Chapter 2 - Brownian motion

2.1 Motivation

2.2 Processes with stationary independent increments

Definition + example: Poisson process

2.3 Multivariate Gaussians and Gaussian processes

- Definitions multivariate Gaussian
- uniquely determined by covariance matrix & mean, computation of characteristic function
- Def Gaussian process, existence of Gauss. process with given covariance kernel and mean

2.4 Definition of standard BM

- equivalence of 2 candidate def (Gaussian / stat indep inc ...)
- def + existence
- a remark on canonical ~~process~~ version & Wiener measure

2.5 Some properties

$Z_t = t B_{1/t}$

(sample path properties without proof: nowhere differentiable - locally Hölder cont. $\gamma < 1/2$ - local law of the iterated logarithm.)

2.4 Definition of Brownian motion

$I = \mathbb{R}_+$

Lemma 2.8 The matrix $(\min(t_i, t_j))_{1 \le i,j \le n}$ is positive semi-definite, for all $n \in \mathbb{N}$ and $0 \le t_1 < \dots < t_n$.

Proof:

a) probabilistic proof: let $Y_1, \dots, Y_n$ be indepent Gaussians with $Y_1 \sim \mathcal{N}(0, t_1)$ and $Y_2 \sim \mathcal{N}(0, t_2 - t_1), \dots, Y_n \sim \mathcal{N}(0, t_n - t_{n-1})$.

Define
$$\begin{pmatrix} \tilde{X}_1 \\ \vdots \\ \tilde{X}_n \end{pmatrix} := \begin{pmatrix} Y_1 \\ Y_1 + Y_2 \\ \vdots \\ Y_1 + \dots + Y_n \end{pmatrix} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 1 & 1 & 0 \dots & 0 \\ & & \ddots & \\ & & & 1 \end{pmatrix} \begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix}$$

Then for $i \le j$ $\quad \mathbb{E}[\tilde{X}_i] = 0$

$$\begin{aligned} \mathrm{Cov}(\tilde{X}_i, \tilde{X}_j) &= \mathbb{E}[\tilde{X}_i \tilde{X}_j] \\ &= \mathbb{E}[\tilde{X}_i^2] + \mathbb{E}[\tilde{X}_i (\tilde{X}_j - \tilde{X}_i)] \\ &= \mathbb{E}[Y_1^2] + \dots + \mathbb{E}[Y_i^2] + \underbrace{\cancel{\mathbb{E}[(Y_1 + \dots + Y_i)(Y_{i+1} + \dots + Y_j)]}}_{=0} \\ &= t_1 + (t_2 - t_1) + \dots + (t_i - t_{i-1}) \\ &= t_i \\ &= \min(t_i, t_j) \end{aligned}$$

Extends to $i > j$ by symmetry

$\rightarrow$ $(\min(t_i, t_j))_{i,j=1,\dots,n}$ is the covariance matrix of a random vector

$\rightarrow$ positive semi-definite. $\square$

Prop. 2.9 $(X_t)_{t\in\mathbb{R}_+}$ real-valued stoch proc.
The following two conditions are equivalent:
(a) (X_t) has stationary independent increments and $X_t \sim \mathcal{N}(0,t)$.
(b) (X_t) is a Gaussian process with $\mathbb{E}[X_t]=0$ and $Cov(X_s, X_t) = \min(s,t)$.

Proof: (a) $\Rightarrow$ (b): $0 \le t_1 < \dots < t_n$ $\quad Y_1 := X_{t_1} - X_0$, $\quad i\geq 2$: $Y_i := X_{t_i} - X_{t_{i-1}}$

$$\begin{pmatrix} X_{t_1} \\ \vdots \\ X_{t_n} \end{pmatrix} = \begin{pmatrix} \cancel{X_{t_1}} \; Y_1 \\ Y_1 + Y_2 \\ \vdots \\ Y_1 + \dots + Y_n \end{pmatrix}$$

~~notice~~ $Y_1 \Rightarrow$

have: • $Y_1, \dots, Y_n$ independent
• $X_0 \sim \mathcal{N}(0,0) \Rightarrow X_0 = 0$ a.s.
$\Rightarrow X_{t_1} - X_0 \overset{a.s.}{=} X_{t_1} \sim \mathcal{N}(0, t_1)$
stationary increments $\rightsquigarrow$ $Y_i \sim \mathcal{N}(0, t_i - t_{i-1})$

