Exercises for Stochastic Processes

In the following all Markov chains are assumed to correspond to standard transition functions and conservative Q-matrices.

1. Show that the invariant measure of an irreducible recurrent Markov chain is unique up to constant multiples.

   You may use the following result without a proof: Let X be an irreducible, recurrent chain on E with transition function p_t and let α be a positive function on E, such that

   $$\sum_{y \in E} p_t(x,y)\alpha(y) = \alpha(x).$$

   Then α is constant.

2. Let p_t be a standard transition function on a countable set E. Define the family of operators P_t, $t \geq 0$, by

   $$P_t f(x) = \sum_{y \in E} p_t(x,y) f(y)$$

   for $f \in C(E)$ (continuous functions).

   (a) Show that P_t is a Feller-Dynkin semi-group if E is finite.

   (b) Show that P_t is a Feller-Dynkin semi-group if E is infinite and, additionally,

   $$\lim_{x \to \infty} p_t(x,y) = 0 \qquad \forall y \in E, t \geq 0. \tag{1}$$

   (c) Take $E = \mathbb{N}_0$ and let the Q-matrix corresponding to p_t, Q, be given by $q(0,1) = 1$, $q(0,0) = -1$, $q(i,i-1) = \delta_i > 0$, $q(i,i) = -\delta_i$ for $i \geq 1$ and $q(i,j) = 0$ otherwise. Prove or disprove:

   $$\sum_{i \in \mathbb{N}_0} \frac{1}{\delta_i} < \infty \implies (1) \textit{ does not hold.}$$

3. Show that the action of the heat kernel (transition function of the Brownian motion) on $C_0(\mathbb{R})$ (continuous functions vanishing at infinity) defines a Feller-Dynkin semi-group.

Deadline: Tuesday, 29.01.2019. Hand in in groups, please!