Exercises for Stochastic Processes

1. (a) Let $(S, \mathfrak{S})$ be a measurable space and let $T \subset \mathbb{R}$ be uncountable. Show that the following identity holds for the product σ-algebra $\mathfrak{S}^T$:

$$\mathfrak{S}^T = \left\{\{f \in S^T \mid (f(t_1), f(t_2), \dots) \in A\} \mid t_1, t_2, \dots \in T, A \in \mathfrak{S}^{\{t_1, t_2, \dots\}}\right\}.$$

("All sets in the product σ-algebra are countably determined.")

(b) Consider the following subsets of the product space S^T:
i) $\{f \in S^T \mid f \textit{ is continuous}\}$,
ii) $\{f \in S^T \mid f \textit{ is bounded}\}$,
iii) $\{f \in S^T \mid f \textit{ is monotone increasing}\}$.
Are these sets measurable with respect to $\mathfrak{S}^T$? Prove or disprove!

2. Let X and Y be two (real-valued) continuous-time processes with right-continuous sample paths defined on the same probability space (Ω, F, P). Furthermore, let X be a modification of Y. Show that the two processes are indistinguishable.

("For processes with right-continuous paths, the notions of modifications and indistinguishability coincide.").

Definition (Stochastic continuity):
We call a real-valued continuous-time process X stochastically continuous if

$$\forall t \in T, \forall \epsilon > 0 : P(\mid X_{t+h} - X_t \mid > \epsilon) \to 0 \textit{ as } h \to 0.$$

Alternatively, this can be formulated as

$$X_{t+h} \xrightarrow{P} X_t \textit{ as } h \to 0,$$

where $\xrightarrow{P}$ denotes *convergence in probability.*

3. (a) Let X be a (real-valued) continuous-time process with continuous sample paths. Show that X is stochastically continuous in the sense of the above definition.

(b) Show that the converse (i.e., "stochastic continuity implies continuity of the sample paths") is not true - by finding a counterexample.

4. Let X be a (real-valued) continuous-time process with continuous sample paths defined on some probability space (Ω, F, P). Show that the following subsets of Ω are measurable with respect to F:

(a) $\{\omega \in \Omega \mid X_t(\omega) = 0 \textit{ for every } t \in T\}$
(b) $\{\omega \in \Omega \mid t \mapsto X_t(\omega) \textit{ is bounded}\}$
(c) $\{\omega \in \Omega \mid t \mapsto X_t(\omega) \textit{ is monotone increasing}\}$.

Deadline: Tuesday, 30.10.2018. Hand in in groups, please!