Note: each $P_t^{(n)}$ has matrix $\mathbb{1} \geq 0$ if $P_t^{(0)}$ has!

Proof of Theorem ~~5.9~~ 5.8. (Sketch)

(1) Define $P_t^{(n)}$ inductively by:

$$P_t^{(0)}(x,y) := 0$$

Liggett's formula notation →
$$P_t^{(n+1)}(x,y) = \delta_{x,y}\, e^{-c(x)t} + \int_0^t e^{-c(x)(t-s)} \sum_{z: z \neq x} q(x,z) P_s^{(n)}(z,y)\, ds$$

(2) Show by induction: $\forall t \geq 0,\ n \geq 0,\ x, y \in E$

Liggett, exercise 2.22

(a) $P_t^{(n)}(x,y) \geq 0 \quad \forall t \geq 0, n \geq 0, x, y \in E$

(b) $\sum_y P_t^{(n)}(x,y) \leq 1$ ← use Q weakly conservative!

(c) $P_t^{(n+1)}(x,y) \geq P_t^{(n)}(x,y)$

(3) Conclude: $P_t^{(n)}(x,y) \nearrow \overline{P}_t(x,y)$ limit exists,

$\sum_y \overline{P}_t(x,y) \leq 1$, solves fixed pt equation

$$\overline{P}_t(x,y) = \delta_{x,y}\, e^{-c(x)t} + \int_0^t e^{-c(x)(t-s)} \sum_{z: z \neq x} q(x,z) \overline{P}_s(z,y)\, ds$$

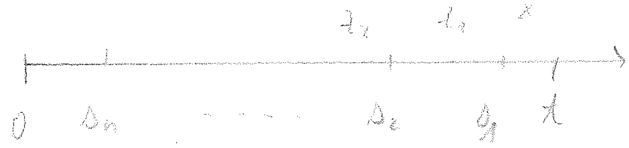
→ deduce differentiability & solution of Kolmogorov backward eqn.

(4) Satisfies Chapman-Kolmogorov:

Show by induction $\forall n \geq 1$

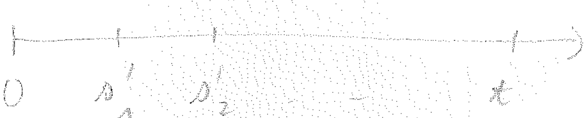
$$P_t^{(n)}(x,y) = \delta_{x,y}\, e^{-c(x)t} + \sum_{k=1}^{n} \int_0^t ds_1 \int_0^{s_1} ds_2 \cdots \int_0^{s_{k-1}} ds_k$$

$$\left\{ \sum_{\substack{z_1, \ldots, z_k \\ x \neq z_1, z_1 \neq z_2, \\ \ldots, z_k \neq y}} e^{-c(x)(t-s_1)} q(x,z_1) e^{-c(z_1)(s_1-s_2)} q(z_1,z_2) \cdots q(z_k,y) e^{-s_k c(y)} \right\}$$



picture unnatural

change variables in integral: $s_k' = t - s_1$



$x \neq z_1, z_1 \neq z_2, \dots, z_k \neq y$

$$P_t^{(n)}(x,y) = \Big\{ \delta_{x,y} e^{-c(x)t} + \sum_{k=1}^{n} \int_{0 \le s_1' < \dots < s_k' \le t} \sum_{z_1, \dots, z_k}^{(1)} e^{-s_1' c(x)} q(x, z_1) \; e^{-(s_2' - s_1') c(z_1)} \dots \, ds_1' \dots ds_n' \Big\}$$

Let $n \uparrow \infty$:

$$\bar{P}_t(x,y) = \delta_{x,y} e^{-c(x)t} + \sum_{k=1}^{\infty} \int_{0 \le s_1' < \dots < s_k' \le t} \dots \quad \dots$$

Similarly:

$$\bar{P}_s(x,y) = \delta_{x,y} e^{-c(x)(t+s-t)} + \sum_{\ell=1}^{\infty} \int_{t \le s_1'' < \dots < s_\ell'' < t+s} \sum_{z_1'', \dots, z_\ell''} e^{-(s_1'' - t) c(z_1'')} \dots \quad \dots$$

Use it to deduce Chapman-Kolmogorov

(5) • Also solves Kolmogorov forward equation:

Show that $P_t^{(n)}$ also satisfies recursion

$$P_t^{(n)}(x,y) = e^{-c(x)t} \delta_{x,y} + \int_0^t \sum_{\substack{z \in E \\ z \neq y}} \left(P_s^{(n)}(x,z) \, q(z,y) \right) e^{-c(y)(t-s)} \, ds.$$

...

