic Processes

1. On $\mathcal{D}(\mathcal{L}) = \{f \in C_0(\mathbb{R}) \mid f' \in C_0(\mathbb{R})\}$ define the operator $\mathcal{L}$ by $(\mathcal{L}f)(x) := f'(x)$. Prove that $\mathcal{L}$ is an infinitesimal generator by showing that it satisfies conditions $i)$, $ii)$ and $iii)$ from Theorem 6.5.

2. (a) For two independent exponentially distributed random variables $S \sim Exp(\alpha)$ and $T \sim Exp(\beta)$, $\alpha \neq \beta$, compute the density function of the sum $S + T$.

   (b) Given a semi-group P_t, let $U(\alpha)$ be the resolvent of P_t. Use part $a)$ to provide a probabilistic interpretation of the so-called resolvent equation:

$$U(\alpha) - U(\beta) = (\beta - \alpha)U(\alpha)U(\beta).$$

3. (a) Show that an infinitesimal generator is a closed operator, in the sense that its graph is a closed subset of $C_0(E) \times C_0(E)$ (with respect to the norm $\|(f, g)\| := \|f\|_\infty + \|g\|_\infty$ on the product space).

   (b) Show that there cannot be two infinitesimal generators $\mathcal{L}_1$, $\mathcal{L}_2$ on the same state space with $\mathcal{D}(\mathcal{L}_1) \subsetneq \mathcal{D}(\mathcal{L}_2)$ and $\mathcal{L}_1 = \mathcal{L}_2$ on $\mathcal{D}(\mathcal{L}_1)$.

4. Let Q be a conservative Q-matrix on a countable space E and define:

$$\mathcal{L}f(x) := \sum_{y \in E} q(x, y)(f(y) - f(x))$$

   on $\mathcal{D}(\mathcal{L}) = \{f \in C_0(E) \mid \sum_{y \in E} q(x, y)(f(y) - f(x))$ *is absolutely convergent for all* $x \in E$ *and* $\mathcal{L}f \in C_0(E)\}$.

   (a) Show that the finiteness of E is a sufficient condition for $\mathcal{L}$ to be an infinitesimal generator.

   (b) For $E = \mathbb{N}_0$ and the Q-matrix Q from Sheet 12, Exercise $2c)$, check for which choices of the sequence $(\delta_i)_{i \in \mathbb{N}_0}$ the properties $i)$, $ii)$ and $iii)$ from Theorem 6.5 hold.

Deadline: Tuesday, 05.02.2019. Hand in in groups, please!