Markov processes

Defining equations

- Simple Markov:

$$\mathbb{E}[f(X_t) \mid \mathcal{F}_s] = \mathbb{E}[f(X_t) \mid X_s] \quad \mathbb{P}\text{-a.s.}$$

- Simple Markov with transition function $(P_t)_{t \geq 0}$:

$$\mathbb{E}[f(X_t) \mid \mathcal{F}_s] = (P_{t-s}f)(X_s). \quad \mathbb{P}\text{-a.s.}$$

- Normal Markov family:
  - $\mathbb{E}_x[f(X_t) \mid \mathcal{F}_s] = (P_{t-s}f)(X_s) \quad \mathbb{P}_x\text{-a.s}, \forall x \in E$
  - $(P_t f)(x) = \mathbb{E}_x[f(X_t)]$ $\qquad (= \mathbb{E}_{X_0}[f(X_{t-s})]$
  - $X_0 = x \quad \mathbb{P}_x\text{-a.s.}, \forall x \in E.$

- Strong Markov family:
  - normal, right-continuous filtration
  - $\mathbb{E}_x[f(X_{\tau+h}) \mid \mathcal{F}_\tau] = (P_h f)(X_\tau)$

    $\mathbb{P}_x$-a.s. on the event $\{\tau < \infty\}$.

    for all $(\mathcal{F}_t)$-stopping times τ.

Heuristic formula for transition function:

$$P_t(x, A) = \mathbb{P}(X_t \in A \mid X_0 = x).$$

$$(= \mathbb{P}(X_{s+t} \in A \mid X_s = x)).$$

Bonus equations

Simple Markov:

$$\mathbb{E}[\, F((X_t)_{t\geq s}) \mid \mathcal{F}_s] = \mathbb{E}[\, F((X_t)_{t\geq s}) \mid X_s]$$

Normal Markov family:

- $$\mathbb{E}_x [\, f(X_{s+h}) \mid \mathcal{F}_s] = \mathbb{E}_{X_s}[\, f(X_h)]$$
- if $Y(\omega) = F((X_t(\omega))_{t\geq 0})$ for some $F \in b\mathcal{E}^{\otimes \mathbb{R}_+}$ (equiv: Y bounded RV, meas. wrt $\sigma(X_t, t\geq 0)$):

  $$\mathbb{E}_x [\, Y \circ \vartheta_s \mid \mathcal{F}_s] = \mathbb{E}_{X_s}[Y]$$