(6) Every standard transition fct with Q-matrix Q satisfies $P_t(x,y) \geq \overline{P}_t(x,y)$:
show by induction $P_t(x,y) \geq P_t^{(n)}(x,y)$ $\forall n \in \mathbb{N}_0$
(trivial for $n=0$, use integral fixed pt eqn for induction step) $(\square)$

Corollary 5.9

Q conservative Q-matrix, $(\overline{P}_t)$ minimal solution
If $(\overline{P}_t)$ is Markov transition fct, then there exists a unique Markov trans. fct. with Q-matrix Q.

Proof: $\overline{P}_t(x,y) \leq P_t(x,y)$ (1)

$$! \; 1 = \sum_y \overline{P}_t(x,y) \leq \sum_y P_t(x,y) \underset{(=)}{\leq} 1$$

$$\Rightarrow \sum_y \overline{P}_t(x,y) = \sum_y P_t(x,y) \qquad (2)$$

(1) and (2) imply that $\overline{P}_t(x,y) = P_t(x,y)$ $\forall x,y$. $\square$

NOTE: we haven't yet said anything about sample path properties!

5.4.2 Probabilistic interpretation of minimal solution

! → Q conservative

Remember

$$\overline{P}_t(x,y) = \delta_{x,y} e^{-c(x)t} + \sum_{k=1}^{\infty} \int_{0 \leq s_1 < \dots < s_k < t} \sum_{z_1, \dots, z_k} \dots$$

Interpretation was:

- exponential random variables $\mathrm{Exp}(c(x))$
- discrete-time jump chain with transition matrix

$$\Pi(x,y) = \begin{cases} 0 & x = y,\ c(x) > 0 \\ \frac{q(x,y)}{c(x)} & x \neq y,\ c(x) > 0 \end{cases} \quad \Big| \quad \Pi(x,y) = \delta_{xy} \text{ if } c(x) = 0.$$

<u>More precisely:</u>

Suppose we are given a prob space $(\Omega, \mathcal{F}, \mathbb{P})$ together with random variables

$Z_n : \Omega \to E \quad n \in \mathbb{N}_0$

$\tau_n : \Omega \to \mathbb{R}_+$

Such that:

(1) $\mu(x) := \mathbb{P}(Z_0 = x) > 0 \quad \forall x \in E$

(2) $(Z_n)_{n\in\mathbb{N}_0}$ is discrete-time Markov with transition matrix $\Pi(x,y)$ i.e.: $\forall n \in \mathbb{N}, x_0, \ldots, x_n \in E$

$$\mathbb{P}_x(Z_0 = x_0, \ldots, Z_n = x_n) = \mu(x_0)\Pi(x_0,x_1)\cdots\Pi(x_{n-1},x_n)$$

(3) Conditional on $Z_0, Z_1, \ldots$, the τ_n's are independent, exponentially distributed, with parameter $c(Z_n)$, i.e.:
for all $z_0, z_1, \ldots \in E$ and $\mathbb{P}_x$-almost all $\omega \in \{Z_0 = z_0, Z_1 = z_1, Z_2 = z_2, \ldots\}$

$$\mathbb{P}(\tau_0 \geq t_0, \tau_1 \geq t_1, \ldots, \tau_n \geq t_n \mid Z_0, Z_1, \ldots)(\omega)$$
$$= \left(\int_{t_0}^{\infty} c(z_0) e^{-c(z_0)t} dt\right) \cdots \left(\int_{t_n}^{\infty} c(z_n) e^{-c(z_n)t} dt\right)$$

(Note: such a tuple $(\Omega, \mathcal{F}, \mathbb{P}, (Z_n)_{n\in\mathbb{N}_0}, (\tau_n)_{n\in\mathbb{N}_0})$ exists!)

