Exercises for Stochastic Processes

1. (a) Conclude the proof of Theorem 5.7 from the lecture by showing that a standard transition function P_t is continuously differentiable assuming the existence of a continuous right derivative (which holds by the conservativeness of the corresponding Q-matrix).

   (b) Using the proof of Theorem 5.7 and part (a) of this exercise show that P_t also satisfies the Kolmogorov forward equations - if, additionally, the corresponding Q-matrix satisfies

$$\sup_{x\in E} c(x) < \infty.$$

For a weakly conservative Q-matrix Q, consider the family of sequences $P_t^{(n)}$ recursively defined by

$$P_t^{(0)}(x,y) := 0,$$
$$P_t^{(n+1)}(x,y) := \delta_{x,y} e^{-c(x)t} + \int_0^t e^{-c(x)(t-s)} \sum_{z:z\neq x} q(x,z) P_s^{(n)}(z,y)ds.$$

2. (a) For all $t \geq 0$, $n \in \mathbb{N}_0$ and $x, y \in E$, show by induction:
   (i) $P_t^{(n)}(x,y) \geq 0$,
   (ii) $\sum_{y\in E} P_t^{(n)}(x,y) \leq 1$,
   (iii) $P_t^{(n+1)}(x,y) \geq P_t^{(n)}(x,y)$.

   (b) Show that $\bar{P}_t(x,y) := \lim_{n\to\infty} P_t^{(n)}(x,y)$ exists, is differentiable in t and solves the Kolmogorov backward equations.

3. Prove Proposition 5.10 from the lecture, i.e., assuming that Q is a weakly conservative Q-matrix show that the identities

   (a) $P_t^{(n)}(x,y) = \mathbb{P}(X_t = y, N_t < n \mid X_0 = x)$
   (b) $\bar{P}_t(x,y) = \mathbb{P}(X_t = y, N_t < \infty \mid X_0 = x)$
   (c) $\sum_{y\in E} \bar{P}_t(x,y) = \mathbb{P}(N_t < \infty \mid X_0 = x)$

   hold, where X is the Markov chain corresponding to Q given by the construction from the lecture and $\bar{P}_t$ is defined as in $2(b)$.

Deadline: Tuesday, 15.01.2019. Hand in in groups, please!