Then conclude as in Lemma 2.8. ✓

(b) $\Rightarrow$ (a): • For $s<t$, $X_t - X_s$ is Gaussian with mean zero and $Var(X_t - X_s) = t + s - 2\min(t,s) = t-s$,
thus $X_t - X_s \sim \mathcal{N}(0, t-s)$ stationary increments ✓

• $X_0 \sim \mathcal{N}(0,0)$ $\quad X_0 = 0$ a.s. $\quad X_t \overset{d}{=} X_t - X_0 \Rightarrow X_t \sim \mathcal{N}(0, t-0)$ ✓

• independent increments:

$$\underbrace{\begin{pmatrix} X_{t_2} - X_{t_1} \\ \vdots \\ X_{t_n} - X_{t_{n-1}} \end{pmatrix}}_{\in \mathbb{R}^{n-1}} = \underbrace{\begin{pmatrix} -1 & +1 & 0 & \cdots & & 0 \\ 0 & -1 & +1 & 0 & \cdots & 0 \\ & & & \ddots & & \\ 0 & \cdots & & 0 & -1 & 1 \end{pmatrix}}_{(n-1)\times n \text{ matrix}} \underbrace{\begin{pmatrix} X_{t_1} \\ \vdots \\ X_{t_n} \end{pmatrix}}_{\in \mathbb{R}^n}$$

$\nwarrow$ Gaussian

$\Rightarrow$ ~~(+4~~ vector of increments is Gaussian.

$\Rightarrow$ by ~~Lemm~~ Corollary 2.5, need only check that increments are uncorrelated:

Let $u < v \leq s < t$, then

$$\mathrm{Cov}(X_v - X_u, X_t - X_s) = \min(v,t) - \min(u,t) - \min(v,s) + \min(u,s)$$

$$= v - u - v + u = 0$$

$\curvearrowright$ uncorrelated ✓ $\square$

<u>Def. 2.10</u> A ~~A~~ real-valued stoch proc $(B_t)_{t \in \mathbb{R}_+}$ is called standard <u>Brownian motion</u> (or <u>Wiener process</u>) if

(i) it has ~~everywhere~~ continuous sample paths

(ii) it satisfies (a) or (b) (hence both) from Prop. 2.9.

Remark: some authors only ask for $\mathbb{P}$-a.s. continuous paths.

<u>Theorem 2.11</u>

Brownian motion exists.

<u>Proof:</u>
- By Lemma 2.8 and Prop. 2.7, there exists a ~~pro~~ Gaussian process $(X_t)_{t \geq 0}$ with $\mathbb{E}[X_t] = 0$ and $\mathrm{Cov}(X_s, X_t) = \min(s,t)$.
- Let $Z \sim \mathcal{N}(0;1)$, then $X_t - X_s \overset{\mathcal{D}}{=} \sqrt{t-s}\, Z$, $s < t$

and for all $\alpha > 0$,

$$\mathbb{E}[\,|X_t - X_s|^\alpha] = |t-s|^{\alpha/2} \underbrace{\mathbb{E}[\,|Z|^\alpha]}_{<\infty}.$$

Choose $\alpha > 2$, write $\frac{\alpha}{2} = 1 + \beta$ with $\beta > 0$,
set $C := \mathbb{E}[\,|Z|^\alpha]$, then

$$\mathbb{E}[\,|X_t - X_s|] \leq C\,|t-s|^{1+\beta} \quad \text{for all } s,t \geq 0.$$

→ condition of Theorem 1.13 satisfied.

→ (X_t) has a modification $(\tilde{X}_t)$ with continuous sample paths.
$(\tilde{X}_t)$ has the same fidis as (X_t) and therefore is Gaussian process, mean-zero, same covariance as $\tilde{X}_t$

→ $(\tilde{X}_t)$ is a Brownian motion.
In particular, Brownian motion exists. □

<u>Remark</u> "canonical version" of Brownian motion

$(B_t)_{t\geq 0}$ is called canonical version of Brownian motion
if (i) the underlying measurable space is

$$\Omega := C(\mathbb{R}_+) = \{\gamma : \mathbb{R}_+ \to \mathbb{R} \mid \gamma \text{ is continuous}\}$$

$$\mathcal{F} := \{A \cap C(\mathbb{R}_+) \mid A \in \mathcal{B}(\mathbb{R})^{\otimes \mathbb{R}_+}\}$$

(trace of the product σ-alg.) Borel

(ii) $B_t(\gamma) = \gamma(t)$ for all $t \geq 0$.