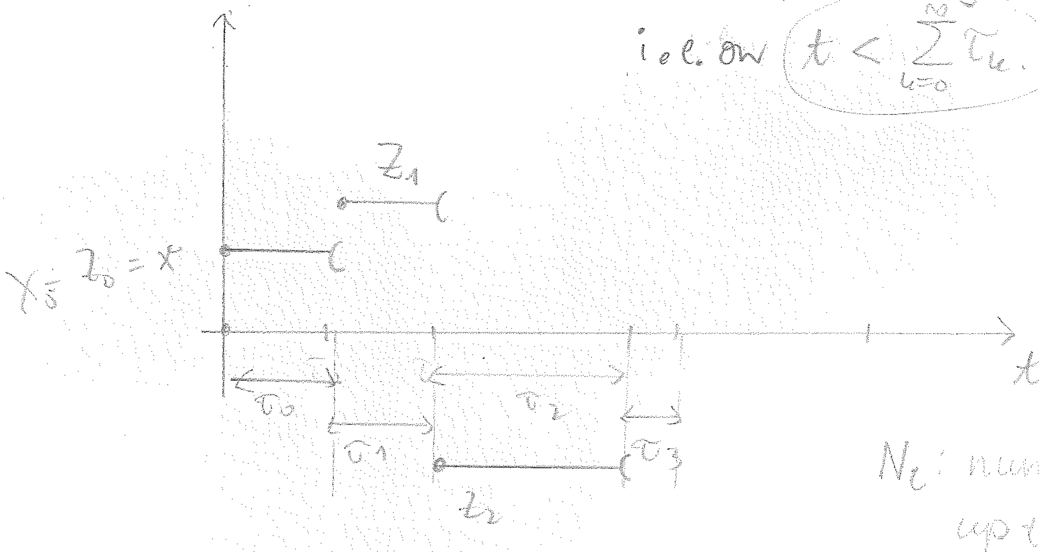
Define

$$N_t := \begin{cases} \min\{n \geq 0 \mid \tau_0 + \ldots + \tau_n \geq t\} & \text{if } \sum_{k=0}^{\infty} \tau_k > t \\ \infty & \text{else} \end{cases}$$

Think τ_0 = time to first jump | N_t = number of
$j \geq 1$ τ_j = time between jump no $j-1$ and no j | jumps up to time t.

and $X_t := Z_{N_t}$ on $\{N_t < \infty\}$

i.e. on $t < \sum_{k=0}^{\infty} \tau_k$.



N_t: number of jumps up to time t

If $\mathbb{P}\left(\sum_{k=0}^{\infty} \tau_k < \infty\right) > 0$: Process with finite lifetime $\zeta := \sum_{k=0}^{\infty} \tau_k$. Can be turned into proper process $(X_t)_{t \geq 0}$ with state space $E \cup \{\partial\}$, $\partial \notin E$ "coffin", "cemetery". $X_t(\omega) := \partial$ if $t \geq \zeta(\omega)$

Proposition 5.10

Q conservative, $(\bar{P}_t)$ minimal solution, $(\Omega, \mathcal{F}, \mathbb{P}, (Z_n), (\tau_n))$ as above; then:

(a) $P_t^{(n)}(x,y) = \mathbb{P}(X_t = y, N_t < n \mid X_0 = x)$

(b) $\bar{P}_t(x,y) = \mathbb{P}(X_t = y, N_t < \infty \mid X_0 = x)$

(c) $\sum_{y \in E} \bar{P}_t(x,y) = \mathbb{P}(N_t < \infty \mid X_0 = x)$

Proof: exercise!

Vocabulary $(Z_n)_{n \in \mathbb{N}}$ is called embedded discrete-time Markov chain.

Theorem 5.11 Q conservative, $(\bar{P}_t)_{t \geq 0}$ minimal solution, $(\Omega, \mathcal{F}, \mathbb{P}, (Z_n)_{n \in \mathbb{N}_0}, (\tau_n)_{n \in \mathbb{N}_0})$

The following are equivalent.

(a) $(\bar{P}_t)$ is Markov (= "stochastic")

(b) $\mathbb{P}(N_t < \infty) = 1 \quad \forall t \geq 0$

(c) $\sum_{k=0}^{\infty} \tau_k = \infty$ a.s.

(d) $\sum_{k=0}^{\infty} \frac{1}{c(Z_k)} = \infty$ a.s.