<u>Fact:</u> canonical version exists and is unique.
Measure $\mathbb{P}$ on $(C(\mathbb{R}_+), \mathcal{F})$ called <u>Wiener measure</u>.

2.5 Some properties of Brownian motion

(mostly without proofs, to be continued in later chapters)

- $(B_t)_{t\geq 0}$ standard BM. Define

  $X_t := B_{t+s} - B_s$, $s>0$ fixed

  $Y_t := \dfrac{B_{ct}}{\sqrt{c}}$, $c>0$.

  Then (X_t) and (Y_t) are also standard BM's

Prop. 2.12 $(B_t)_{t\geq 0}$ standard BM.

$$Z_t(\omega) := \begin{cases} t\, B_{1/t}(\omega), & t>0 \\ 0 & t=0. \end{cases}$$

Then
- $(Z_t)_{t\geq 0}$ is Gaussian with $\mathbb{E}[Z_t]=0$, $Cov(Z_t, Z_s) = \min(s,t)$
- $(Z_t)_{t\geq 0}$ has $\mathbb{P}$-a.s. continuous sample paths.

Proof

- $(Z_t)_{t\geq 0}$ is Gaussian process ✓

  $\mathbb{E}[Z_t] = 0$ for all $t\geq 0$ ✓

  $0<s\leq t$: $Cov(Z_s, Z_t) = st \min\left(\frac{1}{s}, \frac{1}{t}\right) = s\cancel{t}\,\frac{1}{\cancel{t}} = s$

  $0 = s < t$: $Cov(Z_s, Z_t) = Cov(0, Z_t) = 0 = \min(s,t)$

  same for $0=s=t$

  $\Rightarrow (Z_t)_{t\geq 0}$ Gaussian process with mean zero and covariance kernel $\min(s,t)$ ✓

  ($\Rightarrow (Z_t)_{t\geq 0}$ and $(B_t)_{t\geq 0}$ have same fidis)

- $t \mapsto Z_t(\omega)$ continuous in $(0,\infty)$, for all $\omega\in\Omega$
- $\Omega^* := \{\omega\in\Omega \mid \lim_{t\searrow 0} Z_t(\omega) = 0\}$

$$\Omega^* = \bigcap_{m\geq 1} \bigcup_{n\geq 1} \{\omega\in\Omega \mid |Z_t(\omega)|\leq \tfrac{1}{m} \text{ for all } t\in(0;\tfrac{1}{n}]\}$$

$$\Big\downarrow \text{ continuity in } (0;\infty)$$

$$= \underbrace{\bigcap_{m\geq 1} \bigcup_{n\geq 1} \underbrace{\{\omega\in\Omega \mid |Z_t(\omega)|\leq \tfrac{1}{m} \text{ for all } t\in \underbrace{(0;\tfrac{1}{n})\cap\mathbb{Q}}_{\text{countable}}\}}_{\text{measurable}}}$$

$\Rightarrow \Omega^*$ measurable ✓

$(B_t)_{t\geq 0}$ and $(Z_t)_{t\geq 0}$ have same fidis, & involves only a countable family of moments

$$\Rightarrow \mathbb{P}(\Omega^*) = \mathbb{P}\Big(\bigcap_{m\geq 1} \bigcup_{n\geq 1} \{\omega\in\Omega \mid |B_t(\omega)|\leq \tfrac{1}{m} \ \forall t\in(0;\tfrac{1}{n})\cap\mathbb{Q}\}\Big)$$

$$= \mathbb{P}(B_t\to 0 \text{ as } t\to 0)$$

$$= 1. \qquad \square$$

Further properties without proof:

- a.s., the sample paths are nowhere differentiable
  [Liggett, Thm 1.36]
- $\gamma < 1/2 \Rightarrow$ $\mathbb{P}$-a.s., sample paths locally Hölder-continuous with exponent γ, i.e.:

  $\forall T>0$ $\exists$ random variable V_T s.t.
  $$|B_t - B_s| \leq V_T\, |t-s|^{\gamma}$$
  (Scheutzow Skript BM Korollar 5.14)

- $$\limsup_{t\downarrow 0} \frac{B_t}{\sqrt{2t\log\log\frac{1}{t}}} = 1 \quad \text{a.s.}$$
  (Scheutzow Skript BM Satz 5.18)

and many more ...